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Determinant and Permanent of Hessenberg Matrix and Fibonacci Type Numbers

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Abstract

In this paper, we obtain determinants and permanents of some Hessenberg matrices that give the terms of k sequences of generalized order-k Fibonacci numbers for $k = 2$. These results are important, since k sequences of generalized order-k Fibonacci numbers for $k = 2$ are general form of ordinary Fibonacci sequence, Pell sequence and Jacobsthal sequence.

Keywords: *Fibonacci Numbers, Jacobsthal Numbers, k sequences of generalized order-k Fibonacci numbers, Pell Numbers, Hessenberg Matrix.*

1 Introduction

Fibonacci numbers F_n, Pell numbers P_n and Jacobsthal numbers J_n are defined by

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \text{ for } n > 2 \text{ and } F_1 = F_2 = 1, \\ P_n &= 2P_{n-1} + P_{n-2} \text{ for } n > 1 \text{ and } P_0 = 0,\ P_1 = 1, \\ J_n &= J_{n-1} + 2J_{n-2} \text{ for } n > 2 \text{ and } J_1 = J_2 = 1, \end{aligned}$$

respectively.

Generalizations of these sequences have been studied by many researchers.

Er [3] defined k sequences of generalized order-k Fibonacci numbers (kSOkF) as; for $n > 0$, $1 \le i \le k$

$$f_{k,n}^{\,i} = \sum_{j=1}^{k} c_j f_{k,n-j}^{\,i} \tag{1}$$

with boundary conditions for $1 - k \le n \le 0$,

$$f_{k,n}^{\,i} = \begin{cases} 1 \quad , & \text{if } \ i = 1 - n, \\ 0 \quad , & \text{otherwise,} \end{cases}$$

where c_j $(1 \le j \le k)$ are constant coefficients, $f_{k,n}^{\,i}$ is the n-th term of i-th sequence of order k generalization.

Example 1.1 *$f_{k,n}^{\,1}$ and $f_{k,n}^{\,2}$ sequences are*

$$\begin{array}{l} 0, 1, \ c_1, \ c_2 + c_1^2, \ 2c_1c_2 + c_1^3, \ c_2^2 + c_1^4 + 3c_1^2c_2, \ c_1^5 + 3c_1c_2^2 + 4c_1^3c_2, \\ c_2^3 + c_1^6 + 5c_1^4c_2 + 6c_1^2c_2^2, \ c_1^7 + 4c_1c_2^3 + 6c_1^5c_2 + 10c_1^3c_2^2, \\ c_2^4 + c_1^8 + 7c_1^6c_2 + 10c_1^2c_2^3 + 15c_1^4c_2^2, \ldots \end{array}$$

and

$$\begin{array}{l} 1, \ 0, \ c_2, \ c_1c_2, \ c_2^2 + c_1^2c_2, \ 2c_1c_2^2 + c_1^3c_2, \ c_2^3 + c_1^4c_2 + 3c_1^2c_2^2, \\ 3c_1c_2^3 + c_1^5c_2 + 4c_1^3c_2^2, c_2^4 + c_1^6c_2 + 6c_1^2c_2^3 + 5c_1^4c_2^2, \\ 4c_1c_2^4 + c_1^7c_2 + 10c_1^3c_2^3 + 6c_1^5c_2^2, \ldots \end{array}$$

respectively.

A direct consequence of (1) is

$$f_{k,n}^{\,2} = c_2 f_{k,n-1}^{\,1}, \ \text{for } n \ge 0. \tag{2}$$

Remark 1.2 *Let $f_{k,n}^{\,i}$, F_n, P_n and J_n be kSOkF (1), Fibonacci sequence, Pell sequence and Jacobsthal sequence, respectively. Then,*

(i) Substituting $c_1 = c_2 = 1$ for $k = 2$ in (1), we obtain $f_{k,n-1}^{\,1} = F_n$.

(ii) Substituting $c_1 = 2$ and $c_2 = 1$ for $k = 2$ in (1), we obtain $f_{k,n-1}^{\,1} = P_n$.

(iii) Substituting $c_1 = 1$ and $c_2 = 2$ for $k = 2$ in (1), we obtain $f_{k,n-1}^{\,1} = J_n$.

Remark 1.2 shows that $f_{k,n}^1$ is a general form of Fibonacci sequence, Pell sequence and Jacobsthal sequence. Therefore, any result obtained from $f_{k,n}^1$ holds for other sequences mentioned above.

Many researchers studied on determinantal and permanental representations of k sequences of generalized order-k Fibonacci and Lucas numbers. For example, Minc [7] defined an $n \times n$ (0,1)-matrix $F(n,k)$, and showed that the permanents of $F(n,k)$ is equal to the generalized order-k Fibonacci numbers.

The author of [5, 6] defined two $(0,1)$-matrices and showed that the permanents of these matrices are the generalized Fibonacci and Lucas numbers. Öcal et al. [8] gave some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers, and obtained Binet's formula for these sequences. Yılmaz and Bozkurt [9] derived some relationships between Pell and Perrin sequences, and permanents and determinants of a type of Hessenberg matrices.

In this paper, we give some determinantal and permanental representations of k sequences of generalized order-k Fibonacci numbers for $k = 2$ by using various Hessenberg matrices. These results are general form of determinantal and permanental representations of ordinary Fibonacci numbers, Pell numbers and Jacobsthal numbers.

2 The Determinantal Representations

An $n \times n$ matrix $A_n = (a_{ij})$ is called lower Hessenberg matrix if $a_{ij} = 0$ when $j - i > 1$, i.e.,

$$A_n = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix}. \tag{3}$$

Theorem 2.1 *[2] Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and $\det(A_0) = 1$. Then,*

$$\det(A_1) = a_{11}$$

and for $n \geq 2$

$$\det(A_n) = a_{n,n}\det(A_{n-1}) + \sum_{r=1}^{n-1}\left[(-1)^{n-r}a_{n,r}\prod_{j=r}^{n-1}a_{j,j+1}\det(A_{r-1})\right]. \tag{4}$$

Theorem 2.2 *Let $f_{2,n}^{\,1}$ be the first sequence of 2SO2F and $Q_n = (q_{rs})_{n\times n}$ be a Hessenberg matrix defined by*

$$q_{rs} = \begin{cases} i^{|r-s|}.\frac{c_{r-s+1}}{c_2^{(r-s)}} & , \quad \text{if} \quad -1 \le r-s < 2 \ , \\ 0 & , \quad \text{otherwise}, \end{cases}$$

that is

$$Q_n = \begin{bmatrix} c_1 & ic_2 & 0 & 0 & \cdots & 0 \\ i & c_1 & ic_2 & 0 & \cdots & 0 \\ 0 & i & c_1 & ic_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & c_1 & ic_2 \\ 0 & 0 & 0 & 0 & i & c_1 \end{bmatrix}. \tag{5}$$

Then,

$$\det(Q_n) = f_{2,n}^{\,1}, \tag{6}$$

where $c_0 = 1$ and $i = \sqrt{-1}$.

Proof. To prove (6), we use the mathematical induction on m. The result is true for $m = 1$ by hypothesis.

Assume that it is true for all positive integers less than or equal to m, namely $\det(Q_m) = f_{2,m}^{\,1}$. Then, we have

$$\begin{aligned} \det(Q_{m+1}) &= q_{m+1,m+1}\det(Q_m) + \sum_{r=1}^{m}\left[(-1)^{m+1-r} q_{m+1,r}\prod_{j=r}^{m} q_{j,j+1}\det(Q_{r-1})\right] \\ &= c_1\det(Q_m) + \sum_{r=1}^{m-1}\left[(-1)^{m+1-r} q_{m+1,r}\prod_{j=r}^{m} q_{j,j+1}\det(Q_{k,r-1})\right] \\ &\quad + [(-1)q_{m+1,m}q_{m,m+1}\det(Q_{k,m-1})] \\ &= c_1\det(Q_m) + [(-1)ic_2 i\det(Q_{k,m-1})] \\ &= c_1\det(Q_m) + c_2\det(Q_{k,m-1}) \end{aligned}$$

by using Theorem 2.1. From the hypothesis of induction and the definition of 2SO2F, we obtain

$$\det(Q_{m+1}) = c_1 f_{2,m}^{\,1} + c_2 f_{2,m-1}^{\,1} = f_{2,m+1}^{\,1}.$$

Therefore, (6) holds for all positive integers n.

Example 2.3 *We obtain $f_{2,6}^{\,1}$, by using Theorem 2.2*

$$\begin{aligned}
\det(Q_6) &= \det\begin{bmatrix} c_1 & ic_2 & 0 & 0 & 0 & 0 \\ i & c_1 & ic_2 & 0 & 0 & 0 \\ 0 & i & c_1 & ic_2 & 0 & 0 \\ 0 & 0 & i & c_1 & ic_2 & 0 \\ 0 & 0 & 0 & i & c_1 & ic_2 \\ 0 & 0 & 0 & 0 & i & c_1 \end{bmatrix} \\
&= c_2^3 + c_1^6 + 5c_1^4 c_2 + 6c_1^2 c_2^2 \\
&= f_{2,6}^{\,1}.
\end{aligned}$$

Theorem 2.4 *Let $f_{2,n}^{\,1}$ be the first sequence of $2SO2F$ and $B_n = (b_{ij})_{n\times n}$ be a Hessenberg matrix, where*

$$b_{ij} = \begin{cases} -c_2 & , \quad \text{if } \; j = i+1, \\ \frac{c_{i-j+1}}{c_2^{(i-j)}} & , \quad \text{if } \; 0 \le i-j < 2, \\ 0 & , \quad \text{otherwise,} \end{cases}$$

that is

$$B_n = \begin{bmatrix} c_1 & -c_2 & 0 & \cdots & 0 & 0 \\ 1 & c_1 & -c_2 & \cdots & 0 & 0 \\ 0 & 1 & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & -c_2 \\ 0 & 0 & 0 & \cdots & 1 & c_1 \end{bmatrix}$$

Then,

$$\det(B_n) = f_{2,n}^{\,1},$$

where $c_0 = 1$.

Proof. Since the proof is similar to the proof of Theorem 2.2 by using Theorem 2.1, we omit the detail.

Example 2.5 *We obtain $f_{2,4}^{\,1}$ by using Theorem 2.4 that*

$$\begin{aligned}
\det(B_5) &= \det\begin{bmatrix} c_1 & -c_2 & 0 & 0 \\ 1 & c_1 & -c_2 & 0 \\ 0 & 1 & c_1 & -c_2 \\ 0 & 0 & 1 & c_1 \end{bmatrix} \\
&= c_2^2 + c_1^4 + 3c_1^2 c_2 \\
&= f_{2,4}^{\,1}.
\end{aligned}$$

Corollary 2.6 *If we rewrite Theorem 2.2 and Theorem 2.4 for $c_i = 1$, then we obtain* $\det(Q_n) = F_{n+1}$ *and* $\det(B_n) = F_{n+1}$*, respectively, where* F_n*'s are the ordinary Fibonacci numbers.*

Corollary 2.7 *If we rewrite Theorem 2.2 and Theorem 2.4 for $c_1 = 2$ and $c_2 = 1$, then we obtain* $\det(Q_n) = P_{n+1}$ *and* $\det(B_n) = P_{n+1}$*, respectively, where* P_n*'s are the Pell numbers.*

Corollary 2.8 *If we rewrite Theorem 2.2 and Theorem 2.4 for $c_1 = 1$ and $c_2 = 2$, then we obtain* $\det(Q_n) = J_{n+1}$ *and* $\det(B_n) = J_{n+1}$*, respectively, where* J_n*'s are the Jacobsthal numbers.*

3 The Permanent Representations

Let $A = (a_{i,j})$ be an $n \times n$ matrix over a ring. Then, the permanent of A is defined by

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where S_n denotes the symmetric group on n letters.

Theorem 3.1 *[8] Let A_n be $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and $per(A_0) = 1$. Then, $per(A_1) = a_{11}$ and for $n \geq 2$*

$$per(A_n) = a_{n,n} per(A_{n-1}) + \sum_{r=1}^{n-1} \left[a_{n,r} \prod_{j=r}^{n-1} a_{j,j+1} per(A_{r-1}) \right]. \tag{7}$$

Theorem 3.2 *Let $f_{2,n}^{1}$ be the first sequence of $2SO2F$ and $H_n = (h_{rs})$ be an $n \times n$ Hessenberg matrix, where*

$$h_{rs} = \begin{cases} i^{(r-s)} . \frac{c_{r-s+1}}{c_2^{(r-s)}} & , \quad if \quad -1 \leq r-s < 2, \\ 0 & , \quad otherwise, \end{cases}$$

that is

$$H_n = \begin{bmatrix} c_1 & -ic_2 & 0 & \cdots & 0 & 0 \\ i & c_1 & -ic_2 & & 0 & 0 \\ 0 & i & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & -ic_2 \\ 0 & 0 & 0 & \cdots & i & c_1 \end{bmatrix}. \tag{8}$$

Then,

$$per(H_n) = f_{2,n}^{1},$$

where $c_0 = 1$ and $i = \sqrt{-1}$.

Proof. This is similar to the proof of Theorem 2.2 using Theorem 3.1.

Example 3.3 *We obtain* $f_{2,3}^{1}$ *by using Theorem 3.2 that*

$$\begin{aligned} per(H_{4,3}) &= per\begin{bmatrix} c_1 & -ic_2 & 0 \\ i & c_1 & -ic_2 \\ 0 & i & c_1 \end{bmatrix} \\ &= 2c_1c_2 + c_1^3 \\ &= f_{2,3}^{1}. \end{aligned}$$

Theorem 3.4 *Let* $f_{2,n}^{1}$ *be the first sequence of* $2SO2F$ *and* $L_n = (l_{ij})$ *be an* $n \times n$ *lower Hessenberg matrix given by*

$$l_{ij} = \begin{cases} \frac{c_{i-j+1}}{c_2^{(i-j)}} & , \quad \text{if } 0 \leq i-j < 2, \\ 0 & , \quad \text{otherwise,} \end{cases}$$

that is

$$L_n = \begin{bmatrix} c_1 & c_2 & 0 & \cdots & 0 & 0 \\ 1 & c_1 & c_2 & \cdots & 0 & 0 \\ 0 & 1 & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & 0 & \cdots & 1 & c_1 \end{bmatrix}.$$

Then,

$$per(L_n) = f_{2,n}^{1},$$

where $c_0 = 1$.

Proof. This is similar to the proof of Theorem 2.2 by using Theorem 3.1.

Corollary 3.5 *If we rewrite Theorem 3.2 and Theorem 3.4 for* $c_i = 1$, *we obtain* $per(H_n) = F_{n+1}$ *and* $per(L_n) = F_{n+1}$, *respectively, where* F_n*'s are the Fibonacci numbers.*

Corollary 3.6 *If we rewrite Theorem 3.2 and Theorem 3.4 for* $c_1 = 2$ *and* $c_2 = 1$, *we obtain* $per(H_n) = P_{n+1}$ *and* $per(L_n) = P_{n+1}$, *respectively, where* P_n*'s are the Pell numbers.*

Corollary 3.7 *If we rewrite Theorem 3.2 and Theorem 3.4 with* $c_1 = 1$ *and* $c_2 = 2$, *then we obtain* $per(H_n) = J_{n+1}$ *and* $per(L_n) = J_{n+1}$, *respectively, where* J_n*'s are the Jacobsthal numbers.*

3.1 Binet's formula for 2 sequences of generalized order-2 Fibonacci numbers $(2\mathbf{SO}2\mathbf{F})$

Let $\sum\limits_{n=0}^{\infty} a_n z^n$ be the power series of the analytical function f such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{when} \quad f(0) \neq 0$$

and

$$A_n = \begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{bmatrix}_{n\times n}.$$

Then, the reciprocal of $f(z)$ can be written in the following form

$$g(z) = \frac{1}{f(z)} = \sum_{n=0}^{\infty} (-1)^n \det(A_n) z^n,$$

whose radius of converge is $\inf\{|\lambda| : f(\lambda) = 0\}$, [1].

Let

$$p_k(z) = 1 + a_1 z + \cdots + a_k z^k. \tag{9}$$

Then, the reciprocal of $p_k(z)$ is

$$\frac{1}{p_k(z)} = \sum_{n=0}^{\infty} (-1)^n \det(A_{k,n}) z^n,$$

where

$$A_{k,n} = \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_k & a_{k-1} & a_{k-2} & \cdots & 0 \\ 0 & a_k & a_{k-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & a_k & \cdots & a_1 \end{bmatrix}_{n\times n} \quad [8]. \tag{10}$$

Inselberg [4] showed that

$$\det(A_{k,n}) = \sum_{j=1}^{k} \frac{1}{p_k'(\lambda_j)} \left(\frac{-1}{\lambda_j}\right)^{n+1} \quad (n \geq k) \tag{11}$$

if $p_k(z)$ has distinct zeros λ_j for $j \in \{1, 2, \ldots, k\}$; where $p'_k(z)$ is the derivative of polynomial $p_k(z)$ in (9).

Theorem 3.8 *Let $f^1_{2,n}$ be the first sequence of $2SO2F$. Then, for $n \geq 2$ and $(c_1)^2 + 4c_1c_2 > 0$,*

$$f^1_{2,n} = \sum_{j=1}^{k} \frac{1}{p'(\lambda_j)} \left(\frac{-1}{\lambda_j}\right)^{n+1}, \tag{12}$$

where $p(z) = 1 + c_1 z - c_2 z^2$ and $p'(z)$ denotes the derivative of polynomial $p(z)$.

Proof. This is immediate from Theorems 2.4 and (11).

Corollary 3.9 *Let $f^2_{2,n}$ be the second sequences of $2SO2F$. Then,*

$$f^2_{2,n+1} = c_2. \sum_{j=1}^{k} \frac{-1}{p'(\lambda_j)} \left(\frac{1}{\lambda_j}\right)^{n+1}$$

for $n \geq 2$.

Proof. One can easily obtain the proof from (2) and Theorem 3.8.

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References

[1] E.T. Bell, Euler algebra, *Trans. Amer. Math. Soc.*, 25(1923), 135-154.

[2] N.D. Cahill, J.R. D'Errico, D.A. Narayan and J.Y. Narayan, Fibonacci determinants, *College Math. J.*, 33(3) (2002), 221-225.

[3] M.C. Er, Sums of Fibonacci numbers by matrix method, *Fibonacci Quart.*, 23(3) (1984), 204-207.

[4] A. Insenberg, On determinants of Toeplitz-Hessenberg matrices arising in power series, *J. Math. Anal. Appl.*, 63(1978), 347-353.

[5] G.-Y. Lee and S.-G. Lee, A note on generalized Fibonacci numbers, *Fibonacci Quart.*, 33(1995), 273-278.

[6] G.-Y. Lee, k-Lucas numbers and associated bipartite graphs, *Lineer Algebra Appl.*, 320(2000), 51-61.

[7] H. Minc, *Encyclopaedia of Mathematics and its Applications, Permanents, (Vol.6)*, Addison-Wesley Publishing Company, London, (1978).

[8] A.A. Öcal, N. Tuğlu and E. Altınışık, On the representation of k-generalized Fibonacci and Lucas numbers, *Appl. Math. Comput.*, 170(2005), 584-596.

[9] F. Yılmaz and D. Bozkurt, Hessenberg matrices and the Pell and Perrin numbers, *J. Number Theory.*, 131(2011), 1390-1396.