Generalized Van der Laan and Perrin Polynomials, and Generalizations of Van der Laan and Perrin Numbers

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Abstract. In this paper, we present k sequences of generalized Van der Laan polynomials and generalized Perrin polynomials by using generalized Fibonacci and Lucas polynomials. We give some properties of these polynomials. We also obtain generalized order-k Van der Laan numbers, k sequences of generalized order-k Van der Laan numbers, generalized order-k Perrin numbers and k sequences of generalized order-k Perrin numbers. In addition, we examine relationships between them.

1. Introduction

Fibonacci, Lucas, Pell and Perrin numbers have been known for a long time. There are many studies, relations, and applications of them. Generalizations of these numbers have been studied by many researchers.
Miles [14] defined generalized order-k Fibonacci numbers (GOkF) in 1960. Er [1] defined k sequences of the generalized order-k Fibonacci numbers (kSOkF) and gave matrix representation for these sequences in 1984. Kalman [2] obtained a Binet formula for these sequences in 1982. Karaduman [3], Taşçı and Kılıç [16] studied these sequences. Kılıç and Taşçı [6] defined k sequences of the generalized order-k Pell numbers (kSOkP) and obtained sums properties by using matrix method. Kaygısız and Bozkurt [5] studied a generalization of Perrin numbers. Yılmaz and Bozkurt [17] gave some properties of Perrin and Pell numbers.

Meanwhile, MacHenry [7] defined generalized Fibonacci polynomials $(F_{k,n}(t))$, Lucas polynomials $(G_{k,n}(t))$ in 1999, studied these polynomials in [8] and defined matrices $A_{(k)}^{\infty}$ and $D_{(k)}^{\infty}$ in [13]. Studies of MacHenry include most of other studies mentioned above. For example, $A_{(k)}^{\infty}$ is reduced to k sequences of the generalized order-k Fibonacci numbers and $A_{(k)}^{\infty}$ is reduced to k sequences of the generalized order-k Pell numbers when $t_1 = 2$ and $t_i = 1$ (for $2 \leq i \leq k$), respectively. This analogy shows the importance of the matrices $A_{(k)}^{\infty}$ and $D_{(k)}^{\infty}$ and generalized Fibonacci and Lucas polynomials. Based on this idea, Kaygısız and Şahin defined k sequences of the generalized order-k Lucas numbers using $G_{k,n}(t)$ and $D_{(k)}^{\infty}$ in [4].

In this article, we first present k sequences of generalized Van der Laan and Perrin polynomials $(V_{k,n}^i(t)$ and $R_{k,n}^i(t))$ by using generalized Fibonacci and Lucas polynomials. Then, we obtain generalized order-k Van der Laan and Perrin numbers, k sequences of the generalized order-k Van der Laan and Perrin numbers by the help of these polynomials and matrices $A_{(k)}^{\infty}$ and $D_{(k)}^{\infty}$. In addition, we examine relationships between them and explore some of the properties of these sequences. We believe that, our results are important, especially, for those who are interested in well known Fibonacci, Lucas, Pell and Perrin sequences and their generalizations.

MacHenry [7] defined generalized Fibonacci polynomials $F_{k,n}(t)$ and Lucas polynomials $G_{k,n}(t)$ as follows;

$$
\begin{aligned}
&F_{k,n}(t) = 0, \ n < 0, && G_{k,n}(t) = 0, \ n < 0, \\
&F_{k,0}(t) = 1, && G_{k,0}(t) = \ k, \\
&F_{k,n}(t) = \sum_{j=1}^{k} t_j F_{k,n-j}(t), && G_{k,1}(t) = t_1, \\
& && G_{k,n}(t) = G_{k-1,n}(t), \ 1 \leq n \leq k, \\
& && G_{k,n}(t) = \sum_{j=1}^{k} t_j G_{k,n-j}(t), \ n > k
\end{aligned}
$$

where t_i $(1 \leq i \leq k)$ are constant coefficients of the core polynomial

(1.1) $$P(x; t_1, t_2, \ldots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k.$$

In [13], matrices $A_{(k)}^{\infty}$ and $D_{(k)}^{\infty}$ are defined by using the following matrix,

(1.2) $$A_{(k)} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ t_k & t_{k-1} & t_{k-2} & \ldots & t_1 \end{bmatrix}.$$

$A_{(k)}^{\infty}$ is obtained by multiplying $A_{(k)}$ and $A_{(k)}^{-1}$ by the vector t.
Derivative of the core polynomial (1.1) is

$$P'(x) = kx^{k-1} - t_1(k-1)x^{k-2} - \cdots - t_{k-1},$$

which is represented by the vector $(-t_{k-1}, \ldots, -t_1(k-1), k)$. The matrix $D_{(k)}^{\infty}$ is obtained by multiplying $A_{(k)}$ and $A_{(k)}^{-1}$ by the vector $(-t_{k-1}, \ldots, -t_1(k-1), k)$. Right hand column of $A_{(k)}^{\infty}$ contains sequence of the generalized Fibonacci polynomials $F_{k,n}(t)$. In addition, the right hand column of $D_{(k)}^{\infty}$ contains sequence of the generalized Lucas polynomials $G_{k,n}(t)$. Also in [8, 9, 10, 11, 12], authors studied generalized Fibonacci and Lucas polynomials and obtained very useful properties.

For easier reference, we have stated some theorems that will be used in following sections.

Theorem 1.1. [8] Let $F_{k,n}(t)$ and $G_{k,n}(t)$ be the generalized Fibonacci and Lucas polynomials, respectively. Then,

$$\sum_{j=1}^{k} \frac{\partial G_{k,n}(t)}{\partial t_j} t_j = n F_{k,n+1}(t).$$

Theorem 1.2. [13] Let $A_{(k)}$ be a $k \times k$ matrix as in (1.2). Then,

$$\det A_{(k)} = (-1)^{k+1} t_k$$

and

(1.3)
$$\det A_{(k)}^n = (-1)^{n(k+1)} t_k^n.$$

The well-known Cordonnier (Padovan) sequence $\{C_n\}$ is defined recursively by the equation,

$$C_n = C_{n-2} + C_{n-3}, \text{ for } n > 3$$

where $C_1 = 1, C_2 = 1, C_3 = 1$. Van der Laan sequence $\{V_n\}$ is defined recursively by the equation,

$$V_n = V_{n-2} + V_{n-3}, \text{ for } n > 3$$

where $V_1 = 1, V_2 = 0, V_3 = 1$. Perrin sequence $\{R_n\}$ is defined recursively by the equation,

$$R_n = R_{n-2} + R_{n-3}, \text{ for } n > 3$$

where $R_1 = 0, R_2 = 2, R_3 = 3$ [15].

In this paper, we define generalized order-k Van der Laan numbers $v_{k,n}$ and k sequences of the generalized order-k Van der Laan numbers $v_{k,n}^i$ with the help of k sequences of generalized Van der Laan polynomials. Also, we define generalized order-k Perrin numbers $r_{k,n}$ and k sequences of the generalized order-k Perrin numbers $r_{k,n}^i$ with the help of k sequences of generalized Perrin polynomials. In addition, we present some relations between these polynomials and sequences.

Moreover, we show that there is a parallel relationship between Van der Laan and Perrin polynomials (numbers) as Fibonacci and Lucas polynomials (numbers).

2. Generalized Van der Laan and Perrin Polynomials

We define generalized Van der Laan polynomial and k sequences of generalized Van der Laan polynomials by the help of generalized Fibonacci polynomials $(F_{k,n}(t))$ and matrices $A_{(k)}^{\infty}$.

Definition 2.1. Generalized Fibonacci polynomials $(F_{k,n}(t))$ are called generalized Van der Laan polynomials, in the case of $t_1 = 0$ for $k \geq 3$. So, generalized Van der Laan polynomials are

$$
\begin{aligned}
V_{k,n}(t) &= 0, \ n < 0 \\
V_{k,0}(t) &= 1 \\
V_{k,n}(t) &= \sum_{i=2}^{k} t_i V_{k,n-i}(t), \ n > 0
\end{aligned}
$$

For $k \geq 3$, substituting $t_1 = 0$, generalized Fibonacci polynomials $(F_{k,n}(t))$ and matrices $A_{(k)}^{\infty}$ are together reduced to the following polynomials. For $n > 0$ and $1 \leq i \leq k$

$$
(2.1) \qquad V_{k,n}^{i}(t) = \sum_{j=2}^{k} t_j V_{k,n-j}^{i}(t)
$$

with boundary conditions for $1 - k \leq n \leq 0$,

$$
V_{k,n}^{i}(t) = \begin{cases} 1 & \text{if } k = i - n, \\ 0 & \text{otherwise}, \end{cases}
$$

where $V_{k,n}^{i}(t)$ is the n-th term of i-th sequence.

Definition 2.2. The polynomials derived in (2.1) are called k sequences of generalized Van der Laan polynomials.

We note that for $i = k$ and $n \geq 0$, $V_{k,n}^{i}(t) = V_{k,n}(t)$.
In addition,

$$
V_{(k)} = \begin{bmatrix}
0 & 1 & 0 & \dots & 0 \\
0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 1 \\
t_k & t_{k-1} & t_{k-2} & \dots & 0
\end{bmatrix}
$$

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is the generator matrix of k sequences of generalized Van der Laan polynomials. Matrix $V_{(k)}^{\infty}$ is obtained by multiplying $V_{(k)}$ and $V_{(k)}^{-1}$ by the vector $v = (t_k, t_{k-1}, t_{k-2}, \ldots, 0)$.

Note that it is also possible to obtain matrix $V_{(k)}^{\infty}$ from matrix $A_{(k)}^{\infty}$ by substituting $t_1 = 0$.

Let $\widetilde{V}_n$ be generalized Van der Laan matrix, which is obtained by n-th power of $V_{(k)}$ as;

$$(2.2) \qquad \widetilde{V}_n = (V_{(k)})^n = \begin{bmatrix} V_{k,n-k+1}^1(t) & V_{k,n-k+1}^2(t) & \cdots & V_{k,n-k+1}^k(t) \\ \vdots & \vdots & \cdots & \vdots \\ V_{k,n-1}^1(t) & V_{k,n-1}^2(t) & \ddots & V_{k,n-1}^k(t) \\ V_{k,n}^1(t) & V_{k,n}^2(t) & \cdots & V_{k,n}^k(t) \end{bmatrix}.$$

Then, we have $V_{(k)} = \widetilde{V}_1$.

Corollary 2.1. Let $\widetilde{V}_n$ be as in (2.2). Then,

$$\det \widetilde{V}_n = (-1)^{n(k+1)} t_k^n.$$

Proof. It is direct from Theorem 1.3.

We define generalized Perrin polynomials and matrix $R_{(k)}^{\infty}$ by the help of generalized Lucas polynomials $(G_{k,n}(t))$ and matrices $D_{(k)}^{\infty}$.

Definition 2.3. Generalized Lucas polynomials $(G_{k,n}(t))$ are called generalized Perrin polynomials, in case $t_1 = 0$ for $k \geq 3$. So, generalized Perrin polynomials are;

$$
\begin{aligned}
R_{k,0}(t) &= k \\
R_{k,1}(t) &= 0 \\
R_{k,2}(t) &= 2t_2 \\
R_{k,3}(t) &= t_2 R_{k,1}(t) + 3t_3 \\
R_{k,4}(t) &= t_2 R_{k,2}(t) + t_3 R_{k,1}(t) + 4t_4 \\
&\vdots \\
R_{k,k-1}(t) &= t_2 R_{k,k-3}(t) + \cdots + t_{k-1} R_{k,1}(t) + kt_k
\end{aligned}
$$

and for $n \geq k$,

$$R_{k,n}(t) = \sum_{i=2}^{k} t_i R_{k,n-i}(t).$$

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We obtain matrix $R_{(k)}$ by using row vector $(-t_{k-1}, \ldots, -t_2(k-2), 0, k)$. Let k-th row of matrix $R_{(k)}$ be the vector

$$(-t_{k-1}, \ldots, -t_2(k-2), 0, k)$$

and get i-th row of matrix $R_{(k)}$ by

$$(-t_{k-1}, \ldots, -t_2(k-2), 0, k)(V_{(k)})^{-(k-i)}$$

for $1 \leq i \leq k-1$. So, it looks like

$$(2.3) \qquad R_{(k)} = \begin{bmatrix} (-t_{k-1}, \ldots, -t_2(k-2), 0, k).(V_{(k)})^{-(k-1)} \\ (-t_{k-1}, \ldots, -t_2(k-2), 0, k).(V_{(k)})^{-(k-2)} \\ \vdots \\ (-t_{k-1}, \ldots, -t_2(k-2), 0, k).(V_{(k)})^{-1} \\ (-t_{k-1}, \ldots, -t_2(k-2), 0, k) \end{bmatrix}_{k \times k}.$$

For $k \geq 3$, by substituting $t_1 = 0$, generalized Lucas polynomials $(G_{k,n}(t))$ and matrices $D_{(k)}^{\infty}$ are together reduced to polynomials $R_{k,n}^i(t)$. That is, for $n > 0$ and $1 \leq i \leq k$

$$(2.4) \qquad R_{k,n}^i(t) = \sum_{j=2}^{k} t_j R_{k,n-j}^i(t)$$

with boundary conditions for $1 - k \leq n \leq 0$,

$$R_{(k)} = [a_{k+n,i}] = R_{k,n}^i(t).$$

Definition 2.4. The polynomials $R_{k,n}^i(t)$ derived in (2.4) are called k sequences of generalized Perrin polynomials.

For $k \geq 3$, by substituting $t_1 = 0$, matrix $D_{(k)}^{\infty}$ is reduced to matrix $R_{(k)}^{\infty}$. Right hand column of $R_{(k)}^{\infty}$ contains generalized Perrin polynomials $R_{k,n}(t)$. i-th column of matrix $R_{(k)}^{\infty}$ contains i-th sequence of k sequences of generalized Perrin polynomials.

Let $\widetilde{R}_n$ be generalized Perrin matrix obtained by $R_{(k)}.(V_{(k)})^n$ as;

$$(2.5)$$

$$\widetilde{R}_n = R_{(k)}.(V_{(k)})^n = \begin{bmatrix} R_{k,n-k+1}^1(t) & R_{k,n-k+1}^2(t) & \cdots & R_{k,n-k+1}^k(t) \\ \vdots & \vdots & \cdots & \vdots \\ R_{k,n-1}^1(t) & R_{k,n-1}^2(t) & \ddots & R_{k,n-1}^k(t) \\ R_{k,n}^1(t) & R_{k,n}^2(t) & \cdots & R_{k,n}^k(t) \end{bmatrix}.$$

Now, we give four Corollaries by using properties of generalized Fibonacci and Lucas polynomials.

Corollary 2.2. $tr(V_{(k)}^n) = R_{k,n}(t)$, for $n \in \mathbb{Z}$.

Corollary 2.3. For $n \geq 1$,
$$V_{k,n}^k(t) = V_{k,n-1}^{k-1}(t), \; V_{k,n}^1(t) = t_k V_{k,n-1}^k(t), \; R_{k,n}^k(t) = R_{k,n-1}^{k-1}(t) \text{ and }$$
$$R_{k,n}^1(t) = t_k R_{k,n-1}^k(t).$$

Corollary 2.4. For $1 \leq j \leq k$,

$$\sum_{j=1}^k \frac{\partial R_{k,n}^k(t)}{\partial t_j} t_j = n V_{k,n}^k(t).$$

Theorem 2.1. For $1 \leq i \leq k$,

$$R_{k,n}^i(t) = (-t_{k-1}) V_{k,n-k+1}^i(t) + \ldots + (-t_2(k-2)) V_{k,n-2}^i(t) + k V_{k,n}^i(t).$$

Proof. Using (2.2) and (2.5) we obtain

$$
\begin{aligned}
\widetilde{R}_n &= R_{(k)} \widetilde{V}_n \\
&\Rightarrow
\begin{bmatrix}
R_{k,n-k+1}^1(t) & R_{k,n-k+1}^2(t) & \cdots & R_{k,n-k+1}^k(t) \\
\vdots & \vdots & \cdots & \vdots \\
R_{k,n-1}^1(t) & R_{k,n-1}^2(t) & \ddots & R_{k,n-1}^k(t) \\
R_{k,n}^1(t) & R_{k,n}^2(t) & \cdots & R_{k,n}^k(t)
\end{bmatrix}
\end{aligned}
$$

$$
=
\begin{bmatrix}
(-t_{k-1}, \ldots, -t_2(k-2), 0, k).(V_{(k)})^{-(k)} \\
(-t_{k-1}, \ldots, -t_2(k-2), 0, k).(V_{(k)})^{-(k-1)} \\
\vdots \\
(-t_{k-1}, \ldots, -t_2(k-2), 0, k).(V_{(k)})^{-1} \\
(-t_{k-1}, \ldots, -t_2(k-2), 0, k)
\end{bmatrix}
\begin{bmatrix}
V_{k,n-k+1}^1(t) & V_{k,n-k+1}^2(t) & \cdots & V_{k,n-k+1}^k(t) \\
\vdots & \vdots & \cdots & \vdots \\
V_{k,n-1}^1(t) & V_{k,n-1}^2(t) & \ddots & V_{k,n-1}^k(t) \\
V_{k,n}^1(t) & V_{k,n}^2(t) & \cdots & V_{k,n}^k(t)
\end{bmatrix}.
$$

From the above matrix multiplication we get,

$$R_{k,n}^i(t) = (-t_{k-1}) V_{k,n-k+1}^i(t) + \ldots + (-t_2(k-2)) V_{k,n-1}^i(t) + k V_{k,n}^i(t).$$

Example 1. We obtain $R_{4,5}^3(t)$ by using Theorem (2.1)

$$
\begin{aligned}
R_{4,5}^3(t) &= (-t_3)V_{4,5-4+1}^3(t) + (-t_2(k-2))V_{4,5-3+1}^3(t) + kV_{4,5}^3(t) \\
&= (-t_3)V_{4,2}^3(t) + (-t_2(4-2))V_{4,3}^3(t) + kV_{4,5}^3(t) \\
&= (-t_3)t_3 + (-2t_2)(t_4 + t_2^2) + 4(t_3^2 + t_2^3) = 6t_2t_4 + 2t_2^3 + 3t_3^2.
\end{aligned}
$$

Theorem 2.2. For $1 \leq i \leq k$ and positive integers n and m,

$$
V_{k,n+m}^i(t) = \sum_{j=1}^{k} V_{k,m}^j(t) V_{k,n-k+j}^i(t).
$$

Proof. We know that $\widetilde{V}_n = (V_{(k)})^n$. We may rewrite it as

$$
\begin{aligned}
(V_{(k)})^{n+1} &= (V_{(k)})^n (V_{(k)}) = (V_{(k)})(V_{(k)})^n \\
&\Rightarrow \quad \widetilde{V}_{n+1} = \widetilde{V}_n \widetilde{V}_1 = \widetilde{V}_1 \widetilde{V}_n
\end{aligned}
$$

and inductively

(2.6)
$$
\widetilde{V}_{n+m} = \widetilde{V}_n \widetilde{V}_m = \widetilde{V}_m \widetilde{V}_n.
$$

Consequently, any element of $\widetilde{V}_{n+m}$ is obtained by the product of a row of $\widetilde{V}_n$ and a column of $\widetilde{V}_m$; that is

$$
V_{k,n+m}^i(t) = \sum_{j=1}^{k} V_{k,m}^j(t) V_{k,n-k+j}^i(t).
$$

Corollary 2.5. In (2.6), if we take $n = m$, we obtain

$$
(\widetilde{V}_n)^2 = \widetilde{V}_n \widetilde{V}_n = \widetilde{V}_{n+n} = \widetilde{V}_{2n}.
$$

Theorem 2.3. For $1 \leq i \leq k$ and $n \in \mathbb{Z}$,

$$
R_{k,n}^i(t) = kt_k V_{k,n-k}^i(t) + \cdots + 3t_3 V_{k,n-3}^i(t) + 2t_2 V_{k,n-2}^i(t).
$$

Proof.

$$\widetilde{R}_n \quad = \quad R_{(k)}\widetilde{V}_n = R_{(k)}\widetilde{V}_1\widetilde{V}_{n-1}$$

$$\Rightarrow \quad \begin{bmatrix} R^1_{k,n-k+1}(t) & R^2_{k,n-k+1}(t) & \cdots & R^k_{k,n-k+1}(t) \\ \vdots & \vdots & \cdots & \vdots \\ R^1_{k,n-1}(t) & R^2_{k,n-1}(t) & \ddots & R^k_{k,n-1}(t) \\ R^1_{k,n}(t) & R^2_{k,n}(t) & \cdots & R^k_{k,n}(t) \end{bmatrix}$$

$$= \quad \begin{bmatrix} (-t_{k-1},\ldots,-t_2(k-2),0,k).(V_{(k)})^{-(k)} \\ (-t_{k-1},\ldots,-t_2(k-2),0,k).(V_{(k)})^{-(k-1)} \\ \vdots \\ (-t_{k-1},\ldots,-t_2(k-2),0,k).(V_{(k)})^{-1} \\ (-t_{k-1},\ldots,-t_2(k-2),0,k) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ t_k & t_{k-1} & t_{k-2} & \cdots & 0 \end{bmatrix}$$

$$\begin{bmatrix} V^1_{k,n-k}(t) & V^2_{k,n-k}(t) & \cdots & V^k_{k,n-k}(t) \\ \vdots & \vdots & \cdots & \vdots \\ V^1_{k,n-2}(t) & V^2_{k,n-2}(t) & \ddots & V^k_{k,n-2}(t) \\ V^1_{k,n-1}(t) & V^2_{k,n-1}(t) & \cdots & V^k_{k,n-1}(t) \end{bmatrix}$$

$$= \quad \begin{bmatrix} (-t_{k-1},\ldots,-t_2(k-2),0,k).(V_{(k)})^{-(k-1)} \\ (-t_{k-1},\ldots,-t_2(k-2),0,k).(V_{(k)})^{-(k-2)} \\ \vdots \\ (-t_{k-1},\ldots,-t_2(k-2),0,k) \\ (kt_k,\ldots,3t_3,2t_2,0) \end{bmatrix}$$

$$\begin{bmatrix} V^1_{k,n-k}(t) & V^2_{k,n-k}(t) & \cdots & V^k_{k,n-k}(t) \\ \vdots & \vdots & \cdots & \vdots \\ V^1_{k,n-2}(t) & V^2_{k,n-2}(t) & \ddots & V^k_{k,n-2}(t) \\ V^1_{k,n-1}(t) & V^2_{k,n-1}(t) & \cdots & V^k_{k,n-1}(t) \end{bmatrix}.$$

From the above matrix multiplication, we get

$$R^i_{k,n}(t) = kt_k V^i_{k,n-k}(t) + \cdots + 3t_3 V^i_{k,n-3}(t) + 2t_2 V^i_{k,n-2}(t).$$

3 .Generalized order-k Van der Laan and Perrin Numbers

Definition 3.1. For $t_s = 1$, $2 \leq s \leq k$, the generalized Van der Laan polynomial $V_{k,n}(t)$ and $V^\infty_{(k)}$ together are reduced to

$$v_{k,n} = \sum_{j=2}^{k} v_{k,n-j}$$

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with boundary conditions

$$v_{k,1-k} = v_{k,2-k} = \ldots = v_{k,-2} = v_{k,-1} = 0 \text{ and } v_{k,0} = 1,$$

which is called generalized order-k Van der Laan numbers (GOkV).

When $k = 3$, it is reduced to ordinary Van der Laan numbers.

Definition 3.2. For $t_s = 1$, $2 \leq s \leq k$, $V_{k,n}^i(t)$ can be written explicitly as

(3.1)
$$v_{k,n}^i = \sum_{j=2}^{k} v_{k,n-j}^i$$

for $n > 0$ and $1 \leq i \leq k$, with boundary conditions

$$v_{k,n}^i = \begin{cases} 1 & \text{if } i - n = k, \\ 0 & \text{otherwise} \end{cases}$$

for $1 - k \leq n \leq 0$, where $v_{k,n}^i$ is the n-th term of i-th sequence. This generalization is called k sequences of the generalized order-k Van der Laan numbers (kSOkV).

When $i = k = 3$, we obtain ordinary Van der Laan numbers and for any integer k, $v_{k,n}^k = v_{k,n}$.

Example 2. By substituting $k = 3$ and $i = 2$, we obtain the generalized order-3 Van der Laan sequence as;

$$v_{3,-2}^2 = 0, \ v_{3,-1}^2 = 1, \ v_{3,0}^2 = 0, \ v_{3,1}^2 = 1, \ v_{3,2}^2 = 1, \ v_{3,3}^2 = 1, \ v_{3,4}^2 = 2, \ldots$$

We give some properties of kSOkV by using properties of k sequences of generalized Van der Laan polynomials.

Corollary 3.1. Using (3.1), we obtain

$$V_n^\sim = A_1^n$$

where

(3.2) $\quad A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 1 & 1 & 1 & \ldots & 1 & 0 \end{bmatrix}_{k \times k} = \begin{bmatrix} 0 & & \\ 0 & & I \\ 0 & & \\ 1 & \ldots & 1 & 0 \end{bmatrix}_{k \times k}$

where I is a $(k-1) \times (k-1)$ identity matrix and $V_n^{\sim}$ is a matrix as;

$$(3.3) \qquad V_n^{\sim} = \begin{bmatrix} v_{k,n-k+1}^1 & v_{k,n-k+1}^2 & \cdots & v_{k,n-k+1}^k \\ \vdots & \vdots & \cdots & \vdots \\ v_{k,n-1}^1 & v_{k,n-1}^2 & \ddots & v_{k,n-1}^k \\ v_{k,n}^1 & v_{k,n}^2 & \cdots & v_{k,n}^k \end{bmatrix}$$

which is contained by $k \times k$ block of $V_{(k)}^{\infty}$ for $t_i = 1$, $2 \le i \le k$.

Proof. It is clear that $V_1^{\sim} = A_1$ and $V_{n+1}^{\sim} = A_1 V_n^{\sim}$ by (3.1). So, by induction, we have $V_n^{\sim} = A_1^n$.

Corollary 3.2. Let $V_n^{\sim}$ be as in (3.3). Then,

$$\det V_n^{\sim} = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ (-1)^n & \text{if } k \text{ is even.} \end{cases}$$

Proof. Obvious from (1.3).

Corollary 3.3. For $1 \le i \le k$ and any positive integers n and m

$$v_{k,n+m}^i = \sum_{j=1}^{k} v_{k,m}^j v_{k,n-k+j}^i.$$

Proof. Obvious from Theorem (2.2).

Corollary 3.4. For $n > 1 - k$,

$$(3.4) \qquad v_{k,n}^1 = v_{k,n-1}^k = v_{k,n-2}^{k-1}.$$

Proof. It is obvious from (3.1) that these sequences are equal with index iteration.

Lemma 3.1. For $n > 1 - k + i$ and $1 < i \le k$,

$$(3.5) \qquad v_{k,n}^i = v_{k,n}^{i-1} + v_{k,n-i}^k.$$

Proof. Assume for $n > 1 - k + i$, $v_{k,n}^{i} - v_{k,n}^{i-1} = t_n$ and show $t_n = v_{k,n-i}^{k}$. First we obtain initial conditions for t_n by using initial conditions of i-th and $(i-1)$-th sequences of kSOkV simultaneously as follows;

$n \setminus i$	$v_{k,n}^{i}$	$v_{k,n}^{i-1}$	$t_n = v_{k,n}^{i} - v_{k,n}^{i-1}$
$1 - k$	0	0	0
$2 - k$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$i - k - 2$	0	0	0
$i - k - 1$	0	1	-1
$i - k$	1	0	1
$i - k + 1$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	0	0

Since initial conditions of t_n are equal to the initial condition of $v_{k,n}^{k}$ with index iteration, then we have,

$$t_n = v_{k,n-i}^{k}.$$

We give the following Theorem by using generalization of MacHenry in [8].

Theorem 3.1. For $n \geq 1$ and $1 \leq i \leq k$,

$$v_{k,n}^{i} = v_{k,n-1}^{k} + v_{k,n-2}^{k} + \cdots + v_{k,n-i}^{k} = \sum_{m=1}^{i} v_{k,n-m}^{k}.$$

Proof. Writing equality (3.5) recursively, we have

$$
\begin{aligned}
v_{k,n}^{i+1} - v_{k,n}^{i} &= v_{k,n-i-1}^{k} \\
v_{k,n}^{i+2} - v_{k,n}^{i+1} &= v_{k,n-i-2}^{k} \\
&\vdots \\
v_{k,n}^{k-2} - v_{k,n}^{k-3} &= v_{k,n-k+2}^{k} \\
v_{k,n}^{k-1} - v_{k,n}^{k-2} &= v_{k,n-k+1}^{k}
\end{aligned}
$$

and by adding these equations side by side, we obtain

$$v_{k,n}^{k-1} - v_{k,n}^{i} = v_{k,n-k+1}^{k} + v_{k,n-k+2}^{k} + \cdots + v_{k,n-i-2}^{k} + v_{k,n-i-1}^{k}.$$

Then, by using the equation $v_{k,n}^{k-1} = v_{k,n+1}^k$ and (3.1), we obtain

$$
\begin{aligned}
v_{k,n}^i &= v_{k,n+1}^k - (v_{k,n-k+1}^k + v_{k,n-k+2}^k + \cdots + v_{k,n-i-2}^k + v_{k,n-i-1}^k) \\
&= v_{k,n-1}^k + v_{k,n-2}^k + \cdots + v_{k,n-k+1}^k \\
&\quad -(v_{k,n-k+1}^k + v_{k,n-k+2}^k + \cdots + v_{k,n-i-2}^k + v_{k,n-i-1}^k) \\
&= v_{k,n-1}^k + v_{k,n-2}^k + \cdots + v_{k,n-i}^k.
\end{aligned}
$$

Now we initiate the generalized Perrin numbers.

Definition 3.3. For $t_s = 1$, $2 \le s \le k$, the generalized Perrin polynomials $R_{k,n}(t)$ and the matrix $R_{(k)}^\infty$ together are reduced to

(3.6)
$$
r_{k,n} = \sum_{j=2}^{k} r_{k,n-j}
$$

with boundary conditions

$$
r_{k,1-k} = (k-2), \; r_{k,2-k} = \ldots = r_{k,-2} = \; r_{k,-1} = -1 \text{ and } r_{k,0} = k,
$$

which is called generalized order-k Perrin numbers (GOkR).

When $k = 3$, it is reduced to ordinary Perrin numbers; $(1, (-1), 3, 0, 2, 3, 2, 5, 5, 7, \ldots)$ with iterating index by two. We rewrite matrix (2.3) for $t_s = 1$, $2 \le s \le k$ and we obtain

$$
R_{(k1)} = [a_{n,i}]_{k \times k} = \begin{bmatrix} ((-1), \; (-2), \ldots, (k-2), 0, k).(A_1)^{-(k-1)} \\ ((-1), \; (-2), \ldots, (k-2), 0, k).(A_1)^{-(k-2)} \\ \vdots \\ ((-1), \; (-2), \ldots, (k-2), 0, k).(A_1)^{-1} \\ ((-1), \; (-2), \ldots, (k-2), 0, k) \end{bmatrix}.
$$

Definition 3.4. For $t_s = 1$, $2 \le s \le k$, $R_{k,n}^i(t)$ can be written explicitly as

$$
r_{k,n}^i = \sum_{j=2}^{k} r_{k,n-j}^i
$$

for $n > 0$ and $1 \le i \le k$, with boundary conditions

$$
r_{k,n}^i = [a_{k+n,i}]_{k \times k} = R_{(k1)}
$$

for $1 - k \le n \le 0$, where $r_{k,n}^i$ is the n-th term of i-th sequence. This generalization is called k sequences of the generalized order-k Perrin numbers (kSOkR).

Although definitions look similar, the initial conditions of this generalization are different from the generalization in [5]. These initial conditions arise from polynomials.

When $i = k = 3$, we obtain ordinary Perrin numbers and for any integer $k \geq 3$, $r_{k,n}^k = r_{k,n}$.

Corollary 3.5. For $1 \leq i \leq k$,

$$r_{k,n}^i = kv_{k,n}^i - (v_{k,n-k+1}^i + \ldots + (k-2)v_{k,n-2}^i).$$

Corollary 3.6. For $1 \leq i \leq k$,

$$r_{k,n}^i(t) = kv_{k,n-k}^i + \cdots + 3v_{k,n-3}^i + 2v_{k,n-2}^i.$$

Conclusion 3.1. There are a number of studies on Fibonacci and Lucas numbers and on their generalizations. In this paper, we showed that these studies can be transferred to the Van der Laan and Perrin numbers. Since our definitions of these numbers are polynomial based, it can be applied to a great number of areas.

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