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Sums of products of hypergeometric Bernoulli numbers

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ABSTRACT

We give a formula for sums of products of hypergeometric Bernoulli numbers. This formula is proved by using special values of multiple analogues of hypergeometric zeta functions.

1. Introduction and main results

The Bernoulli numbers B_n $(n = 0, 1, 2, \dots)$ are rational numbers defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \quad \bigl(|x| < 2\pi\bigr).$$

The following well-known formula for sums of two products of Bernoulli numbers is called *Euler's formula*:

$$\sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = -nB_{n-1} - (n-1)B_n \quad (n \geqslant 1). \tag{1.1}$$

Since $B_{2n+1} = 0$ for $n \geqslant 1$, Eq. (1.1) can be written as

$$\sum_{i=1}^{n-1} \binom{2n}{2i} B_{2i} B_{2n-2i} = -(2n+1)B_{2n} \quad (n \geqslant 2). \tag{1.2}$$

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These formulas (1.1) and (1.2) have been generalized in many directions. Dilcher [5] gave formulas for sums of N products of Bernoulli numbers ($N = 1, 2, 3, \ldots$), which generalize (1.2). Agoh and Dilcher [2,3] considered the following types of sums of products of Bernoulli numbers:

$$\sum_{\substack{i_1 \geqslant 0, \ldots, i_r \geqslant 0 \\ i_1 + \cdots + i_r = n}} \frac{n!}{i_1! \cdots i_r!} B_{m_1+i_1} \cdots B_{m_r+i_r}$$

for non-negative integers m_i ($1 \leqslant i \leqslant r$) and gave some formulas for them. Petojević [17] and Petojević and Srivastava [18] studied other types of sums of products of Bernoulli numbers. Many results on sums of products of analogues of Bernoulli numbers are also known. For example, sums of products of Carlitz's q-Bernoulli numbers [14,19] and sums of products of Kronecker's double series [15] were studied.

In the article [5], Dilcher also gave formulas for sums of products of Bernoulli polynomials and Euler polynomials. Here we state his formula on Bernoulli polynomials $B_n(x)$:

Theorem 1.1. *(See [5, Theorem 3].) Let $(x_1, \ldots, x_r) \in \mathbb{R}^r$ and $y = x_1 + \cdots + x_r$. Then the following identity holds*:

$$\sum_{\substack{i_1 \geqslant 0, \ldots, i_r \geqslant 0 \\ i_1 + \cdots + i_r = n}} \frac{n!}{i_1! \cdots i_r!} B_{i_1}(x_1) \cdots B_{i_r}(x_r)$$
$$= (-1)^{r-1} r \binom{n}{r} \sum_{i=0}^{r-1} \sum_{k=0}^{i} \binom{r-i-1+k}{k} \begin{bmatrix} r \\ r-i+k \end{bmatrix} (-y)^k \frac{B_{n-i}(y)}{n-i} \quad (n \geqslant r), \tag{1.3}$$

where $\left[\begin{smallmatrix} r \\ k \end{smallmatrix}\right]$ are unsigned Stirling numbers of the first kind defined by

$$x(x+1)\cdots(x+r-1) = \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} x^k. \tag{1.4}$$

Since $B_n(0) = B_n$, we obtain the following formula for sums of r products of the ordinary Bernoulli numbers by setting $x_1 = \cdots = x_r = 0$ in (1.3):

$$\sum_{\substack{i_1 \geqslant 0, \ldots, i_r \geqslant 0 \\ i_1 + \cdots + i_r = n}} \frac{n!}{i_1! \cdots i_r!} B_{i_1} \cdots B_{i_r} = (-1)^{r-1} \binom{n}{r} \sum_{i=0}^{r-1} \begin{bmatrix} r \\ r-i \end{bmatrix} \frac{r}{n-i} B_{n-i} \quad (n \geqslant r). \tag{1.5}$$

We note that this formula (1.5) was proved by Vandiver [20, Eq. (140)] and it gives Euler's formula (1.1) when $r = 2$.

It is known that formulas like (1.1), (1.2) and (1.3) can be proved by the method of multiple zeta functions (e.g., [4,6,7]). More precisely, these formulas can be obtained by expressing a certain multiple zeta function in two ways and comparing special values of them at non-negative integers. Chen [4, Theorems 3–5] gave some formulas, which include (1.3), for sums of products of generalized Bernoulli polynomials and Euler polynomials by this method. The main results of this paper, which is stated below, will be proved by essentially the same method.

Table 1
Values of $B_{N,n}$.

n	0	1	2	3	4	5	6	7	8
$B_{1,n}$	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$
$B_{2,n}$	1	$-\frac{1}{3}$	$\frac{1}{18}$	$\frac{1}{90}$	$-\frac{1}{270}$	$-\frac{5}{1134}$	$-\frac{1}{5670}$	$\frac{7}{2430}$	$\frac{13}{7290}$
$B_{3,n}$	1	$-\frac{1}{4}$	$\frac{1}{40}$	$\frac{1}{160}$	$\frac{1}{5600}$	$-\frac{1}{896}$	$-\frac{13}{19200}$	$\frac{7}{76800}$	$\frac{7453}{14784000}$
$B_{4,n}$	1	$-\frac{1}{5}$	$\frac{1}{75}$	$\frac{3}{875}$	$\frac{13}{26250}$	$-\frac{19}{78750}$	$-\frac{239}{918750}$	$-\frac{289}{3093750}$	$\frac{5689}{108281250}$
$B_{5,n}$	1	$-\frac{1}{6}$	$\frac{1}{126}$	$\frac{1}{504}$	$\frac{1}{2646}$	$-\frac{1}{31752}$	$-\frac{431}{4889808}$	$-\frac{31}{598752}$	$-\frac{262}{35756721}$

For a positive integer N, Howard [12,13] defined generalized Bernoulli numbers $B_{N,n}$ ($n = 0, 1, 2, \dots$) as

$$\frac{x^N/N!}{e^x - T_{N-1}(x)} = \sum_{n=0}^{\infty} \frac{B_{N,n}}{n!} x^n, \tag{1.6}$$

where $T_{N-1}(x)$ is the Taylor polynomial of e^x of degree $N-1$, i.e. $T_{N-1}(x) = \sum_{m=0}^{N-1} x^m/m!$. When $N = 1$, the numbers $B_{1,n}$ are nothing but the ordinary Bernoulli numbers B_n. We list the numbers $B_{N,n}$ for $1 \leqslant N \leqslant 5$ and $0 \leqslant n \leqslant 8$ in Table 1. Howard [13] himself referred to $B_{N,n}$ as $A_{N,n}$ and gave many congruences about them. For example, he proved the congruence $2B_{2,n} \equiv 1 \pmod 4$ for $n > 1$ [12, Theorem 4.1]. These numbers $B_{N,n}$ were revisited by Hassen and Nguyen [10,11] in the study of hypergeometric zeta functions, which are defined in the next section, and they call $B_{N,n}$ *hypergeometric Bernoulli numbers*.

Here we explain why these numbers $B_{N,n}$ are called *hypergeometric* Bernoulli numbers. For real numbers a and b with $b > 0$, the confluent hypergeometric function ${}_1F_1(a, b; x)$ is defined by the following infinite series:

$${}_1F_1(a, b; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \tag{1.7}$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1) & (n \geqslant 1), \\ 1 & (n = 0). \end{cases} \tag{1.8}$$

The confluent hypergeometric function ${}_1F_1(a, b; x)$ is a degenerate form of Gauss's hypergeometric function ${}_2F_1(a, b, c; x)$, and which arises as a solution of a certain differential equation called *Kummer's equation* (cf. [1, Chapter 13]). By definition, we have $e^x - T_{N-1}(x) = x^N {}_1F_1(1, N+1; x)/N!$ for any positive integer N. Hence Eq. (1.6) can be expressed as

$$\frac{1}{{}_1F_1(1, N+1; x)} = \sum_{n=0}^{\infty} \frac{B_{N,n}}{n!} x^n \tag{1.9}$$

and we may call $B_{N,n}$ hypergeometric Bernoulli numbers.

As a generalization of the left-hand side of (1.1) and (1.5), we set the sums of products of hypergeometric Bernoulli numbers as

$$S_{N,r}(n) := \sum_{\substack{i_1 \geqslant 0, \dots, i_r \geqslant 0 \\ i_1 + \cdots + i_r = n}} \frac{n!}{i_1! \cdots i_r!} B_{N,i_1} \cdots B_{N,i_r} \tag{1.10}$$

for positive integers N and r. Then Euler's formula (1.1) can be written as

$$S_{1,2}(n) = -nB_{n-1} - (n-1)B_n \quad (n \geqslant 1). \tag{1.11}$$

The purpose of this paper is to give formulas similar to (1.11) for general $S_{N,r}(n)$. The following is the main result of this paper, and it will be proved by the method of multiple zeta functions.

Main Theorem. *Let N and r be positive integers. For any integer $n \geqslant r-1$, we have*

$$S_{N,r}(n) = \frac{1}{N^{r-1}} \sum_{i=0}^{r-1} A_r^{(N)}\bigl(i; 1 + N(r-1) - n\bigr)(-1)^i \binom{n}{i} i! B_{N,n-i}, \tag{1.12}$$

where $A_r^{(N)}(i;s) \in \mathbb{Q}[s]$ $(0 \leqslant i \leqslant r-1)$ are polynomials defined by the following recurrence relation:

$$\begin{aligned} &A_1^{(N)}(0;s) = 1, \\ &A_r^{(N)}(i;s) = \frac{s-1}{r-1} A_{r-1}^{(N)}(i; s-N) + A_{r-1}^{(N)}(i-1; s-N+1) \quad (r \geqslant 2). \end{aligned} \tag{1.13}$$

Here $A_r^{(N)}(i;s)$ are defined to be zero for $i \leqslant -1$ and $i \geqslant r$.

It can be proved by induction on r that the degree of $A_r^{(N)}(i; 1 + N(r-1) - n)\binom{n}{i}$ is $r-1$ as a polynomial of n. Therefore we obtain the following corollary.

Corollary. *Let N and r be positive integers. The number $S_{N,r}(n)$ has the following expression:*

$$S_{N,r}(n) = \sum_{i=0}^{r-1} F_i(n) B_{N,n-i} \quad (n \geqslant r-1), \tag{1.14}$$

where each $F_i(X) \in \mathbb{Q}[X]$ is a polynomial of degree $r-1$, which depends only on N and r.

This paper is organized as follows. In Section 2 we review hypergeometric zeta functions defined by Hassen and Nguyen, and define their multiple analogues called multiple hypergeometric zeta functions. In Section 3 we prove the Main Theorem by the method of multiple hypergeometric zeta functions defined in Section 2. In the last Section 4 we give some examples. We give formula (1.5) from our Main Theorem, and give explicit formulas for sums of r products of hypergeometric Bernoulli numbers for $r = 2, 3$ and 4.

2. Multiple hypergeometric zeta functions

Let r be a positive integer and s be a complex variable. We consider the following multiple zeta function:

$$Z_r(s) = \sum_{m_1 \geqslant 1, \ldots, m_r \geqslant 1} \frac{1}{(m_1 + \cdots + m_r)^s} \quad \bigl(\Re(s) > r\bigr). \tag{2.1}$$

The right-hand side of (2.1) is absolutely convergent for $\Re(s) > r$. When $r = 1$, the function $Z_1(s)$ is nothing but the classical Riemann zeta function $\zeta(s)$. By the usual method, we can obtain the integral representation

$$Z_r(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{(e^x-1)^r}\, dx \quad \bigl(\Re(s) > r\bigr), \tag{2.2}$$

where $\Gamma(s)$ is the gamma function. This is a simple generalization of the integral representation of the Riemann zeta function:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x-1}\, dx \quad \bigl(\Re(s) > 1\bigr). \tag{2.3}$$

We note that the function $Z_r(s)$ is a special case of the Barnes multiple zeta function $\zeta_r(s;\alpha,\underline{\omega})$ defined by

$$\zeta_r(s;\alpha,\underline{\omega}) = \sum_{m_1 \geqslant 0,\dots,m_r \geqslant 0} \frac{1}{(\omega_1 m_1 + \cdots + \omega_r m_r + \alpha)^s} \quad \bigl(\Re(s) > r\bigr) \tag{2.4}$$

for $\underline{\omega} = (\omega_1,\dots,\omega_r) \in \mathbb{C}^r$ and $\alpha \in \mathbb{C}$ with $\Re(\omega_i) > 0$ $(1 \leqslant i \leqslant r)$ and $\Re(\alpha) > 0$. In fact, it is clear that $Z_r(s) = \zeta_r(s; r, (1,\dots,1))$. The right-hand side of (2.4) is also absolutely convergent for $\Re(s) > r$. It is known that the Barnes multiple zeta function $\zeta_r(s;\alpha,\underline{\omega})$ can be holomorphically continued to the whole plane except for simple poles at $s = 1, 2, \dots, r$. Moreover, its values at negative integers can be expressed in terms of Bernoulli numbers (see, e.g., [16, Theorem 1]).

For a positive integer N and a complex variable $s \in \mathbb{C}$ with $\Re(s) > 1$, Hassen and Nguyen [11] defined hypergeometric zeta functions $\zeta_N(s)$ as

$$\zeta_N(s) = \frac{\Gamma(N+1)}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{x^N {}_1F_1(1, N+1; x)}\, dx, \tag{2.5}$$

or equivalently,

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}\, dx. \tag{2.6}$$

The right-hand side of (2.6) is convergent for $\Re(s) > 1$. When $N = 1$, the right-hand side of (2.6) coincides with the integral representation (2.3) of the Riemann zeta function, i.e. $\zeta_1(s) = \zeta(s)$. Hassen and Nguyen [11] proved that hypergeometric zeta functions can be meromorphically continued to the whole plane and values of hypergeometric zeta functions at non-negative integers are expressed by hypergeometric Bernoulli numbers $B_{N,n}$.

Theorem 2.1. *(See [11, Theorem 3.3].) The function $\zeta_N(s)$ is analytic on the whole plane except for simple poles at $\{2-N, 3-N, \dots, 1\}$ whose residues are*

$$\operatorname*{Res}_{s=n} \zeta_N(s) = (2-n)\binom{N}{2-n} B_{N,1-n} \quad (2-N \leqslant n \leqslant 1).$$

Furthermore, for negative integers n less than $2-N$, we have

$$\zeta_N(n) = (-1)^{-n-N+1} \binom{1-n}{N}^{-1} B_{N,1-n}.$$

Remark 1. The right-hand side of (2.5) can be defined for any real number $N > 0$, hence $\zeta_N(s)$ can be defined for any real number $N > 0$. Hassen and Nguyen [9] focused particularly on the function $\zeta_{\frac{1}{2}}(s)$ and gave some analytic properties of it. In this paper we only consider the case where N is a positive integer.

Now we introduce multiple hypergeometric zeta functions. Let N and r be positive integers. For $s \in \mathbb{C}$ with $\Re(s) > 1 + N(r-1)$, multiple hypergeometric zeta functions are defined as

$$\zeta_{N,r}(s) = \frac{\Gamma(N+1)^r}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{(x^N {}_1F_1(1, N+1; x))^r}\, dx. \tag{2.7}$$

The right-hand side of (2.7) is absolutely convergent if $\Re(s) > 1 + N(r-1)$. When $r = 1$, the function $\zeta_{N,1}(s)$ is the ordinary hypergeometric zeta function $\zeta_N(s)$. When $N = 1$, the function $\zeta_{1,r}(s)$ is equal to $Z_r(s)$ because $x {}_1F_1(1, 2; x) = e^x - 1$. Therefore we can say that our functions $\zeta_{N,r}(s)$ are multiple analogues of hypergeometric zeta functions.

To investigate the sums of products of hypergeometric Bernoulli numbers, it is convenient to treat the following modified zeta function:

$$\tilde{\zeta}_{N,r}(s) := \frac{1}{\Gamma(2-s-N)\Gamma(N+1)^r} \zeta_{N,r}(s). \tag{2.8}$$

Since $\Gamma(1-s)\Gamma(s) = \pi / \sin(\pi s)$, the function $\tilde{\zeta}_{N,r}(s)$ has an expression

$$\tilde{\zeta}_{N,r}(s) = \frac{\sin(\pi(s+N-1))}{\pi} \int_0^\infty \frac{x^{s+N-2}}{(x^N {}_1F_1(1, N+1; x))^r}\, dx \tag{2.9}$$

for $\Re(s) > 1 + N(r-1)$.

Let us show that the function $\tilde{\zeta}_{N,r}(s)$ can be continued to an entire function by the contour integral method. For a complex variable s, we put

$$I_{N,r}(s) = \frac{1}{2\pi i} \int_\gamma \left(w^N {}_1F_1(1, N+1; w)\right)^{-r} (-w)^{s+N-1} \frac{dw}{w}. \tag{2.10}$$

Here the contour γ is taken to be along the real axis from ∞ to $\delta > 0$, counterclockwise around the circle of radius δ with center at the origin, and then along the real axis from δ to ∞. We let $-w$ have argument $-\pi$ when we are going towards the origin and argument π when we are going towards ∞. Moreover we suppose that δ is sufficiently small such that there are no roots of $w^N {}_1F_1(1, N+1; w)$ inside the circle of radius δ with center at the origin except for the trivial zero $w = 0$. Then the integral (2.10) converges for all s and $I_{N,r}(s)$ defines an entire function. We remark that the integral in (2.10) is independent of the choice of δ by Cauchy's theorem.

Proposition 2.2.

(i) *We have $I_{N,r}(s) = \tilde{\zeta}_{N,r}(s)$ for $\Re(s) > 1 + N(r-1)$. Therefore, the function $\tilde{\zeta}_{N,r}(s)$ can be continued to an entire function.*

(ii) *For an integer n, we have*

$$I_{N,r}\big(1 - N + (Nr - n)\big) = \begin{cases} (-1)^{Nr+n} S_{N,r}(n)/n! & (n \geqslant 0), \\ 0 & (n < 0). \end{cases} \tag{2.11}$$

Proof. (i) We decompose the integral path as follows:

$$\begin{aligned} I_{N,r}(s) &= \frac{1}{2\pi i}\int_{\infty}^{\delta} \bigl(x^N {}_1F_1(1,N+1;x)\bigr)^{-r} e^{(s+N-1)(\log x-\pi i)}\frac{dx}{x} \\ &\quad + \frac{1}{2\pi i}\int_{|w|=\delta} \bigl(w^N {}_1F_1(1,N+1;w)\bigr)^{-r}(-w)^{s+N-1}\frac{dw}{w} \\ &\quad + \frac{1}{2\pi i}\int_{\delta}^{\infty} \bigl(x^N {}_1F_1(1,N+1;x)\bigr)^{-r} e^{(s+N-1)(\log x+\pi i)}\frac{dx}{x}. \end{aligned} \tag{2.12}$$

Under the condition $\Re(s) > 1 + N(r-1)$, the second term of (2.12) vanishes when δ tends to zero. Therefore we have

$$\begin{aligned} I_{N,r}(s) &= \frac{e^{\pi i(s+N-1)} - e^{-\pi i(s+N-1)}}{2\pi i}\int_0^{\infty} \frac{x^{s+N-2}}{(x^N {}_1F_1(1,N+1;x))^r}\,dx \\ &= \frac{\sin(\pi(s+N-1))}{\pi}\int_0^{\infty} \frac{x^{s+N-2}}{(x^N {}_1F_1(1,N+1;x))^r}\,dx \\ &= \tilde{\zeta}_{N,r}(s) \end{aligned}$$

and the assertion holds.

(ii) When s is an integer, the first and third terms of (2.12) cancel each other. Therefore

$$\begin{aligned} I_{N,r}\bigl(1-N+(Nr-n)\bigr) &= \frac{1}{2\pi i}\int_{|w|=\delta} \frac{1}{{}_1F_1(1,N+1;w)^r}(-1)^{Nr-n}w^{-n}\frac{dw}{w} \\ &= \frac{(-1)^{Nr-n}}{2\pi i}\int_{|w|=\delta}\sum_{m=0}^{\infty}\frac{S_{N,r}(m)}{m!}w^{m-n}\frac{dw}{w}. \end{aligned} \tag{2.13}$$

By the residue theorem, this value is equal to $(-1)^{Nr+n}S_{N,r}(n)/n!$ for $n \geqslant 0$ and equal to zero for $n < 0$. $\square$

Since $\zeta_{N,r}(s) = \Gamma(2-s-N)\Gamma(N+1)^r\tilde{\zeta}_{N,r}(s)$, we can obtain the following theorem on the function $\zeta_{N,r}(s)$. This theorem is a generalization of Theorem 2.1.

Theorem 2.3.

(i) *The function $\zeta_{N,r}(s)$ defined by (2.7) can be holomorphically continued to the whole plane except for possible simple poles at $s = (1-N)+1, (1-N)+2, \dots, (1-N)+Nr$. The residue of $\zeta_{N,r}(s)$ at $s = (1-N)+n$ $(1 \leqslant n \leqslant Nr)$ is*

$$\frac{(N!)^r}{(n-1)!(Nr-n)!}S_{N,r}(Nr-n). \tag{2.14}$$

(ii) *Let u be an integer with $u \geqslant 0$. Then we have*

$$\zeta_{N,r}(1-N-u) = \frac{(-1)^u (N!)^r u!}{(Nr+u)!} S_{N,r}(Nr+u).$$

Proof. (i) By definition, we have $\zeta_{N,r}(s) = \Gamma(2-s-N)\Gamma(N+1)^r \tilde{\zeta}_{N,r}(s)$. The gamma factor $\Gamma(2-s-N)$ has simple poles at $s = (1-N)+i$ for $i \geqslant 1$ and $\zeta_{N,r}(s)$ vanishes when $s = (1-N)+i$ for $i \geqslant Nr+1$ by Proposition 2.2(ii). Therefore $\zeta_{N,r}(s)$ has possible simple poles at $s = (1-N)+1$, $(1-N)+2, \dots, (1-N)+Nr$.

The residue of $\zeta_{N,r}(s)$ at $s = 1-N+n$ $(1 \leqslant n \leqslant Nr)$ is equal to

$$\begin{aligned}
&\lim_{s \to 1-N+n} \big(s-(1-N+n)\big)\zeta_{N,r}(s) \\
&= \lim_{s \to 1-N+n} \big(s-(1-N+n)\big)\Gamma(2-s-N)(N!)^r \tilde{\zeta}_{N,r}(s) \\
&= \lim_{s \to 1-N+n} -(n+1-s-N)\frac{\Gamma(n+2-s-N)}{(2-s-N)\cdots(n+1-s-N)}(N!)^r \tilde{\zeta}_{N,r}(s) \\
&= \frac{(-1)^n}{(n-1)!}(N!)^r \tilde{\zeta}_{N,r}(1-N+n).
\end{aligned}$$

By Proposition 2.2(ii) again, this value is equal to

$$\frac{(N!)^r}{(n-1)!(Nr-n)!} S_{N,r}(Nr-n).$$

(ii) When $s = 1-N-u$ $(u \geqslant 0)$, we obtain from Proposition 2.2(ii) that

$$\begin{aligned}
\zeta_{N,r}(1-N-u) &= \Gamma\big(2-(1-N-u)-N\big)\Gamma(N+1)^r \tilde{\zeta}_{N,r}(1-N-u) \\
&= \frac{(-1)^u (N!)^r u!}{(Nr+u)!} S_{N,r}(Nr+u)
\end{aligned}$$

and this completes the proof. □

Remark 2. Since $B_{N,0} = 1$, we have $S_{N,r}(0) = 1$ for any N and r. Thus the function $\zeta_{N,r}(s)$ has a *proper* simple pole at $s = 1+N(r-1)$ with residue $(N!)^r/(Nr-1)!$. The author does not know whether or not the residue (2.14) of $\zeta_{N,r}(s)$ at $s = 1-N+n$ vanishes for some N, r and n.

3. Proof of the Main Theorem

In this section we prove our Main Theorem. First we give the following recurrence relation of $\tilde{\zeta}_{N,r}(s)$:

Lemma 3.1. *For $r \geqslant 2$, we have*

$$\tilde{\zeta}_{N,r}(s) = \frac{(-1)^N}{N}\left(\frac{s-1}{r-1}\tilde{\zeta}_{N,r-1}(s-N) + \tilde{\zeta}_{N,r-1}(s-N+1)\right). \tag{3.1}$$

Proof. It suffices to show (3.1) for $\Re(s) > 1 + N(r-1)$ because both sides of (3.1) are holomorphically continued to the whole plane. We recall the following properties of the confluent hypergeometric function:

$$\frac{d}{dx}\big(x^N {}_1F_1(1, N+1; x)\big) = Nx^{N-1} {}_1F_1(1, N; x), \tag{3.2}$$

$${}_1F_1(1, N; x) = 1 + \frac{x}{N} {}_1F_1(1, N+1; x) \tag{3.3}$$

for any $N > 0$. Then, for any $s \in \mathbb{C}$ with $\Re(s) > 1 + N(r-1)$, we have

$$\begin{aligned}
&\int_0^\infty \frac{x^{s+N-2}}{(x^N {}_1F_1(1, N+1; x))^r}\, dx \\
&\quad = \frac{1}{N}\left(\int_0^\infty \frac{Nx^{N-1} {}_1F_1(1, N; x)}{(x^N {}_1F_1(1, N+1; x))^r} x^{s-1}\, dx - \int_0^\infty \frac{1}{(x^N {}_1F_1(1, N+1; x))^{r-1}} x^{s-1}\, dx\right) \\
&\quad = \frac{1}{N}\int_0^\infty \left(\frac{1}{-r+1}\big(x^N {}_1F_1(1, N+1; x)\big)^{-r+1}\right)' x^{s-1}\, dx \\
&\qquad - \frac{1}{N}\int_0^\infty \frac{x^{s-1}}{(x^N {}_1F_1(1, N+1; x))^{r-1}}\, dx.
\end{aligned}$$

Calculating the first term by integration by parts, we obtain that

$$\begin{aligned}
&\int_0^\infty \frac{x^{s+N-2}}{(x^N {}_1F_1(1, N+1; x))^r}\, dx \\
&\quad = \frac{1}{N}\int_0^\infty \frac{s-1}{r-1} \frac{x^{s-2}}{(x^N {}_1F_1(1, N+1; x))^{r-1}}\, dx - \frac{1}{N}\int_0^\infty \frac{x^{s-1}}{(x^N {}_1F_1(1, N+1; x))^{r-1}}\, dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
\tilde{\zeta}_{N,r}(s) &= \frac{\sin(\pi(s+N-1))}{\pi}\int_0^\infty \frac{x^{s+N-2}}{(x^N {}_1F_1(1, N+1; x))^r}\, dx \\
&= \frac{s-1}{r-1} \cdot \frac{(-1)^N \sin(\pi((s-N)+N-1))}{N\pi}\int_0^\infty \frac{x^{(s-N)+N-2}}{(x^N {}_1F_1(1, N+1; x))^{r-1}}\, dx \\
&\quad - \frac{(-1)^{N-1} \sin(\pi((s-N+1)+N-1))}{N\pi}\int_0^\infty \frac{x^{(s-N+1)+N-2}}{(x^N {}_1F_1(1, N+1; x))^{r-1}}\, dx
\end{aligned}$$

$$= \frac{(-1)^N}{N}\left(\frac{s-1}{r-1}\tilde{\zeta}_{N,r-1}(s-N) + \tilde{\zeta}_{N,r-1}(s-N+1)\right)$$

and this completes the proof. □

Proposition 3.2. *For $r \geqslant 1$, we have*

$$\tilde{\zeta}_{N,r}(s) = \frac{(-1)^{N(r-1)}}{N^{r-1}} \sum_{i=0}^{r-1} A_r^{(N)}(i;s)\tilde{\zeta}_{N,1}\big(s - N(r-1) + i\big). \tag{3.4}$$

Proof. We prove (3.4) by induction on r. The case $r = 1$ is clear. We assume that the case $r-1$ holds. Then, by Lemma 3.1, we have

$$\begin{aligned}
\tilde{\zeta}_{N,r}(s) &= \frac{(-1)^N}{N}\left(\frac{s-1}{r-1}\tilde{\zeta}_{N,r-1}(s-N) + \tilde{\zeta}_{N,r-1}(s-N+1)\right) \\
&= \frac{(-1)^N}{N}\left(\frac{s-1}{r-1}\frac{(-1)^{N(r-2)}}{N^{r-2}}\sum_{i=0}^{r-2} A_{r-1}^{(N)}(i;s-N)\tilde{\zeta}_{N,1}\big(s-N-N(r-2)+i\big)\right. \\
&\quad \left. + \frac{(-1)^{N(r-2)}}{N^{r-2}}\sum_{i=0}^{r-2} A_{r-1}^{(N)}(i;s-N+1)\tilde{\zeta}_{N,1}\big(s-N+1-N(r-2)+i\big)\right) \\
&= \frac{(-1)^{N(r-1)}}{N^{r-1}}\left(\frac{s-1}{r-1}\sum_{i=0}^{r-2} A_{r-1}^{(N)}(i;s-N)\tilde{\zeta}_{N,1}\big(s-N(r-1)+i\big)\right. \\
&\quad \left. + \sum_{i=1}^{r-1} A_{r-1}^{(N)}(i-1;s-N+1)\tilde{\zeta}_{N,1}\big(s-N(r-1)+i\big)\right) \\
&= \frac{(-1)^{N(r-1)}}{N^{r-1}}\sum_{i=0}^{r-1}\left(\frac{s-1}{r-1}A_{r-1}^{(N)}(i;s-N) + A_{r-1}^{(N)}(i-1;s-N+1)\right) \\
&\quad \times \tilde{\zeta}_{N,1}\big(s-N(r-1)+i\big) \\
&= \sum_{i=0}^{r-1}\frac{(-1)^{N(r-1)}}{N^{r-1}} A_r^{(N)}(i;s)\tilde{\zeta}_{N,1}\big(s-N(r-1)+i\big).
\end{aligned}$$

Hence the case r also holds and this completes the proof. □

We are now in a position to prove the Main Theorem.

Proof of the Main Theorem. By Proposition 3.2, we have

$$\begin{aligned}
&\tilde{\zeta}_{N,r}\big(1+N(r-1)-n\big) \\
&\quad = \frac{(-1)^{N(r-1)}}{N^{r-1}}\sum_{i=0}^{r-1} A_r^{(N)}\big(i;1+N(r-1)-n\big)\,\tilde{\zeta}_{N,1}\big(1-(n-i)\big) \\
&\quad = \frac{(-1)^{N(r-1)}}{N^{r-1}}\sum_{i=0}^{r-1} A_r^{(N)}\big(i;1+N(r-1)-n\big)(-1)^{N+n-i}\frac{B_{N,n-i}}{(n-i)!}
\end{aligned} \tag{3.5}$$

for $n \geqslant r-1$. On the other hand, by Proposition 2.2(ii), we have

$$\tilde{\zeta}_{N,r}\bigl(1+N(r-1)-n\bigr) = I_{N,r}\bigl(1+N(r-1)-n\bigr) = (-1)^{n+Nr} S_{N,r}(n)/n!. \tag{3.6}$$

By comparing (3.5) and (3.6), we obtain that

$$S_{N,r}(n) = \frac{1}{N^{r-1}} \sum_{i=0}^{r-1} A_r^{(N)}\bigl(i; 1+N(r-1)-n\bigr)(-1)^i \binom{n}{i} i! B_{N,n-i} \tag{3.7}$$

for $n \geqslant r-1$ and this proves (1.12). $\square$

4. Examples

In this section we give some examples. Let us first consider the case $N=1$ in the Main Theorem. In this case we can give the known formula (1.5).

We recall some properties of Stirling numbers, which are easily proved by (1.4):

$$\begin{bmatrix} r \\ r \end{bmatrix} = 1 \quad (r \geqslant 1), \tag{4.1}$$

$$\begin{bmatrix} r \\ k \end{bmatrix} + r\begin{bmatrix} r \\ k+1 \end{bmatrix} = \begin{bmatrix} r+1 \\ k+1 \end{bmatrix} \quad (r \geqslant 1,\ k \geqslant 0) \tag{4.2}$$

(cf. [8]). These formulas are used to prove the following lemma.

Lemma 4.1. *For $r \geqslant 1$ and $0 \leqslant i \leqslant r-1$, we have*

$$A_r^{(1)}(i; s) = \begin{bmatrix} r \\ r-i \end{bmatrix} \frac{\Gamma(s)}{(r-1)!\Gamma(s-r+1+i)}. \tag{4.3}$$

Proof. We prove the lemma by induction on r. By $A_1^{(1)}(0; s) = 1$ and (4.1), Eq. (4.3) holds for $r=1$ and $i=0$. We assume that (4.3) holds for some r and all $i = 0, 1, \ldots, r-1$. Then, by the recurrence relation (1.13) and the inductive assumption, we have

$$\begin{aligned} A_{r+1}^{(1)}(i; s) &= \frac{s-1}{r} A_r^{(1)}(i; s-1) + A_r^{(1)}(i-1; s) \\ &= \frac{s-1}{r} \begin{bmatrix} r \\ r-i \end{bmatrix} \frac{\Gamma(s-1)}{(r-1)!\Gamma((s-1)-r+1+i)} \\ &\quad + \begin{bmatrix} r \\ r-(i-1) \end{bmatrix} \frac{\Gamma(s)}{(r-1)!\Gamma(s-r+1+(i-1))} \\ &= \left(\begin{bmatrix} r \\ r-i \end{bmatrix} + r\begin{bmatrix} r \\ r-i+1 \end{bmatrix} \right) \frac{\Gamma(s)}{r!\Gamma(s-(r+1)+1+i)}. \end{aligned}$$

Using (4.2), we obtain

$$A_{r+1}^{(1)}(i;s)=\begin{bmatrix} r+1 \\ (r+1)-i \end{bmatrix}\frac{\Gamma(s)}{r!\Gamma(s-(r+1)+1+i)},$$

and this means (4.3) holds for the case $r+1$. □

Now we can deduce formula (1.5). In fact, we obtain from Lemma 4.1 that

$$\begin{aligned} A_r^{(1)}(i;r-n) &= \begin{bmatrix} r \\ r-i \end{bmatrix}\frac{\Gamma(r-n)}{(r-1)!\Gamma(-n+1+i)} \\ &= \begin{bmatrix} r \\ r-i \end{bmatrix}\frac{1}{(r-1)!}(-n+i+1)\cdots(-n+r-1) \\ &= \begin{bmatrix} r \\ r-i \end{bmatrix}\frac{(-1)^{r-i+1}}{(r-1)!}(n-i-1)\cdots(n-r+1) \\ &= \begin{bmatrix} r \\ r-i \end{bmatrix}\frac{(-1)^{r-i+1}}{(r-1)!}\frac{(n-i-1)!}{(n-r)!} \end{aligned}$$

for $n \geqslant r$. By this equation and our Main Theorem, we get

$$\begin{aligned} S_{1,r}(n) &= \sum_{i=0}^{r-1} A_r^{(1)}(i;r-n)(-1)^i\binom{n}{i} i! B_{n-i} \\ &= \sum_{i=0}^{r-1}\begin{bmatrix} r \\ r-i \end{bmatrix}(-1)^{r+1}\binom{n}{r}\frac{r}{n-i}B_{n-i} \end{aligned}$$

and this proves formula (1.5).

We end this paper with examples of formulas for sums of products of hypergeometric Bernoulli numbers:

$$\begin{aligned} S_{N,2}(n) &= -\frac{1}{N}\big((n-N)B_{N,n}+nB_{N,n-1}\big) \quad (n \geqslant 1). \\ S_{N,3}(n) &= \frac{1}{2N^2}\big((n-N)(n-2N)B_{N,n} \\ &\quad -n(4N-3n+2)B_{N,n-1}+2n(n-1)B_{N,n-2}\big) \quad (n \geqslant 2). \\ S_{N,4}(n) &= -\frac{1}{6N^3}\big((n-N)(n-2N)(n-3N)B_{N,n} \\ &\quad +\big(6n^3-n^2(22N+8)+3n(2N+1)(3N+1)\big)B_{N,n-1} \\ &\quad -n(n-1)(-11n+18N+15)B_{N,n-2}+6n(n-1)(n-2)B_{N,n-3}\big) \quad (n \geqslant 3). \end{aligned}$$

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