SOOCHOW JOURNAL OF MATHEMATICS
Volume 23, No. 4, pp. 359-363, October 1997

ON PARTLY BILATERAL AND PARTLY UNILATERAL GENERATING RELATIONS

BY

M. KAMARUJJAMA, M. A. HUSSAIN AND F. AFTAB

Abstract. The main object of the present paper is to derive a new class of generating functions which is partly bilateral and partly unilateral. Some interesting special cases of generating functions involving the product of three generalized hypergeometric polynomials are also considered.

1. Introduction

Hubbell and Srivastava [3] introduced an interesting generalization of generalized hypergeometric polynomials, defined by

$$\omega_N^{\nu}(x)=(\nu)_N\sum_{k=0}^{\infty}\frac{\Omega_k x^{N-2k}}{(1-\nu-N)_k}, \tag{1.1}$$

where $\{\Omega_n\}_{n=0}^{\infty}$ is a suitably bounded sequence and the parameters ν and N are unrestricted, in general.

In fact by suitably choosing the coefficients Ω_n in (1.1), the generalized polynomial can be applied to numerous other hypergeometric polynomials. The following special cases of (1.1) are given below :

(i) Setting $\Omega_k=\frac{(-r)_{Rk}(\alpha_1)_k\cdots(\alpha_p)_k}{k!(\beta_1)_k\cdots(\beta_q)_k}$;

$$\omega_N^{-\nu}(x)=(-\nu)_N x^N\mathcal{L}_{r,R}\Big[(\alpha_p);(\beta_q),(1+\nu-N);x^{-2}\Big], \tag{1.2}$$

Received December 5, 1996; revised April 18, 1997.
AMS Subject Classification. Primary 33C45; secondary 33C99.
Key words. generalized hypergeometric polynomials and functions, extended Jacobi and Laguerre polynomials, Gegenbauer polynomial.

where the generalized hypergeometric polynomial is given by [8; p. 107 (1.12)].

$$\mathcal{L}_{r,R}\Big[(\alpha_p);(\beta_q),1+\nu-N:z\Big] = {}_{R+p}F_{q+1}\left[\begin{matrix}\Delta(R;-r),(\alpha_p); \\ \\ 1+\nu-N,(\beta_q);\end{matrix}\; z\right], \tag{1.3}$$

with, as usual, $\Delta(N,\mu)$ represents the array of N parameters $(\mu+j-1)/N, \quad j = 1,2,\ldots N;\ N \geq 1$.

(ii) Choosing

$$\Omega_K = \frac{(-r)_{Rk}(\lambda+r)_{Rk}(\alpha_1)_k\cdots(\alpha_p)_k}{k!(\beta_1)_k\cdots(\beta_q)_k};$$

$$\omega_N^{-\nu}(x) = (-\nu)_N x^N \mathcal{L}^{\lambda}_{r,R}\Big[(\alpha_p);(\beta_q),1+\nu-N;x^{-2}\Big], \tag{1.4}$$

where the generalized hypergeometric polynomials is defined by [8; p. 107 (1.11)].

$$\mathcal{F}^{\lambda}_{r,R}\Big[(\alpha_p);(\beta_q),1+\nu-N:x\Big] = {}_{2R+p}F_{q+1}\left[\begin{matrix}\Delta(R;-r),\Delta(R;\lambda+r),(\alpha_p); \\ 1+\nu-N,\quad(\beta_q);\end{matrix}\; x\right]. \tag{1.5}$$

Pathan and Yasmeen [4] modified Exton's [2; p. 147 (3)] result by defining

$$\begin{aligned} m^* &= \max\{0,-m\} \quad \text{and} \\ F_n^m(x) &= L_n^m(x)/(m+n)! = \frac{1}{n!}\sum_{r=m^*}^{n}\frac{(-n)_r x^r}{(m+r)!r!}, \qquad \text{if } n \geq m^* \\ &= 0 \quad \text{if } 0 \leq n < m^* \quad (\text{i.e. if } n+m \leq 0 \leq n). \end{aligned} \tag{1.6}$$

All factorials occurring in this definition have meaning.

Now an interesting double generating function can be written as

$$\exp(s+t-xt/s) = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} s^m t^n F_n^m(x), \tag{1.7}$$

by using the modified definition of $F_n^m(x)$. The fact that generating relation of the type (1.7) for many classes of polynomials are generally not known, suggests that a set of generating relations also exists which may be obtained in a similar manner. In an attempt to obtain such relations, we have found a new generating relation involving the product of three generalized polynomials which is partly bilateral and partly unilateral.

$$\omega_M^\nu(y)\omega_N^\mu(z)\omega_R^\eta(-\frac{xz}{y})$$
$$= (\nu)_M(\mu)_N(\eta)_R y^{M-R} z^{N+R}(-x)^R \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{y^{-2m}z^{-2n}}{(1-\nu-M)_m(1-\mu-N)_n}$$
$$\sum_{k=0}^{\infty}\frac{(\mu+N-\eta)_k\Omega_k'''\Omega_{m+k}'\Omega_{n-k}''}{(1-\nu-M+m)_k(1-\eta-R)_k}(-x^{-2})^k, \tag{1.8}$$

provided that both sides of (1.8) exist.

2. Derivation of the Main Generating Function

If the function

$$V(x,y,z) = \omega_M^\nu(y)\omega_N^\mu(z)\omega_R^\eta(-\frac{xz}{y})$$

is expanded as a double series of powers y and z, we have

$$V(x,y,z)$$
$$= (\nu)_M(\mu)_N(\eta)_R \sum_{k=0}^{\infty}\frac{\Omega_k'''(-x)^{R-2k}}{(1-\eta-R)_k}\sum_{i=0}^{\infty}\frac{\Omega_i' y^{M-R-2i+2k}}{(1-\nu-M)i}\sum_{j=0}^{\infty}\Omega_j''\frac{z^{N+R-2j-2k}}{(1-\mu-N)j}.$$

Replace $i-k$ and $j+k$, respectively, by m and n, then after rearrangement justified by the absolute convergence as the above series. We are thus led finally to the generating relation (1.8).

Equation (1.8) gives many generating function for well known polynomials. We presenting some interesting special cases here.

3. Special Cases

On setting

$$\Omega_k''' = \frac{(-r)_{M_{3k}}((e_u))_k}{k!((f_v))_k}, \qquad \Omega_{m+k}' = \frac{(-P)_{M_1(m+k)}((a_i))_{m+k}}{(m+k)!((b_j))_{m+k}},$$
$$\Omega_{n-k}'' = \frac{(-q)_{M_2(n-k)}((c_l))_{n-k}}{(n-k)!((d_s))_{n-k}},$$

in (1.8) and using a relation (1.2) we get the following generating relation involving a product of three generalized hypergeometric polynomials:

$$\mathcal{L}_{p,M_1}[(a_i);(b_j),1+\nu-M:y^{-2}]$$
$$\mathcal{L}_{q,M_2}[(c_t);(d_s),1+\mu-N;z^{-2}]$$
$$\mathcal{L}_{r,M_3}[(e_u);(f_v),1+\eta-R;(\frac{xz}{y})^{-2}]$$
$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(-p)_{M_1m}(-q)_{M_2n}((a_i))_m((c_t))_n y^{-2m}z^{-2n}}{(1+\nu-M)_m(1+\mu-N)_n((b_j))_m((d_s))_n m!n!}$$
$$_{M_1+M_3+i+s+u+2}F_{M_2+j+l+v+3}\left[\begin{matrix}\Delta(m_1;-p+M_1m), & \Delta(M_3;-r), & (a_i)+m\\ \Delta(M_2;1+q-M_2n), & (b_j)+n, & 1-(c_l)-n\end{matrix}\right.$$
$$\left.\begin{matrix}1-(d_s)-n, & e_u, & \mu+N-n, & -n;\\ (f_v), & 1+\eta-R, & 1+\nu-M+m, & m+1;\end{matrix}(-1)^{M_2+l-s}x^{-2}\right]. \quad (3.1)$$

Again in (1.8), setting

$$\Omega_k'''=\frac{(-r)_{M_3k}(\lambda_3+r)_{M_3k}((e_u))_k}{k!((f_u))_k},$$
$$\Omega_{m+k}'=\frac{(-P)_{M_1(m+k)}(\lambda_1+p)_{M_1(m+k)}((a_i))_{m+k}}{(m+k)!((b_j))_{m+k}},$$
$$\Omega_{n-k}''=\frac{(-q)_{M_2(n-k)}(\lambda_2+q)_{M_2(n-k)}((c_t))_{n-k}}{(n-k)!((d_s))_{n_k}}.$$

Using a relation (1.4) and replacing y^{-2}, z^{-2} and x^{-2} respectively by y, z and $-x$, we get

$$\mathcal{F}_{M_1}^{\lambda_1}[(a_i);(b_j),1+\nu-M:y]\cdot\mathcal{F}_q^{\lambda_2}[(c_l);(d_s),1+\mu-N:z]\cdot$$
$$F_{r,M_3}^{\lambda_3}[(e_u);(f_\nu),1+\eta-R;-\frac{xz}{y}]$$
$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(-p)_{M_1m}(-q)_{M_2n}(\lambda_1+p)_{M_1m}(\lambda_2+q)_{M_2n}((c_t))_n Y^m Z^n}{(1+\nu-M)_m(1+\mu-N)_n((b_j))_m((d_s))_n m!n!}$$
$$_{2M_1+2M_3+i+s+u+2}F_{2M_2+j+l+v+3}\left[\begin{matrix}\Delta(M_1;-p+M_1m), & \Delta(M_1;\lambda_1+p+M_1m),\\ \Delta(M_2;1+q-M_2n), & \Delta(M_2;1-\lambda_2-q-M_2n),\end{matrix}\right.$$
$$\begin{matrix}\Delta(M_3;-r),\Delta(M_3;\lambda_3+r),(a_i)+m, & 1-(d_s)-n,\\ (b_j)+m, \quad 1-(c_l)-n, \quad (f_v), & 1+\nu-M+m,\end{matrix}$$

$$\left.\begin{matrix} e_u, & \mu+N-n,-n; & \\ & & (-1)^{l-s-1}x \\ 1+n-R, & m+1; & \end{matrix}\right] . \tag{3.2}$$

Acknowledgment

The authors are highly thankful to Prof. H. M. Srivastava of the University of Victoria, Victoria, Canada, for his kind help and many valuable suggestions in the preparation and improvement of this paper in the present form.

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Department of Mathematics, G. L. A. College, Ranchi University, Daltonganj-822102, Palamau (Bihar), India.

P. G. Department of Mathematics, H. D. Jain College, Veer Kunwar Singh University, Arrah 802 301, Bihar, Inida.

Department of Mathematics, Oriental College, Magadh University, Patna 800 008, Bihar, India.