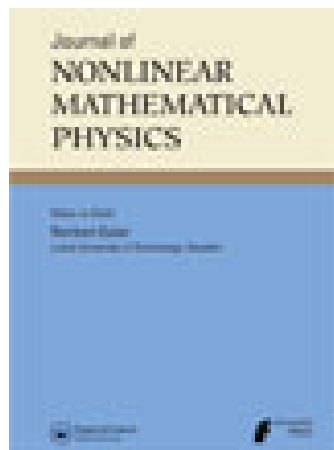


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A note on q-Bernoulli numbers and polynomials

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A note on q -Bernoulli numbers and polynomials

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Abstract

Recently, B. A. Kupershmidt have constructed a reflection symmetries of q -Bernoulli polynomials (see [9]). In this paper we give another construction of a q -Bernoulli polynomials, which form Barnes' multiple Bernoulli polynomials at $q = 1$, cf. [1, 13, 14]. By using q -Volkenborn integration, we can also investigate the properties of the reflection symmetries of these q -Bernoulli polynomials.

1 Introduction

Let \mathbb{Z} , \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will be denoted by the ring of integers, the ring of p -adic integers, the field of p -adic numbers and the completion of algebraic closure of \mathbb{Q}_p and let ν_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q -extension, q is variously considered as an indeterminate, a complex number or a p -adic number. If $q \in \mathbb{C}_p$, then we normally assume $|q - 1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. In this paper, we use the notation:

$$[x] = [x : q] = \frac{1 - q^x}{1 - q}.$$

Hence, $\lim_{q \rightarrow 1} [x] = x$ for any x with $|x|_p \leq 1$ in the present p -adic case. Let d be a fixed integer and let p be a fixed prime number. We set

$$\begin{aligned} X &= \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, cf. [2, 3, 4, 5, 6, 7, 8]. For any positive integer N , we set

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]}$$

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and this can be extended to a distribution on X . This distribution yields an integral for each non-negative integer m :

$$\int_{\mathbb{Z}_p} [x]^m d\mu_q(x) = \int_X [a]^m d\mu_q(a) = \frac{1}{(1-q)^m} \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{i+1}{[i+1]},$$

cf. [5, 20].

The multiple Barnes' Bernoulli polynomials were defined by

$$\left(\prod_{j=1}^r \frac{w_j}{e^{w_j t} - 1} \right) t^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x|w_1, w_2, \dots, w_r) \frac{t^n}{n!}, \quad (1.1)$$

for each $w_j > 0, 0 < t < 1$, cf. [1, 13, 14, 20].

The numbers $B_n^{(r)}(w_1, w_2, \dots, w_r) = B_n^{(r)}(0|w_1, w_2, \dots, w_r)$ are called Barnes' multiple Bernoulli numbers. Barnes employed a representation for his multiple zeta function in terms of contour integrals, generalizing the Hankel integral representation for the gamma function, cf. [1, 13, 14]. These Barnes' multiple zeta function interpolates Barnes' multiple Bernoulli numbers at negative integers, cf. [1, 10, 11, 12, 13, 14, 20]. In [13], K Ota have studied Kummer-type congruences for derivatives of these' Barnes' multiple Bernoulli polynomials. The identities for Barnes' multiple Bernoulli polynomials are now intensively studied by mathematicians and physicists, cf. [7, 10, 11, 12, 13, 15, 16, 17, 19, 20]. As a side remark, we note that special values of Barnes' multiple zeta function at positive integers have come to the foreground in the recent years, both in connection with theoretical physics (Feynman diagrams) and the theory of mixed Tate motives, cf. [10, 14, 15, 16, 17, 18, 19, 20]. The computation of Feynman diagrams has confounded physicists with classes of integrals that are usually hard to be evaluated, both analytically and numerically, cf. [10, 11, 12, 14, 15, 16, 17].

Throughout this paper, we assume that $\alpha_1, \dots, \alpha_k$ are taken in the positive integers and let $w \in \mathbb{Z}_p$. Now, we can rewrite the multiple q -Bernoulli numbers, polynomials as follows, cf. [4]:

$$\beta_n^{(r)}(w, q|\alpha_1, \alpha_2, \dots, \alpha_r) = \underbrace{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [w + \alpha_1 x_1 + \cdots + \alpha_r x_r]^n d\mu_q(x_1) \cdots d\mu_q(x_r),$$

and

$$\beta_n^{(r)}(q|\alpha_1, \alpha_2, \dots, \alpha_r) = \underbrace{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^n d\mu_q(x_1) \cdots d\mu_q(x_r).$$

Originally q -Bernoulli numbers and polynomials were introduced by L. Carlitz in 1948 [2], but they do not seem to be the most natural ones; in particular, they don't appear as the values at non positive integers of various Riemann and Hurwitz q -zeta functions (see [9: p. 412]). In [8], Koblitz have studied the q -analogue of p -adic L -function which are interpolating q -Bernoulli numbers at negative integers and some properties of Carlitz's q -Bernoulli numbers are studies by many authors, cf. [2, 3, 7, 8, 15, 16, 17, 18, 19, 20,

21]. In the recent paper (see [9]), B A Kupershmidt constructed a reflection symmetries of q -Bernoulli polynomials. These properties are seem to be interesting and worthwhile in the areas of number theory and mathematical physics. In this paper we give another construction of a q -Bernoulli polynomials, which are Barnes' multiple Bernoulli polynomials at $q = 1$. By using q -Volkenborn integration, we can also investigate the properties of the reflection symmetries of q -Bernoulli polynomials. Finally, we give the new explicit formulas which are related to these numbers.

2 An extension of q -Bernoulli numbers and polynomials

For $h \in \mathbb{Z}$, we define the extension of Changhee q -Bernoulli polynomials, numbers as follows:

$$\begin{aligned} & \beta_n^{(h,r)}(w, q | \alpha_1, \alpha_2, \dots, \alpha_r) \\ &= \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{r \text{ times}} q^{\sum_{i=1}^r (h-i)x_i} [w + \alpha_1 x_1 + \dots + \alpha_r x_r]^n d\mu_q(x_1) \dots d\mu_q(x_r), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} & \beta_n^{(h,r)}(q | \alpha_1, \alpha_2, \dots, \alpha_r) \\ &= \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \dots + \alpha_r x_r]^n q^{\sum_{i=1}^r (h-i)x_i} d\mu_q(x_1) \dots d\mu_q(x_r). \end{aligned} \tag{2.2}$$

These can be written as

$$\beta_n^{(h,r)}(w, q | \alpha_1, \alpha_2, \dots, \alpha_r) = \sum_{j=0}^n \binom{n}{j} q^{wj} \beta_j^{(h,r)}(q | \alpha_1, \alpha_2, \dots, \alpha_r) [w]^{n-j}.$$

By (2.2), we have

$$\begin{aligned} & \beta_n^{(h,r)}(q | \alpha_1, \alpha_2, \dots, \alpha_r) = \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \dots + \alpha_r x_r]^n q^{\sum_{i=1}^r (h-i)x_i} d\mu_q(x_1) \dots d\mu_q(x_r) \\ &= \frac{1}{(1-q)^n} \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(j\alpha_1 + h)(j\alpha_2 + h - 1) \dots (j\alpha_r + h - r + 1)}{[j\alpha_1 + h][j\alpha_2 + h - 1] \dots [j\alpha_r + h - r + 1]}. \end{aligned} \tag{2.3}$$

Therefore we obtain the following :

Theorem 1. For any $n \geq 0$, we have

$$\beta_n^{(h,r)}(w, q | \alpha_1, \alpha_2, \dots, \alpha_r) = \frac{1}{(1-q)^n} \sum_{j=0}^n \binom{n}{j} (-q^w)^j \prod_{l=1}^r \left(\frac{j\alpha_l + h - l + 1}{[j\alpha_l + h - l + 1]} \right).$$

Remark 1. Note that

$$\begin{aligned}\beta_0^{(2,1)}(q|1) &= \frac{2}{[2]}, & \beta_1^{(2,1)}(q|1) &= \frac{2q+1}{[2][3]}, & \beta_2^{(2,1)}(q|1) &= \frac{2q^2}{[3][4]}, \\ \beta_3^{(2,1)}(q|1) &= -\frac{q^2(q-1)(2[3]+q)}{[3][4][5]}, \dots, \\ \beta_0^{(h,1)}(q|1) &= \frac{h}{[h]}, \\ \beta_1^{(h,1)}(q|1) &= -\frac{(1+q+\dots+q^{h-1})+q(1+q+\dots+q^{h-2})+\dots+q^{h-1}}{[h][h+1]} \dots \\ \beta_0^{(2,2)}(q|1,1) &= \frac{2!}{[2][1]}, & \beta_1^{(2,2)}(q|1,1) &= -\frac{2(q+2)}{[2][3]}, \\ \beta_2^{(2,2)}(q|1,1) &= -\frac{2((q-1)^2+5q)}{[3][4]}, \dots, \\ \beta_0^{(r,r)}(q|\underbrace{1, \dots, 1}_{r \text{ times}}) &= -\frac{r!}{[r][r-1] \dots [2][1]}.\end{aligned}$$

Remark 2. By the definition of $\beta_n^{(h,r)}(q|\alpha_1, \alpha_2, \dots, \alpha_r)$, we note that

$$\beta_n^{(h,r)}(0, q|\alpha_1, \alpha_2, \dots, \alpha_r) = \beta_n^{(h,r)}(q|\alpha_1, \alpha_2, \dots, \alpha_r).$$

By (2.1), (2.2), it is easy to see that

$$\begin{aligned}& \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \dots + \alpha_r x_r]^n q^{\sum_{i=1}^r (h-i)x_i} d\mu_q(x_1) \dots d\mu_q(x_r) \\ &= (q-1) \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \dots + \alpha_r x_r]^{n+1} q^{\sum_{i=1}^r (h-\alpha_i-i)x_i} d\mu_q(x_1) \dots d\mu_q(x_r) \\ &+ \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \dots + \alpha_r x_r]^n q^{\sum_{i=1}^r (h-\alpha_i-i)x_i} d\mu_q(x_1) \dots d\mu_q(x_r).\end{aligned}$$

Thus we have

$$\beta_m^{(h,r)}(q|\underbrace{1, 1, \dots, 1}_{r \text{ times}}) = (q-1)\beta_m^{(h-1,r)}(q|\underbrace{1, 1, \dots, 1}_{r \text{ times}}) + \beta_m^{(h-1,r)}(q|\underbrace{1, 1, \dots, 1}_{r \text{ times}}).$$

It is easy to see that

$$\begin{aligned}
 & \underbrace{\int_X \cdots \int_X}_{r \text{ times}} [w + \alpha_1 x_1 + \cdots + \alpha_r x_r]^n q^{\sum_{i=1}^r (h-i)x_i} d\mu_q(x_1) \cdots d\mu_q(x_r) \\
 &= [d]^{n-r} \sum_{i_1, i_2, \dots, i_r=0}^{d-1} q^{(i_1+\dots+i_r)h-i_2-2i_3-\dots-(r-1)i_r} \\
 &\times \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} \left[\frac{w + \alpha_1 i_1 + \cdots + \alpha_r i_r}{d} + \alpha_1 x_1 + \cdots + \alpha_r x_r; q^d \right]^n \\
 &\times q^{x_1(h-1)d+\dots+x_r(h-r)d} d\mu_{q^d}(x_1) \cdots d\mu_{q^d}(x_r). \tag{2.4}
 \end{aligned}$$

From (2.2), (2.4), we have the following:

Theorem 2. For any positive integer n , we have

$$\begin{aligned}
 \beta_n^{(h,r)}(w, q|\alpha_1, \alpha_2, \dots, \alpha_r) &= [d]^{n-r} \sum_{i_1, i_2, \dots, i_r=0}^{d-1} q^{(i_1+\dots+i_r)h-i_2-2i_3-\dots-(r-1)i_r} \\
 &\times \beta_n^{(h,r)}\left(\frac{w + \alpha_1 i_1 + \cdots + \alpha_r i_r}{d}, q^d|\alpha_1, \dots, \alpha_r\right).
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 \beta_n^{(h,r)}(wd, q|\alpha_1, \alpha_2, \dots, \alpha_r) &= [d]^{n-r} \sum_{i_1, i_2, \dots, i_r=0}^{d-1} q^{(i_1+\dots+i_r)h-i_2-2i_3-\dots-(r-1)i_r} \\
 &\times \beta_n^{(h,r)}\left(w + \frac{\alpha_1 i_1 + \cdots + \alpha_r i_r}{d}, q^d|\alpha_1, \dots, \alpha_r\right).
 \end{aligned}$$

Remark 3. Note that

$$\lim_{q \rightarrow 1} \beta_n^{(h,r)}(w, q|\alpha_1, \alpha_2, \dots, \alpha_r) = B_n^{(r)}(w|\alpha_1, \alpha_2, \dots, \alpha_r), \text{ (see Eq. (1.1)) .}$$

Hence, $\beta_n^{(h,r)}(w, q|\alpha_1, \alpha_2, \dots, \alpha_r)$ can be considered by the q -analogue of Barnes' multiple Bernoulli polynomials. Now, we will give the inverse formula of Eq. (2.3).

Indeed we see

$$\begin{aligned}
 & \sum_{i=0}^n \binom{n}{i} (q-1)^i \beta_i^{(h,r)}(q|\alpha_1, \alpha_2, \dots, \alpha_r) \\
 &= \sum_{i=0}^n \binom{n}{i} (q-1)^i \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^i q^{\sum_{j=1}^r (h-j)x_j} d\mu_q(x_1) \cdots d\mu_q(x_r) \\
 &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} q^{n(\alpha_1 x_1 + \cdots + \alpha_r x_r)} q^{\sum_{j=1}^r (h-j)x_j} d\mu_q(x_1) \cdots d\mu_q(x_r) \\
 &= \frac{(n\alpha_1 + h)(n\alpha_2 + h - 1) \cdots (n\alpha_r + h - r + 1)}{[n\alpha_1 + h][n\alpha_2 + h - 1] \cdots [n\alpha_r + h - r + 1]}.
 \end{aligned}$$

Therefore we obtain the following:

Theorem 3. For $h \in \mathbb{Z}$, we have

$$\sum_{i=0}^m \binom{m}{i} (q-1)^i \beta_i^{(h,r)}(q|\alpha_1, \alpha_2, \dots, \alpha_r) = \prod_{j=1}^r \left(\frac{m\alpha_j + h - j + 1}{[m\alpha_j + h - j + 1]} \right).$$

Let χ be a Dirichlet character with conductor $d \in \mathbb{Z}_{\geq \nu}$. Then we define the generalized Changhee q -Bernoulli numbers as follows: For $m \geq 0$,

$$\begin{aligned} & \beta_{m,\chi}^{(h,r)}(q|\alpha_1, \alpha_2, \dots, \alpha_r) \\ &= \underbrace{\int_X \cdots \int_X}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^m q^{\sum_{j=1}^r (h-j)x_j} \left(\prod_{j=1}^r \chi(x_j) \right) d\mu_q(x_1) \cdots d\mu_q(x_r). \end{aligned} \quad (2.5)$$

By simple calculation, we see that

$$\begin{aligned} & \underbrace{\int_X \cdots \int_X}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^m q^{\sum_{j=1}^r (h-j)x_j} \left(\prod_{j=1}^r \chi(x_j) \right) d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= [d]^{m-r} \sum_{i_1, \dots, i_r=0}^{d-1} q^{hi_1 + \cdots + (h-r+1)i_r} \left(\prod_{j=1}^r \chi(x_j) \right) \\ & \times \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} \left[\frac{\alpha_1 i_1 + \cdots + \alpha_r i_r}{d} + \alpha_1 x_1 + \cdots + \alpha_r x_r; q^d \right]^m \\ & \times q^{\sum_{j=1}^r x_j (h-j)d} d\mu_{q^d}(x_1) \cdots d\mu_{q^d}(x_r). \end{aligned} \quad (2.6)$$

By (2.1), (2.6), we have the following :

Theorem 4. For $h \in \mathbb{Z}$, we have

$$\begin{aligned} & \beta_{m,\chi}^{(h,r)}(q|\alpha_1, \alpha_2, \dots, \alpha_r) = [d]^{m-r} \sum_{i_1, \dots, i_r=0}^{d-1} q^{hi_1 + \cdots + (h-r+1)i_r} \\ & \times \left(\prod_{j=1}^r \chi(x_j) \right) \beta_m^{(h,r)} \left(\frac{\alpha_1 i_1 + \cdots + \alpha_r i_r}{d}, q^d | \alpha_1, \alpha_2, \dots, \alpha_r \right). \end{aligned}$$

Remark. By using our formulae in the case of $q = 1$, we can obtain many new formulas which are related to the multiple Barnes' Bernoulli numbers.

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