Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 22, 1089 – 1094

Extended Laguerre Polynomials

Adnan Khan

National College of Business Administration and Economics
Gulberg-III, Lahore, Pakistan
adnankhantariq@gmail.com

G. M. Habibullah

National College of Business Administration and Economics
Gulberg-III, Lahore, Pakistan
mustafa1941@yahoo.com

Abstract

In this paper, by using generalized hypergeometric functions of the type ${}_2F_2$, an extension of the Laguerre polynomials is introduced and similar to those relating to the Laguerre polynomials, a number of generating functions and recurrence relations for this extended Laguerre polynomials have been determined.

Mathematics Subject Classification: 33C45, 05A15, 11B37

Keywords: Polynomials, Generating Functions, Recurrence relations.

1 Introduction

This note concerns with the polynomials $A_{2,n}(x)$ based on ${}_2F_2$ similar to the Laguerre polynomials dealt with in [4] and [12]. Most of the classical results on the Laguerre polynomials can be generalized straight away by using relation involving hypergeometric functions. A large number of relevant properties of the Laguerre orthogonal polynomials, its extensions and its applications are available in books and journals. In this regard we can refer numerous recent works e.g. [1], [3], [5], [6], [9], [10], and [11].

There is a wide range of applications of the Laguerre polynomials in many areas including in permutation statistics. The moments of the measure for these polynomials are the generating functions for permutations according to eight different statistics. Gurland *et al.* [7] has considered a discrete distribution in which the probabilities are expressible by the Laguerre polynomials, is formulated in terms of a probability generating function involving three parameters. Many authors (see [2], [8], and [13] for details) have studied problems of permutation polynomials modulo m, polynomials with integer coefficients that can induce bijections.

We follow the well known techniques to describe the properties of the extended Laguerre polynomials $A_{2,n}(x)$ defined by

$$A_{2,n}(x) = {}_2F_2\left(\frac{-n}{2}, \frac{-n+1}{2}; \frac{1}{2}, 1; x^2\right),$$

where n is any non-negative integer. Rewriting it in series form, we have

$$A_{2,n}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \left[\frac{\left(\frac{-n}{2}\right)_k \left(\frac{-n+1}{2}\right)_k}{\left(\frac{1}{2}\right)_k (1)_k}\right] \frac{x^{2k}}{(2k)!}.$$

By direct evaluation and using Lemma 5, pp 22 of [12], we obtain

$$A_{2,n}(x) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \left[\frac{1}{(n-2k)!(2k)!}\right] \frac{x^{2k}}{(2k)!}. \tag{1.1}$$

Also note that

$$\sum_{n=0}^{\infty} \left[\frac{A_{2,n}(x)}{n!}\right] t^n = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\left[\frac{n}{2}\right]} \left[\frac{1}{(n-2k)!(2k)!}\right] \frac{x^{2k}}{(2k)!}\right] t^n \tag{1.2}$$

which leads to the generating function

$$\sum_{n=0}^{\infty} \frac{A_{2,n}(x) t^n}{n!} = e^t \, {}_0F_2\left(-; \frac{1}{2}, 1; \left(\frac{xt}{2}\right)^2\right). \tag{1.3}$$

2 Main Results

In this section, we prove main results and determine recurrence relations for the extended Laguerre polynomials $A_{2,n}(x)$.

Theorem 2.1:

If c is any positive integer, and n is any non-negative integer then

$$\sum_{n=0}^{\infty}\frac{(c)_n A_{2,n}(x)t^n}{n!}=\frac{1}{(1-t)^c}\,{}_2F_2\left(\frac{c}{2},\frac{c+1}{2};\frac{1}{2},1;\left(\frac{xt}{1-t}\right)^2\right). \quad (2.1)$$

Proof:

From Equation (1.2), we note that

$$\sum_{n=0}^{\infty}(c)_n\left[\frac{A_{2,n}(x)}{n!}\right]t^n=\sum_{n=0}^{\infty}(c)_n\left[\sum_{k=0}^{\left[\frac{n}{2}\right]}\left[\frac{1}{(n-2k)!(2k)!}\right]\frac{x^{2k}}{(2k)!}\right]t^n.$$

By using Lemma 11(8), pp. 57 of [12], we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(c)_n A_{2,n}(x)t^n}{n!}&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c)_{n+2k}\,t^{n+2k}}{n!(2k)!}\frac{x^{2k}}{(2k)!}\\&=\sum_{k=0}^{\infty}\left[\sum_{n=0}^{\infty}\frac{(c+2k)_n t^n}{n!}\right]\left[\frac{(c)_{2k}}{(2k)!}\right]\frac{(xt)^{2k}}{(2k)!}\\&=\frac{1}{(1-t)^c}\sum_{k=0}^{\infty}\left[\frac{(c)_{2k}}{(2k)!}\right]\frac{1}{(2k)!}\left(\frac{xt}{1-t}\right)^{2k},\end{aligned}$$

so that

$$\sum_{n=0}^{\infty}\frac{(c)_n A_{2,n}(x)t^n}{n!}=\frac{1}{(1-t)^c}\,{}_2F_2\left(\frac{c}{2},\frac{c+1}{2};\frac{1}{2},1;\left(\frac{xt}{1-t}\right)^2\right).$$

With $c=1$, it reduces to

$$\sum_{n=0}^{\infty}A_{2,n}(x)t^n=\frac{1}{1-t}\exp\left(\frac{xt}{1-t}\right)^2. \quad (2.2)$$

Theorem 2.2:

If $n\geq 1$, then

$$xDA_{2,n}(x)=nA_{2,n}(x)-nA_{2,n-1}(x),\qquad D=\frac{d}{dx}. \quad (2.3)$$

Proof:

From Equation (1.3)

$$\sum_{n=0}^{\infty}\frac{A_{2,n}(x)t^n}{n!}=e^t\,{}_0F_2\left(—;\frac{1}{2},1;\left(\frac{xt}{2}\right)^2\right).$$

Set $$\sigma_{2,n}(x)=\frac{A_{2,n}(x)}{n!}\quad.$$

Suppose that

$$\psi\left(\frac{x^2t^2}{2}\right)={}_0F_2\left(—;\frac{1}{2},1;\left(\frac{xt}{2}\right)^2\right).$$

Then $$F=e^t\psi\left(\frac{x^2t^2}{2}\right)=\sum_{n=0}^{\infty}\sigma_{2,n}(x)t^n, \tag{2.4}$$

provided that the series is uniformly convergent.
Partial derivatives of F leads to

$$x\frac{\partial F}{\partial x}-t\frac{\partial F}{\partial t}=-tF\ . \tag{2.5}$$

Furthermore together with $\frac{\partial F}{\partial x}=\sum_{n=0}^{\infty}\sigma'_{2,n}(x)t^n$ and $t\frac{\partial F}{\partial t}=\sum_{n=0}^{\infty}n\sigma_{2,n}(x)t^n$,
Equation (2.5), then yields

$$x\sum_{n=0}^{\infty}\sigma'_{2,n}(x)t^n-\sum_{n=0}^{\infty}n\sigma_{2,n}(x)t^n=-\sum_{n=1}^{\infty}\sigma_{2,n-1}(x)t^n.$$

It then follows that $\sigma'_{2,0}(x)=0$, and for $n>1$,

$$x\sigma'_{2,n}(x)-n\sigma_{2,n}(x)=-\sigma_{2,n-1}(x)$$
$$xDA_{2,n}(x)=nA_{2,n}(x)-nA_{2,n-1}(x).$$

Theorem 2.3:
If $n\geq 2$, then

$$DA_{2,n}(x)=2DA_{2,n-1}(x)-DA_{2,n-2}(x)+2xA_{2,n-2}(x). \tag{2.6}$$

Proof: Let

$$F=A(t)\exp\left[x^2\left(\frac{t}{1-t}\right)^2\right]=\sum_{n=0}^{\infty}y_{2,n}(x)t^n, \tag{2.7}$$

so that $$\frac{\partial F}{\partial x}=2x\left(\frac{t}{1-t}\right)^2A(t)\exp\left[x^2\left(\frac{t}{1-t}\right)^2\right]=\sum_{n=0}^{\infty}y'_{2,n}(x)t^n. \tag{2.8}$$

$$\left(1-t\right)^{2}\frac{\partial F}{\partial x}=2xt^{2}A(t)\exp\left[x^{2}\left(\frac{t}{1-t}\right)^{2}\right]=2xt^{2}F\ . \tag{2.9}$$

Consequently

$$\sum_{n=0}^{\infty}y_{2,n}'\left(x\right)t^{n}-2\sum_{n=1}^{\infty}y_{2,n-1}'\left(x\right)t^{n}+\sum_{n=2}^{\infty}y_{2,n-2}'\left(x\right)t^{n}=2x\sum_{n=2}^{\infty}y_{2,n-2}\left(x\right)t^{n}.$$

It thus follows that $y_{2,0}'\left(x\right)=0,\ y_{2,1}'\left(x\right)=0,$ and for $n>2,$

$$DA_{2,n}\left(x\right)=2DA_{2,n-1}\left(x\right)-DA_{2,n-2}\left(x\right)+2xA_{2,n-2}\left(x\right).$$

Theorem 2.4:

If $n\geq 2$, then

$$DA_{2,n}\left(x\right)=2x\sum_{k=0}^{n-2}(n-k-1)A_{2,k}\left(x\right). \tag{2.10}$$

Proof:

By using Equation (2.8), we obtain

$$\sum_{n=0}^{\infty}y_{2,n}'\left(x\right)t^{n}=2x\left[\sum_{n=0}^{\infty}{}^{n+1}C_{n}t^{n+2}\right]\left[\sum_{n=0}^{\infty}y_{2,n}\left(x\right)t^{n}\right]$$

$$=2x\sum_{n=2}^{\infty}\sum_{k=0}^{n-2}(n-k-1)y_{2,k}\left(x\right)t^{n}\ .$$

Hence, we get $\quad DA_{2,n}\left(x\right)=2x\sum_{k=0}^{n-2}(n-k-1)A_{2,k}\left(x\right).$

Similarly we can show that if $n\geq 3$, then

$$nA_{2,n}\left(x\right)=(n-2)A_{2,n-3}\left(x\right)-(3n-4-2x^{2})A_{2,n-2}\left(x\right)+(3n-2)A_{2,n-1}\left(x\right).$$

References

[1] A. Akbary, D. Ghioca and Q. Wang, On permutation polynomials of prescribed shape, Finite Fields and Their Applications, 15(2009), 195 – 206.

[2] E. Aksoy, A. Cesmelioglu, W. Meidl and A. Topuzoglu. On the Carlitz rank of permutation polynomials, Finite Fields and Their Applications, 15(2009), 428 – 440.

[3] S. Alam and A. K. Chongdar, On generating functions of modified Laguerre polynomials, Rev. Real Acad. De Ciencias Zaragoza, 62(2007), 91 – 98.

[4] G. Andrews, R. Askey and R. Roy. Special Functions, Cambridge University Press, 1999.

[5] K. Y. Chen and H. M. Srivastava, A limit relationship between Laguerre and Hermite polynomials. Integral Transforms and Special Functions, 16(2005), 75 – 80.

[6] E. H. Doha, H. M. Ahmed and S. I. El-Soubhy, Explicit formulae for the coefficients of integrated expansions of Laguerre and Hermite polynomials and their integrals, Integral Transforms and Special Functions, 20 (2009), 491 – 503.

[7] J. Gurland, E. E. Chen and F. M. Hernandez, A new discrete distribution involving Laguerre polynomials, Communications in Statistics - Theory and Methods, 12(1983), 1987 – 2004.

[8] I. Krasikov and A. Zarkh, Equioscillatory property of the Laguerre polynomials, J. of Approximation Theory, 162(2010), 2021 – 2047.

[9] D. W. Lee, Properties of multiple Hermite and multiple Laguerre polynomials by the generating function, Integral Transforms and Special Functions, 18(2007), 855 – 869.

[10] K. S. Nisar and M. A. Khan, A note on Binomial and Trinomial operator representations of certain polynomials, Int. J. Math. Analysis, 5(2011), 667 – 674.

[11] V. Radulescu, Rodrigues-type formulae for Hermite and Laguerre polynomials, An. St. Uni. Ovidius constanta, 16(2008), 109 – 116.

[12] E. D. Rainville, Special Functions, The Macmillan Company. New York, 1960.

[13] Q. Wang, On inverse permutation polynomials, Finite Fields and Their Applications, 15(2009), 207 – 213.

Received: August, 2011