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Generalized Binet Formulas, Lucas Polynomials, and Cyclic Constants

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Abstract. Generalizations of Binet's theorem are used to produce generalized Pell sequences from two families of silver means. These Pell sequences are also generated from the family of Fibonacci polynomials. A family of Pell-Lucas sequences are also generated from the family of Lucas polynomials and from another generalization of Binet's formula. A periodic set of cyclic constants are generated from the Lucas polynomials. These cyclic constants are related to the Gauss-Wantzel proof of the constructibility by compass and straightedge of regular polygons.

1. Introduction

In the study of dynamical systems and chaos theory (SCHROEDER, 1991), hydrocarbon chemistry (HOSOYA, 2005), studies of plant growth (see KAPPRAFF in another article in this issue), and the geometry of regular polygons (KAPPRAFF *et al.*, 2005), the golden mean and its close relatives make their appearance through certain sequences, graphs, and irrational numbers known as silver means. By sequences, we are referring to the Fibonacci sequence, 1, 1, 2, 3, 5, 8, ..., the Lucas sequence 1, 3, 4, 7, 11, 18, ... and their generalizations to Pell and Pell-Lucas sequences. Fibonacci sequences have also been generalized in other ways. KAPPRAFF (2002) discusses their generalizations to n-Fibonacci sequences. Kappraff *et al.*, in another article in this issue, generalize the Fibonacci sequence, beginning with 11 to sequences beginning with 111, 1111, ... The ratio of successive terms of these sequences approach the diagonals of regular n-gons with unit edge for n odd just as the diagonal of a regular pentagon with unit edge is the golden mean, these diagonals are generalizations of the golden mean to the diagonals of other regular n-gons form elaborate algebraic systems that we call "golden fields." Two families of silver means generalize the golden mean in a different way. As KAPUSTA has shown in this issue, silver means are the natural result of a tightly woven system of geometry of the circle and the square.

Le Corbusier pondered the mysteries of the golden mean and was moved to comment:

Behind the wall, the gods play; they play with numbers of which the universe is made up.

We have found that four families of polynomials are fundamental to this elegant mathematical system and helps to shed light on these mysteries. The first two families are called Fibonacci polynomials (F) of the first and second kind (KOSHY, 2001; KAPPRAFF, 2002). They originate from Pascal's triangle reconfigured so that their rows sum to the Fibonacci numbers. KOSHY (2001) refers to them as Fibonacci polynomials (see Appendix A). The *F polynomials of the first kind*, $K_n^{(1)}(x)$, have all positive coefficients. The *F polynomials of the second kind*, $K_n^{(2)}(x)$, have alternating signs. These polynomials are discussed in "Golden Fields" by Kappraff *et al.* in another article in this issue. The other two families of polynomials are the Lucas polynomials of the first and second kind. They are derived from a generalized form of Pascal's triangle as shown by Hosoya in another article in this issue and in Appendix B. The coefficients of the *Lucas polynomials of the first kind*, $L_n^{(1)}(x)$, sum to the Lucas numbers while the coefficients of *Lucas polynomials of the second kind*, $L_n^{(2)}(x)$, alternate in signs. KAPPRAFF and ADAMSON (2002) have shown that $L_n^{(2)}(x)$ represents the operator that generates the Mandelbrot set at the extreme point on the real axis in a region of full-blown chaos. Based on this insight, generalized Mandelbrot sets that resemble the standard one but somewhat more complex, were shown to be related to the other $L_n^{(2)}(x)$ polynomials of the Lucas family. In fact both the $K_n^{(2)}(x)$ and the $L_n^{(2)}(x)$ polynomials work together. When roots of the $K_n^{(2)}(x)$ are replaced into the $L_n^{(2)}(x)$ polynomials they lead to periodic trajectories of all lengths as KAPPRAFF and ADAMSON (2002) and KAPPRAFF *et al.* (2005), have shown Since we will not refer to Lucas polynomials of the first kind in this paper we omit the superscript and denote Lucas polynomials of the second kind by $L_n(x)$.

In this paper we introduce these sequences and numbers as players on a mathematical stage. Two families of silver means are introduced. By way of generalizations of Binet's Formula, these are used to generate Pell and Pell-Lucas sequences. These sequences are also generated directly from the F polynomials of the first and second kinds and the L polynomials of the second kind. Finally, Binet's formula is generalized still further by use of DeMoivre's theorem to a family of cosines that generate, from $L_n(x)$, a sequence of cyclic constants. These cyclic constants reproduce the energy levels of a cyclically conjugated hydrocarbon molecules, on the one hand, and information about the constructability of regular polygons with compass and straightedge according to the Gauss-Wantzel Theorem.

2. Generalized Binet's Formula and Silver Means of the Second Kind

Consider the solution T_N to the equation,

$$x - \frac{1}{x} = N \text{ for } N = 1. \tag{1}$$

T_N is called the *N-th silver mean of the first kind*, where for $T_1 = \tau$ for $\tau = (1+\sqrt{5})/2$, the *golden mean*. For arbitrary N as stated in KAPPRAFF (2002),

$$T_N = e^{\operatorname{arcsin} h \frac{N}{2}},$$

and when expanded as a continued fraction (see appendix B of Growth of Plants by KAPPRAFF, 2005) it takes the elegant form,

$$T_N = 1 + 1/N + 1/N + 1/N + 1/\ldots$$

KAPUSTA gives a geometric construction of the silver means of the first kind in another paper in this issue.

The well known *Binet's formula* states that,

$$F_k = \frac{\tau^k - (-\tau)^{-k}}{\tau + \tau^{-1}} \tag{2}$$

where F_k is the k-th number of the *Fibonacci sequence*,

$$1, 1, 2, 3, 5, 8, 13, \ldots \tag{3}$$

from which the ratio of successive terms approaches τ. It is also known that the following generalized form of Binet's formula holds for all values of N (KOSHY, 2001; KAPPRAFF and ADAMSON, 2003),

$$G_k^{(N)} = \frac{T_N^k - (-T_N)^{-k}}{T_N + T_N^{-1}} \tag{4}$$

where $G_k^{(N)}$ is the k-th term of a *generalized Pell sequence* satisfying the recursion,

$$G_{k+1}^{(N)} = NG_k^{(n)} + G_{k-1}^{(n)} \text{ where } G_0^{(N)} = 1 \text{ and } G_1^{(N)} = N. \tag{5}$$

When $N = 2$, $T_2 = \theta = 1 + \sqrt{2}$, the *silver mean*, and $G_k^{(2)}$ is the *Pell sequence*,

$$1, 2, 5, 12, 29, 70, \ldots \tag{6}$$

The ratio of successive terms in Sequence (6) approaches θ, whereas the ratio of successive terms of the generalized Pell sequences approach the N-th silver means. The first nine members of the family of *Fibonacci polynomials of the first kind*, $K_n^{(1)}(x)$, are listed in Appendix A. When N replaces x in the sequence of *Fibonacci polynomials of the first kind*, $K_n^{(1)}(x)$, the N-th generalized Pell sequence of Eq. (5) results since the recursion formula that generates $K_n^{(1)}(x)$ is,

$$K_{k+1}^{(1)}(x) = xK_k^{(1)}(x) + K_{k-1}^{(1)}(x) \text{ where } K_1^{(1)} = 1 \text{ and } K_2^{(1)} = x. \tag{7}$$

Consider another generalization of Binet's theorem. Denote by S_N, the solution to the equation,

$$x + \frac{1}{x} = N \text{ for } N = 3. \tag{8}$$

S_N is called the *N-th silver mean of the second kind* and is given in KAPPRAFF (2002) by the formula,

$$S_N = e^{\operatorname{arccos} h \frac{N}{2}}$$

and can be expanded in another elegant family of continued fractions as shown by KAPPRAFF (2002),

$$S_N = 1 + 1/N + 1/N + 1/N + 1/...$$

Binet's formula generates,

$$H_k^{(N)} = \frac{S_N^k - \left(-S_N\right)^{-k}}{S_N + S_N^{-1}} \tag{9}$$

where $H_k^{(N)}$ is the k-th term of a *generalized Pell sequence* satisfying the recursion,

$$H_{k+1}^{(N)} = NH_k^{(n)} - H_{k-1}^{(n)} \text{ where } H_0^{(N)} = 1 \text{ and } H_1^{(N)} = N \text{ for } N \geq 3. \tag{10}$$

The sequence for $N = 3$ is:

$$1, 3, 8, 21, 55, \ldots \tag{11}$$

The ratio of successive terms of this sequence approaches $S_3 = \tau^2$. If N is replaced in the Fibonacci polynomials of the second kind, $K_n^{(2)}(x)$, given by equation (3b) of "Golden Fields" by KAPPRAFF *et al.*, in another article in this issue, the family of generalized Pell sequences, $H_k^{(N)}$ is obtained for since the recursion relation governing the $K_n^{(1)}(x)$ is,

$$K_{k+1}^{(2)}(x) = xK_k^{(2)}(x) - K_{k-1}^{(2)}(x) \text{ where } K_1^{(2)} = 1 \text{ and } K_2^{(2)} = x. \tag{12}$$

We have proven in Appendix A to "Golden Fields" that $K_n^{(2)}$ are related to the derivatives of the Chebyshev polynomials. The ratio of successive terms of these $H_k^{(N)}$ sequences approach the silver means of the second kind, S_N.

3. Lucas Polynomials and Generalized Binet's Formula, and Silver Means of the Second Kind

Consider the sequence of *Lucas polynomials of the second kind* described by HOSOYA

(2005), KOSHY(2001), KAPPRAFF(2005), and defined by the recursion,

$$L_{k+1}(x) = xL_k(x) - L(x) \text{ where } L_1 = 1 \text{ and } L_2 = x. \tag{13}$$

The Lucas polynomials are generated from a generalized form of Pascal's equation as described in Appendix B where the first seven members of the family are listed. They are related to the Chebyshev Polynomials and satisfy the following important relation,

$$L_m(2\cos\theta) = 2\cos m\theta. \tag{14}$$

By setting $x = N$ for $N \geq 3$ in the Lucas polynomials, we generate a family of sequences with the following recursion,

$$L_{k+1}^{(N)} = NL_k^{(N)} - L_{k-1}^{(N)} \text{ where } a_0 = 2 \text{ and } a_1 = N. \tag{15}$$

For example, $N = 3$ results in the Pell-Lucas sequence,

$$L_k^{(3)}\text{: } 2, 3, 7, 18, 47, \ldots, \tag{16}$$

alternate terms from the Lucas sequence: 2, 1, 3, 4, 7, 11, 18, 29, ...

We are also able to generate Pell-Lucas sequences, $L_k^{(N)}$, directly from the following generalization of Binet's formula,

$$L_k^{(N)} = S_N^k + S_N^{-k} \text{ for } N = 3. \tag{17}$$

where $L_k^{(N)}$ is the k-th term of the N-th Lucas-Pell sequence. For example, replacing $S_3 = \tau^2$ into Eq. (17) results in Sequence (16).

The Pell and Pell-Lucas sequences have many interesting properties that have been extensively written about in the literature (KOSHY, 2001; KAPPRAFF, 2002). We will not go into them here.

4. Cyclic Constants in the Lucas Polynomials

Consider, the principal rot of unity, $E_N = \exp((2\pi)/(N)i)$ satisfying the equation,

$$x^N = 1.$$

Using *De Moivre's Theorem*,

$$\begin{aligned} C_j^{(N)} &= E_N^j + E_N^{-j} = 2\cos\frac{2\pi j}{N}, \text{ and} \\ S_j^{(N)} &= E_N^j - E_N^{-j} = 2i\sin\frac{2\pi j}{N}. \end{aligned} \tag{18}$$

In other words, E_N satisfies the equations,

$$x+\frac{1}{x}=2\cos\frac{2\pi}{N} \quad \text{and} \quad x-\frac{1}{x}=2i\sin\frac{2\pi}{N}. \tag{19}$$

There is an analogy between Eqs. (1), (8) and (18), on the one hand, and Binet's formula given by Eqs. (9) and (17).

If $x = 2\cos(2\pi/N)$ is now replaced in the Lucas polynomials $L_k(x)$ instead of N, a periodic sequence with period N results as a consequence of Eq. (14). For example, if $x = 2\cos(2\pi/7)$ the following sequence of *cyclic constants* results,

$$\begin{aligned}
C_0^7 &= 2\\
C_1^7 &= 2\cos\frac{2\pi}{7} = 1.24\\
C_2^7 &= 2\cos\frac{2\pi 2}{7} = -0.445\\
C_3^7 &= 2\cos\frac{2\pi 3}{7} = -1.801\\
C_4^7 &= 2\cos\frac{2\pi 4}{7} = -1.801\\
C_5^7 &= 2\cos\frac{2\pi 5}{7} = -0.445\\
C_6^7 &= 2\cos\frac{2\pi 6}{7} = 1.24\\
C_7^7 &= 2.
\end{aligned} \tag{20}$$

In Huckel molecular orbital method, Eq. (20) represents the energy levels of a cyclic conjugated hydrocarbon molecule (STREITWEISER, 1961; HEILBRONNER and BOCK, 1968; TANG *et al.*, 1986).

In general the sequence for any odd value of N is:

$$C_j^{(N)} = 2\cos\frac{2\pi j}{N} \quad \text{where} \quad C_{N-j}^{(N)} = C_j^{(n)} \quad \text{for} \quad 0 \le j \le k = \frac{N-1}{2}.$$

It is interesting that these periodic sequences arise from movement around the adjacent vertices of the regular N-gon, whereas replacing $2\cos(2\pi/7)$ into the Lucas polynomials, seen as the operator $x \mapsto L_k(x)$, gives rise to periodic trajectories $x_j = 2\cos k^j(2\pi/7)$. For example if $k = 2$ the operator is, $x \mapsto x^2 - 2$ and $2^j = 1, 2, 4, 8, \ldots$, i.e., the trajectory point progresses around the vertices in a geometric progression. This represents the operator of the Mandelbrot set at the leftmost point on the real axis, $c = -2$, a point of full-blown chaos. This is discussed in great detail in KAPPRAFF and ADAMSON (2005) and by Kappraff *et al.* in another paper in this issue.

5. Gauss' Theory of the Non-constructibility of Regular Polygons

Gauss showed that the heptagon cannot be constructed using an unmarked straightedge and compass whereas a pentagon and a 17-gon can. But also the nonagon can also not be constructed although Gauss did not prove this. It appears that in his proof of the constructability of certain regular polygons Gauss unknowingly derived one set of cyclic polynomials in his proof. We refer to the case for $N = 7$ in BOLD (1982) The Gauss-Wantzel proof is rather technical calling upon the elements of Galois theory. However the essence of the proof is to show that if the cyclotomic equation of $x^N = 1$ has a factor that is an irreducible polynomial of degree higher than 2 then the polygon is not constructible. To show that a regular n-gon is constructible, it is sufficient to show that the real part of the roots, $\cos(2\pi k/N)$ for $1 \le k \le N - 1$ of the cyclotomic polynomial, must be roots of quadratic factors. All N-gons for N even are constructible as are certain odd valued ones such as $N = 5$, 15, and 17. In fact, it has been proven that N-gons for N prime are constructible if N is a Fermat prime of the form $N = 1 + 2^{2^n}$. For example $N = 5$, 17, and 65537 for $n = 1, 2,$ and 3 have been shown to be constructible. Following BOLD (1982), we shall demonstrate that $2\cos(2\pi/N)$ is the root of irreducible cubics for $N = 7$ and 9, and therefore they are not constructible.

Let's consider the case for $N = 7$ and the solutions to,

$$x^7 = 1 \tag{21}$$

the *roots of unity*, R^k for $0 \le k \le 6$ where $R = \exp((2\pi/7)i) = \cos(2\pi/7) + i\sin(2\pi/7)$. These points lie at the vertices of a regular heptagon in the complex plane. Excluding the root, $x = 1$, leads to the *cyclotomic equation* for the other six roots,

$$\frac{x^7 - 1}{x - 1} = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0. \tag{22}$$

Represent the general cubic by,

$$(x - r_1)(x - r_2)(x - r_3) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - r_1 r_2 r_3.$$

Let,

$$\begin{aligned} r_1 &= R + R^{-1} = R + R^6 = 2\cos\frac{2\pi}{7} \\ r_2 &= R^2 + R^{-2} = R^2 + R^5 = 2\cos 2\frac{2\pi}{7} \\ r_3 &= R^3 + R^{-3} = R^3 + R^4 = 2\cos 4\frac{2\pi}{7} \end{aligned} \tag{23}$$

where we have used the fact that $R^{-k} = R^{N-k}$ and $2\cos j(2\pi/N) = 2\cos(N-j)(2\pi/N)$. But,

$$\begin{aligned} &r_1 + r_2 + r_3 = R^1 + R^2 + R^3 + R^4 + R^5 + R^6 = -1 \\ &r_1 r_2 + r_1 r_3 + r_2 r_3 = -2 \\ &r_1 r_2 r_3 = 1. \end{aligned} \tag{24}$$

Therefore the cyclotomic polynomial has led to the irreducible cubic polynomial,

$$x^3 + x^2 - 2x - 1. \tag{25}$$

Therefore the heptagon is non-constructible. Notice that r_1, r_2, r_3 are the cyclic constants $2\cos k(2\pi/7)$ for $k = 1, 2, 4$.

Next consider the nonagon which was not studied by Gauss. Its vertices are determined by the nine roots of unity,

$$x^9 = 1. \tag{26}$$

We find that,

$$\left(x^9 - 1\right) = \left(x^3 - 1\right)\left(x^6 + x^3 + 1\right)$$

therefore, $x^6 + x^3 + 1$ has the same roots as $x^9 - 1$ except for the cubic roots of unity. But the cubic roots of unity are R^3, R^6, and R^9 where $R = \exp((2\pi/9)i) = \cos(2\pi/9) + i\sin(2\pi/9)$, since each of their cubes equals 1. As a result, the other six powers of R must be the roots of $x^6 + x^3 + 1$. In the manner of Gauss, we pair these powers as follows,

$$\begin{aligned} &r_1 = R + R^{-1} = R + R^8 = 2\cos\frac{2\pi}{9} \\ &r_2 = R^2 + R^{-2} = R^2 + R^7 = 2\cos 2\frac{2\pi}{9} \\ &r_3 = R^4 + R^{-4} = R^4 + R^5 = 2\cos 4\frac{2\pi}{9} \end{aligned} \tag{27}$$

and find that,

$$r_1 + r_2 + r_3 = R^1 + R^2 + R^4 + R^5 + R^7 + R^8 = a.$$

But because the sum of these six roots must equal the coefficient of x^5 in the polynomial $x^6 + x^3 + 1$ it follows that $a = 0$.

From a similar analysis, $r_1 r_2 + r_1 r_3 + r_2 r_3 = -3$ and $r_1 r_2 r_3 = 1$ resulting in the irreducible cubic polynomial,

$$x^3 - 3x + 1.$$

It follows that the nonagon is non-constructible. Once again the roots of this polynomial are the cyclic constants for $k = 1$, 2, and 4.

6. Conclusion

We have generalized Binet's formula to the two families of silver means and used these formulas to generate generalized Pell and Pell-Lucas sequences. Binet's formula was again generalized, with the help of DeMoivre's theorem, to produce a sequence of cyclic constants, $2\cos j(2\pi/N)$, with period N when $x = 2\cos(2\pi/N)$ is replaced successively into the Lucas polynomials. These cyclic constants arise naturally in the Gauss-Wantzel proof of the nonconstructibility of certain N-gons.

Appendix A: The Fibonacci Polynomials

Consider Pascal's triangle.

Table A1. Pascal's triangle.

1	1
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
1 5 10 10 5 1	32
1 6 15 20 15 6 1	64
…	

The integers of Pascal's triangles can be reorganized so that its diagonals become rows that sum to the Fibonacci numbers as in Table A1. We refer to this as the Fibonacci-Pascal triangle.

Table A2. Fibonacci-Pascal triangle.

		Sum	
1	1^0	1	1
2	1^1	1	x
3	1^2 1^0	2	x^2+1
4	1^3 2^1	3	x^3+2x
5	1^4 3^2 1^0	5	x^4+3x^2+1
6	1^5 4^3 3^1	8	x^5+4x^3+3x
7	1^6 5^4 6^2 1^0	13	$x^6+5x^4+6x^2+1$
8	1^7 6^5 10^3 4^1	21	$x^7+6x^5+10x^3+4x$
9	1^8 7^6 15^4 10^2 1^0	34	$x^8+7x^6+15x^4+10x^2+1$
	etc.		*etc.*

The superscripts in Table A1 are exponents, and the integers are coefficients of a sequence of polynomials called Fibonacci polynomials of the first kind, $K_n^{(1)}(x)$, discussed in KOSHY (2001). Inserting $x = 1$ into these polynomials yields the Fibonacci numbers. If $x = 2$, the *Pell sequence* is obtained in which the ratio of successive terms approaches the *silver mean* (the positive solutions of $x - (1/x) = 2$). The other integers result in sequences whose ratios approach higher order *silver means of the first kind*, T_N (solutions of $x - (1/x) = N$). where the sequences satisfy the recursion, $a_{n+1} = Na_n + a_{n-1}$ for integer values of N. If the signs of these polynomials alternate, they are the Fibonacci polynomials of the second kind, $K_n^{(2)}$, which satisfy the recursion relation,

$$K_n^{(2)}(x) = xK_{n-1}^{(2)}(x) - K_{n-2}^{(2)}(x) \text{ starting with } K_1^{(2)} = 1 \text{ and } K_2^{(2)} = x. \tag{A1}$$

These polynomials play an important role in "Golden Fields" by Kappraff *et al.* in another article in this issue. Replacing integer values of into the Fibonacci polynomials with alternating signs yields another family of generalized Pell sequences satisfying the recursion, $a_{n+1} = Na_n - a_{n-1}$ in which the ratio of successive terms approach higher order silver means of the second kind (the positive solutions of $x + (1/x) = N$).

The Lucas polynomials, $L_m(x)$, can be expressed in terms of the derivatives of the Chebyshev polynomials, $T_m(x)$,

$$K_n(x) = \frac{2}{n}\frac{dT_n\left(\frac{x}{2}\right)}{dx}.$$

The proof is given in Appendix A of "Golden Fields."

Appendix B: The Lucas Polynomials

Consider a modified version of Pascal's triangle.

Table B1. A generalized Pascal's triangle.

$$\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & 2 & & 1 & & & \\
 & & 2 & & 3 & & 1 & & \\
 & 2 & & 5 & & 4 & & 1 & \\
2 & & 7 & & 9 & & 5 & & 1 \\
 & & & & \dots & & & &
\end{array}$$

Reorganizing Table B1 so that the diagonals become columns yields,

Table B2. Lucas-Pascal's triangle.

					Sum	
0	2^0				2	2
1	1^1				1	x
2	1^2	2^0			3	$x^2 - 2$
3	1^3	3^1			4	$x^3 - 3x$
4	1^4	4^2	2^0		7	$x^4 - 4x^2 + 2$
5	1^5	5^3	5^1		11	$x^5 - 5x^3 + 5x$
6	1^6	6^4	9^2	2^0	18	$x^6 - 6x^4 + 9x^2 - 2$
7	1^7	7^5	14^3	7^1	29	$x^7 - 7x^5 + 14x^3 - 7x$
	etc.					etc.

The integers in each row of the Table B2 sum to the Lucas Numbers. The superscripts in the table are exponents, and the integers are coefficients of a sequence of polynomials with alternating signs known as the Lucas Polynomials of the second kind, $L_n(x)$, which play an important role in "Golden Fields" by Kappraff *et al.* in another article in this issue. Beginning with $L_0 = 2$ and $L_1 = x$, the Lucas polynomials with alternating signs are generated by the recursive formula:

$$L_n(x) = xL_{n-1}(x) - L_{n-2}(x).$$

The Lucas polynomials, $L_m(x)$, can be expressed in terms of the Chebyshev polynomials, $T_m(x)$, of the first kind,

$$\frac{1}{2}L_m(2x) = T_m(x).$$

The recursion relation for T_m is,

$$T_m = 2xT_{m-1} - T_{m-2}$$

which is related to a recursion for the Pells sequence.

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