



# Bernoulli Polynomials and Convolution Sums

Aeran Kim\*

Department of Mathematics, Chonbuk National University,  
Chonju, Chonbuk 561-756, South Korea.

Original Research  
Article

Received: 02 September 2013  
Accepted: 18 October 2013  
Published: 07 November 2013

## Abstract

In this paper, we study the summations  $\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a))$  and  $\sum_{\substack{(a,b,c,d) \in \mathbb{N}^4 \\ a+b+c+d=n}} (f(a+b+c) + f(b+c+d) + f(c+d+a))$ . Especially, for the first summation we obtain some relations related to Bernoulli and Euler polynomials.

Keywords: Convolution sum; Bernoulli polynomials  
2010 Mathematics Subject Classification: 11A05

## 1 Introduction

The study of arithmetical identities is classical in number theory and such investigations have been carried out by several mathematicians including the legend Srinivasa Ramanujan. For  $n \in \mathbb{N}$  with  $l \in \mathbb{N} \cup \{0\}$ , we define some notations which also appear in many areas of number theory:

$$\sigma_l(n) := \sum_{d|n} d^l, \quad S_l(n) = \sum_{i=1}^{n-1} i^l \quad (n \neq 1), \quad S_l(1) = 0,$$

and  $[x]$  called the *floor* of  $x$ , denotes the greatest integer that does not exceed  $x$ .

The Bernoulli polynomials  $B_m(x)$ , which are usually defined by the exponential generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!},$$

play an important role in various areas of mathematics, including number theory and the theory of finite differences (see [2]). The Bernoulli polynomials satisfy the following well-known identities :

$$B_m(x+1) - B_m(x) = mx^{m-1} \quad (m \in \mathbb{N})$$

\*Corresponding author: E-mail: ae\_ran\_kim@hotmail.com

and

$$\sum_{j=0}^n j^m = \frac{B_{m+1}(n+1) - B_{m+1}(0)}{m+1} \quad (m \in \mathbb{N}), \tag{1.1}$$

where  $\mathbb{N}$  denotes the set of positive integers. We set  $B_m := B_m(0)$ . It is obvious, from the way the polynomials  $B_m(x)$  are constructed, that all the  $B_m$  are rational numbers. It can be shown that  $B_{2m+1} = 0$  for  $m \geq 1$ , and is alternatively positive and negative for even  $m$ .

Let  $n$  be a positive integer and let  $f : \mathbb{Z} \rightarrow \mathbb{C}$  be an even function. In 1858 Liouville stated the surprising result that the sum called the third identity of Liouville

$$\begin{aligned} & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} (f(a-b) - f(a+b)) \\ &= f(0)(\sigma_1(n) - \sigma_0(n)) + \sum_{\substack{d \in \mathbb{N} \\ d|n}} \left(1 + \frac{2n}{d} - d\right) f(d) - 2 \sum_{\substack{d \in \mathbb{N} \\ d|n}} \left(\sum_{v=1}^d f(v)\right) \end{aligned}$$

can be evaluated in terms of sums over the positive integers  $d$  dividing  $n$  (see ([5], p. 247)). In 2000 Huard, Ou, Spearman and Williams (see [7]) proved a far reaching generalization of Liouville's Identity as Proposition 1.1. We introduce Proposition 1.1 which is the motivation of our results Lemma 2.1 and Lemma 2.3.

**Proposition 1.1.** ([7], Theorem 13.1) *Let  $f : \mathbb{Z}^4 \rightarrow \mathbb{C}$  be such that*

$$f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b) \tag{1.2}$$

for all integers  $a, b, x$  and  $y$ . Let  $n \in \mathbb{N}$ . Then

$$\begin{aligned} & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} (f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y) \\ & \quad - f(a, a+b, y-x, y) + f(b-a, b, x, x+y) - f(a+b, b, x, x-y)) \\ &= \sum_{\substack{d \in \mathbb{N} \\ d|n}} \sum_{\substack{x \in \mathbb{N} \\ x < d}} (f(0, \frac{n}{d}, x, d) + f(\frac{n}{d}, 0, d, x) + f(\frac{n}{d}, \frac{n}{d}, d-x, -x) \\ & \quad - f(x, x-d, \frac{n}{d}, \frac{n}{d}) - f(x, d, 0, \frac{n}{d}) - f(d, x, \frac{n}{d}, 0)), \end{aligned} \tag{1.3}$$

where the sum on the left hand side of (1.3) is over all positive integers  $a, b, x, y$  satisfying  $ax + by = n$ , the inner sum on the right hand side is over all positive integers  $x < d$ , and the outer sum on the right hand side is over all positive integers  $d$  dividing  $n$ . (See also ([3], Theorem 1))

In this paper we will find a special formula of the summation

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a))$$

and so, by using this, we will present some relations related to Bernoulli polynomial and the generalized Riemann zeta function :

**Theorem 1.1.** Let  $n \in \mathbb{N}$ . If  $l \in \mathbb{N}$ , then we have

$$\begin{aligned}
 A(l, n) &:= \sum_{i=0}^l \binom{l}{i} \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \\
 &\quad \times \binom{l+1-i+j}{m} B_m \sum_{k=1}^{n-2} (n-k)^{i-j} (n-k-1)^{l+1-i+j-m} \\
 &= \frac{(l+1)(B_{l+2}(n) - B_{l+2}) - (l+2)(B_{l+1}(n) - B_{l+1})}{(l+2)(l+1)}.
 \end{aligned}$$

Furthermore, if  $l \in \mathbb{N} \cup \{0\}$ , then we obtain

$$A(l, n) = S_{l+1}(n) - S_l(n).$$

**Corollary 1.2.** Let  $t, l \in \mathbb{N} \cup \{0\}$ . Then we have

(a)

$$\sum_{l=0}^t A(l, n) = S_{t+1}(n) - S_0(n).$$

(b) Let  $\overline{A(l, n)} = S_{l+1}(n) + S_l(n)$ . So

$$\sum_{l=0}^t A(l, n) \overline{A(l, n)} = S_{t+1}^2(n) - S_0^2(n).$$

**Corollary 1.3.** Let  $l, n \in \mathbb{N}$ . Then we have

$$\begin{aligned}
 &2^{l+1} A\left(l, \left\lceil \frac{n+1}{2} \right\rceil\right) - A(l, n) + 2^{l+1} S_{l+1}\left(\left\lceil \frac{n+1}{2} \right\rceil\right) \\
 &= \frac{E_{l+1}(0) - E_l(0) - (-1)^n (E_{l+1}(n) - E_l(n))}{2},
 \end{aligned}$$

where Euler polynomials  $E_l(x)$  are defined by  $\sum_{l=0}^{\infty} E_l(x) \frac{t^l}{l!} = \frac{2e^{xt}}{e^t+1}$ .

## 2 Proofs of Lemma 2.1, Theorem 1.1 and Corollary 1.2

Using the four basic theta functions, one is led to the formula (see ([7], p. 14)) :

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=2n \\ a,b,x,y \text{ odd}}} \left( (a+b)^{2k} - (a-b)^{2k} \right) = \sum_{\substack{m \in \mathbb{N} \\ m|n \\ \frac{n}{m} \text{ odd}}} 2^{2k} m^{2k+1}, \quad (k, n \in \mathbb{N}). \tag{2.1}$$

There are several arithmetical formulae due to Liouville. A classical one of Liouville very often used in elementary number theory (completely avoiding the hard analysis) is that (see ([4], p. 144)) : For  $n \in \mathbb{N}$  and for  $f : \mathbb{Z} \rightarrow \mathbb{C}$  an even function, the identity

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=2n \\ a,b,x,y \text{ odd}}} (f(a+b) - f(a-b)) = \sum_{\substack{m \in \mathbb{N} \\ m|n \\ \frac{n}{m} \text{ odd}}} (f(2m) - f(0)) \tag{2.2}$$

holds.

**Lemma 2.1.** Let  $n \geq 3$  be a positive integer. Let  $f : \mathbb{Z} \rightarrow \mathbb{C}$  be a function. Then

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a)) = 3 \sum_{k=1}^{n-2} (n-k-1) f(n-k).$$

*Proof.* We note that

$$\begin{aligned} & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a)) \\ &= 3 \sum_{a+b+c=n} f(a+b) = 3 \sum_{k \geq 2} f(k) \sum_{\substack{a+b+c=n \\ a+b=k}} 1 \\ &= 3 \{(n-2)f(n-1) + (n-3)f(n-2) + \dots + (n-(n-1))f(2)\} \\ &= 3 \sum_{k=1}^{n-2} (n-k-1) f(n-k). \end{aligned}$$

□

We can see the arithmetic function  $F_l(n)$  given by

$$F_l(n) := \begin{cases} 1, & l \mid n, \\ 0, & l \nmid n \end{cases}$$

for  $l, n \in \mathbb{N}$  in ([7], p. 32). Using this function we introduce Example 2.2 and Example 2.4.

**Example 2.2.** Let us consider the function  $f(x) = F_2(x)$  in Lemma 2.1. If  $n$  is even, then the left hand side of Lemma 2.1 is

$$\begin{aligned} & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ n \in \mathbb{N} \\ a+b+c=2n}} (f(a+b) + f(b+c) + f(c+a)) \\ &= 3 \sum_{a+b+c=2n} F_2(a+b) = 3 \sum_{\substack{a+b+c=2n \\ 2 \mid (a+b)}} 1 \end{aligned}$$

and the right hand side of Lemma 2.1 is

$$\begin{aligned} & 3 \sum_{k=1}^{2n-2} (2n-k-1) f(2n-k) = 3 \sum_{k=1}^{2n-2} (2n-k-1) F_2(2n-k) \\ &= 3 \sum_{\substack{k=1 \\ 2 \mid (2n-k)}}^{2n-2} (2n-k-1) = 3 \sum_{\substack{k=1 \\ 2 \mid k}}^{2n-2} (2n-k-1) \\ &= 3 \sum_{k=1}^{n-1} (2n-2k-1) = 3(n-1)^2, \end{aligned}$$

where we use  $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ . These facts show that

$$\sum_{\substack{a+b+c=2n \\ 2|(a+b)}} 1 = (n-1)^2 \tag{2.3}$$

Also, if  $n$  is odd, then we obtain

$$\begin{aligned} & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ n \in \mathbb{N} \\ a+b+c=2n-1}} (f(a+b) + f(b+c) + f(c+a)) \\ &= 3 \sum_{a+b+c=2n-1} F_2(a+b) = 3 \sum_{\substack{a+b+c=2n-1 \\ 2|(a+b)}} 1 \\ &= 3 \sum_{k=1}^{2n-3} (2n-2-k) F_2(2n-1-k) = 3 \sum_{\substack{k=1 \\ 2|(2n-1-k)}}^{2n-3} (2n-2-k) \\ &= 3 \sum_{\substack{k=1 \\ 2 \nmid k}}^{2n-3} (2n-2-k) = 3 \left\{ \sum_{k=1}^{2n-3} (2n-2-k) - \sum_{\substack{k=1 \\ 2|k}}^{2n-3} (2n-2-k) \right\} \\ &= 3 \left\{ \sum_{k=1}^{2n-3} (2n-2-k) - \sum_{k=1}^{n-2} (2n-2-2k) \right\} = 3(n-1)^2. \end{aligned} \tag{2.4}$$

Therefore, by (2.3) and (2.4), we conclude that

$$\sum_{\substack{n \in \mathbb{N} \\ a+b+c=n \\ 2|(a+b)}} 1 = \left( \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \right)^2.$$

We can prove Lemma 2.3 in a similar manner as in the proof of Lemma 2.1 as follows :

**Lemma 2.3.** *Let  $n \geq 4$  be a positive integer. Let  $f : \mathbb{Z} \rightarrow \mathbb{C}$  be a function. Then*

$$\sum_{\substack{(a,b,c,d) \in \mathbb{N}^4 \\ a+b+c+d=n}} (f(a+b+c) + f(b+c+d) + f(c+d+a)) = \frac{3}{2} \sum_{k=1}^{n-1} (k-1)(k-2)f(k).$$

*Proof.* We observe that

$$\begin{aligned} & \sum_{\substack{(a,b,c,d) \in \mathbb{N}^4 \\ a+b+c+d=n}} (f(a+b+c) + f(b+c+d) + f(c+d+a)) \\ &= 3 \sum_{a+b+c+d=n} f(a+b+c) = 3 \sum_{k \geq 3} f(k) \sum_{\substack{a+b+c+d=n \\ a+b+c=k}} 1 \\ &= 3 \sum_{k=3}^{n-1} f(k) \{1 + 2 + \dots + (k-2)\} = 3 \sum_{k=3}^{n-1} f(k) \cdot \frac{(k-1)(k-2)}{2} \\ &= \frac{3}{2} \sum_{k=1}^{n-1} (k-1)(k-2)f(k). \end{aligned}$$

□

**Example 2.4.** We apply the function  $f(x) = F_2(x)$  in Lemma 2.3. If  $n$  is even, then the left hand side of Lemma 2.3 is

$$\begin{aligned} & \sum_{\substack{(a,b,c,d) \in \mathbb{N}^4 \\ n \in \mathbb{N} \\ a+b+c+d=2n}} (f(a+b+c) + f(b+c+d) + f(c+d+a)) \\ &= 3 \sum_{a+b+c+d=2n} F_2(a+b+c) = 3 \sum_{\substack{a+b+c+d=2n \\ 2|(a+b+c)}} 1 \end{aligned}$$

and the right hand side of Lemma 2.3 is

$$\begin{aligned} & \frac{3}{2} \sum_{k=1}^{2n-1} (k-1)(k-2)f(k) = \frac{3}{2} \sum_{k=1}^{2n-1} (k-1)(k-2)F_2(k) \\ &= \frac{3}{2} \sum_{\substack{k=1 \\ 2|k}}^{2n-1} (k-1)(k-2) = \frac{3}{2} \sum_{k=1}^{n-1} (2k-1)(2k-2) \\ &= \frac{3}{2} \sum_{k=1}^{n-1} \{4k^2 - 6k + 2\} = \frac{1}{2} \{4n^3 - 15n^2 + 17n - 6\}, \end{aligned}$$

where we use  $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$ . These facts lead us to

$$\sum_{\substack{a+b+c+d=2n \\ 2|(a+b+c)}} 1 = \frac{1}{6} \{4n^3 - 15n^2 + 17n - 6\}. \tag{2.5}$$

Similarly, if  $n$  is odd, then we have

$$\begin{aligned} & \sum_{\substack{(a,b,c,d) \in \mathbb{N}^4 \\ n \in \mathbb{N} \\ a+b+c+d=2n-1}} (f(a+b+c) + f(b+c+d) + f(c+d+a)) \\ &= 3 \sum_{a+b+c+d=2n-1} F_2(a+b+c) = 3 \sum_{\substack{a+b+c+d=2n-1 \\ 2|(a+b+c)}} 1 \\ &= \frac{3}{2} \sum_{k=1}^{2n-2} (k-1)(k-2)F_2(k) = \frac{3}{2} \sum_{\substack{k=1 \\ 2|k}}^{2n-2} (k-1)(k-2) \\ &= \frac{3}{2} \sum_{k=1}^{n-1} (2k-1)(2k-2) = \frac{1}{2} \{4n^3 - 15n^2 + 17n - 6\}. \end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6), we conclude that

$$\sum_{\substack{n \in \mathbb{N} \\ a+b+c+d=n \\ 2|(a+b+c)}} 1 = \frac{1}{6} \left\{ 4 \left[ \frac{n+1}{2} \right]^3 - 15 \left[ \frac{n+1}{2} \right]^2 + 17 \left[ \frac{n+1}{2} \right] - 6 \right\}.$$

**Proof of Theorem 1.1.** Let  $l \in \mathbb{N} \cup \{0\}$ . We apply  $f(x) = x^l$  in Lemma 2.1. Then the left hand side is

$$\begin{aligned} & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a)) \\ &= 3 \sum_{a+b+c=n} f(a+b) = 3 \sum_{a+b+c=n} (a+b)^l \\ &= 3 \sum_{a+b+c=n} \sum_{i=0}^l \binom{l}{i} a^{l-i} b^i \\ &= 3 \sum_{i=0}^l \binom{l}{i} \sum_{a+b+c=n} a^{l-i} b^i c^0. \end{aligned} \tag{2.7}$$

Here, we can observe that

$$\begin{aligned} & \sum_{a+b+c=n} a^{l-i} b^i c^0 \\ &= \sum_{a+b=n-1} a^{l-i} b^i 1^0 + \sum_{a+b=n-2} a^{l-i} b^i 2^0 + \dots + \sum_{a+b=2} a^{l-i} b^i (n-2)^0 \\ &= \sum_{a+b=n-1} a^{l-i} b^i + \sum_{a+b=n-2} a^{l-i} b^i + \dots + \sum_{a+b=2} a^{l-i} b^i. \end{aligned} \tag{2.8}$$

Then we can calculate the first term of (2.8) as follows:

$$\begin{aligned} & \sum_{a+b=n-1} a^{l-i} b^i = \sum_{a=1}^{n-2} a^{l-i} \{(n-1) - a\}^i \\ &= \sum_{a=1}^{n-2} a^{l-i} \sum_{j=0}^i \binom{i}{j} (n-1)^{i-j} (-1)^j a^j \\ &= \sum_{j=0}^i \binom{i}{j} (n-1)^{i-j} (-1)^j \sum_{a=1}^{n-2} a^{l-i+j} \\ &= \sum_{j=0}^i \binom{i}{j} (n-1)^{i-j} (-1)^j \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^m \binom{l+1-i+j}{m} \\ & \quad \times B_m (n-2)^{l+1-i+j-m} \\ &= \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \binom{l+1-i+j}{m} B_m \\ & \quad \times (n-1)^{i-j} (n-2)^{l+1-i+j-m}, \end{aligned}$$

where we use the Faulhaber's sum (see [1]) as

$$\sum_{k=1}^n k^p = \frac{1}{p+1} \sum_{j=0}^p (-1)^j \binom{p+1}{j} B_j n^{p+1-j}$$

with the Bernoulli numbers  $B_j$ . Also we can apply the same method into the other terms of (2.8). So (2.8) becomes

$$\begin{aligned}
 & \sum_{a+b+c=n} a^{l-i} b^i c^0 \\
 &= \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \binom{l+1-i+j}{m} B_m \\
 & \quad \times \left\{ (n-1)^{i-j} (n-2)^{l+1-i+j-m} + (n-2)^{i-j} (n-3)^{l+1-i+j-m} \right. \\
 & \quad \left. + \dots + (n-(n-2))^{i-j} (n-(n-1))^{l+1-i+j-m} \right\} \\
 &= \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \binom{l+1-i+j}{m} B_m \\
 & \quad \times \sum_{k=1}^{n-2} (n-k)^{i-j} (n-k-1)^{l+1-i+j-m}.
 \end{aligned}$$

So (2.7) can be written as

$$\begin{aligned}
 & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a)) \\
 &= 3 \sum_{i=0}^l \binom{l}{i} \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \\
 & \quad \times \binom{l+1-i+j}{m} B_m \sum_{k=1}^{n-2} (n-k)^{i-j} (n-k-1)^{l+1-i+j-m}.
 \end{aligned} \tag{2.9}$$

On the other hand, the right hand side of Lemma 2.1 is

$$\begin{aligned}
 & 3 \sum_{k=1}^{n-2} (n-k-1)(n-k)^l \\
 &= 3 \sum_{k=1}^{n-2} \left\{ (n-k)^{l+1} - (n-k)^l \right\} \\
 &= 3 \left[ \left\{ (n-1)^{l+1} - (n-1)^l \right\} + \left\{ (n-2)^{l+1} - (n-2)^l \right\} \right. \\
 & \quad \left. + \dots + \left\{ 3^{l+1} - 3^l \right\} + \left\{ 2^{l+1} - 2^l \right\} \right] \\
 &= 3 \left\{ \sum_{k=1}^{n-1} k^{l+1} - \sum_{k=1}^{n-1} k^l \right\}.
 \end{aligned} \tag{2.10}$$

Equating (2.9) and (2.10), we obtain



$$\begin{aligned}
 A(l, n) &:= \sum_{i=0}^l \binom{l}{i} \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \\
 &\quad \times \binom{l+1-i+j}{m} B_m \sum_{k=1}^{n-2} (n-k)^{i-j} (n-k-1)^{l+1-i+j-m} \\
 &= \sum_{k=1}^{n-1} k^{l+1} - \sum_{k=1}^{n-1} k^l.
 \end{aligned} \tag{2.11}$$

For  $l \in \mathbb{N} \cup \{0\}$ , we can write  $A(l, n) = S_{l+1}(n) - S_l(n)$ .

Now, let  $l \in \mathbb{N}$ . Then, by (1.1) and  $B_l = B_l(0)$ , the right hand side of (2.11) is

$$\begin{aligned}
 &\sum_{k=1}^{n-1} k^{l+1} - \sum_{k=1}^{n-1} k^l \\
 &= \frac{B_{l+2}(n) - B_{l+2}(0)}{l+2} - \frac{B_{l+1}(n) - B_{l+1}(0)}{l+1} \\
 &= \frac{(l+1)(B_{l+2}(n) - B_{l+2}) - (l+2)(B_{l+1}(n) - B_{l+1})}{(l+2)(l+1)},
 \end{aligned} \tag{2.12}$$

so that we have

$$A(l, n) = \frac{(l+1)(B_{l+2}(n) - B_{l+2}) - (l+2)(B_{l+1}(n) - B_{l+1})}{(l+2)(l+1)}.$$

□

*Remark 2.1.* We recall the Riemann zeta function and the generalized Riemann(or Hurwitz) zeta function are given by  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$  and  $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(k+q)^s}$ , respectively. Equation (2.10) can be written as

$$\begin{aligned}
 &3 \sum_{k=1}^{n-2} (n-k-1)(n-k)^l \\
 &= 3 \sum_{k=1}^{n-2} \left\{ (n-k)^{l+1} - (n-k)^l \right\} \\
 &= 3 \left\{ \zeta(-l, n) - \zeta(-l-1, n) - \zeta(-l) + \zeta(-l-1) \right\}.
 \end{aligned} \tag{2.13}$$

Indeed, observe that

$$\begin{aligned}
 &\sum_{k=1}^{n-2} \left\{ (n-k)^{l+1} - (n-k)^l \right\} \\
 &= \left\{ 1^{l+1} + 2^{l+1} + 3^{l+1} + \dots + (n-2)^{l+1} + (n-1)^{l+1} \right\} \\
 &\quad - \left\{ 1^l + 2^l + 3^l + \dots + (n-2)^l + (n-1)^l \right\}
 \end{aligned}$$

and

$$\begin{aligned} \zeta(-l-1) - \zeta(-l-1, n) &= \sum_{n=1}^{\infty} n^{l+1} - \sum_{k=0}^{\infty} (k+n)^{l+1} \\ &= \left\{ 1^{l+1} + 2^{l+1} + \dots + (n-1)^{l+1} + n^{l+1} + (n+1)^{l+1} + \dots \right\} \\ &\quad - \left\{ n^{l+1} + (n+1)^{l+1} + (n+2)^{l+1} + (n+3)^{l+1} + \dots \right\} \\ &= 1^{l+1} + 2^{l+1} + 3^{l+1} + \dots + (n-2)^{l+1} + (n-1)^{l+1}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \zeta(-l) - \zeta(-l, n) &= \sum_{n=1}^{\infty} n^l - \sum_{k=0}^{\infty} (k+n)^l \\ &= 1^l + 2^l + 3^l + \dots + (n-2)^l + (n-1)^l. \end{aligned}$$

Thus we can rewrite (2.11) as

$$A(l, n) = \zeta(-l, n) - \zeta(-l-1, n) - \zeta(-l) + \zeta(-l-1).$$

**Example 2.5.** If  $l = 3$  in Theorem 1.1, then we have

$$A(3, n) = \frac{1}{60}n(12n^4 - 45n^3 + 50n^2 - 15n - 2).$$

**Proof of Corollary 1.2.** (a) From Theorem 1.1, we obtain

$$\begin{aligned} \sum_{l=0}^t A(l, n) &= \sum_{l=0}^t \left\{ \sum_{k=1}^{n-1} k^{l+1} - \sum_{k=1}^{n-1} k^l \right\} = \sum_{k=1}^{n-1} \sum_{l=0}^t \{k^{l+1} - k^l\} \\ &= \sum_{k=1}^{n-1} \{(k-1) + (k^2 - k) + (k^3 - k^2) + \dots + (k^t - k^{t-1}) + (k^{t+1} - k^t)\} \\ &= \sum_{k=1}^{n-1} \{k^{t+1} - 1\} = S_{t+1}(n) - S_0(n). \end{aligned}$$

(b) By the definition of  $\overline{A(l, n)} = S_{l+1}(n) + S_l(n)$ , we have

$$\begin{aligned} \sum_{l=0}^t A(l, n) \overline{A(l, n)} &= \sum_{l=0}^t \{S_{l+1}^2(n) - S_l^2(n)\} \\ &= (S_1^2(n) - S_0^2(n)) + (S_2^2(n) - S_1^2(n)) + (S_3^2(n) - S_2^2(n)) + \dots \\ &\quad + (S_t^2(n) - S_{t-1}^2(n)) + (S_{t+1}^2(n) - S_t^2(n)) \\ &= S_{t+1}^2(n) - S_0^2(n). \end{aligned}$$

□

**Proof of Corollary 1.3.** In ([6], (2.1), Theorem 2.1) we can see that

$$T_l(n) := \sum_{i=0}^{n-1} (-1)^i i^l,$$

$$T_l(n) = \frac{E_l(0) - (-1)^n E_l(n)}{2} = 2^{l+1} S_l \left( \left[ \frac{n+1}{2} \right] \right) - S_l(n),$$

for  $l, n \in \mathbb{N}$ . These facts lead us to

$$\begin{aligned} & T_{l+1}(n) - T_l(n) \\ &= \frac{E_{l+1}(0) - (-1)^n E_{l+1}(n)}{2} - \frac{E_l(0) - (-1)^n E_l(n)}{2} \\ &= \frac{E_{l+1}(0) - E_l(0) - (-1)^n (E_{l+1}(n) - E_l(n))}{2}. \end{aligned} \tag{2.14}$$

On the other hand,  $T_{l+1}(n) - T_l(n)$  can be rewritten as

$$\begin{aligned} & T_{l+1}(n) - T_l(n) \\ &= \left\{ 2^{l+2} S_{l+1} \left( \left[ \frac{n+1}{2} \right] \right) - S_{l+1}(n) \right\} - \left\{ 2^{l+1} S_l \left( \left[ \frac{n+1}{2} \right] \right) - S_l(n) \right\} \\ &= 2^{l+1} \left\{ S_{l+1} \left( \left[ \frac{n+1}{2} \right] \right) - S_l \left( \left[ \frac{n+1}{2} \right] \right) \right\} + 2^{l+1} S_{l+1} \left( \left[ \frac{n+1}{2} \right] \right) \\ &\quad - \{ S_{l+1}(n) - S_l(n) \} \\ &= 2^{l+1} A \left( l, \left[ \frac{n+1}{2} \right] \right) + 2^{l+1} S_{l+1} \left( \left[ \frac{n+1}{2} \right] \right) - A(l, n), \end{aligned} \tag{2.15}$$

where we use Theorem 1.1. Equating (2.14) and (2.15), we obtain the proof. □

### 3 Conclusions

In this paper, we study a special case summation  $\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a))$  and we show this summation is related to Bernoulli and Euler polynomials.

### Competing Interests

The author declares that no competing interests exist.

### References

- [1] Encyclopedia. Faulhaber's sum. [http://en.wikipedia.org/wiki/Faulhaber's\\_formula](http://en.wikipedia.org/wiki/Faulhaber's_formula). (Last accessed on 1 October 2013 at 09:10).
- [2] Chen WYC, Fu AM, Zhang IF. Faulhaber's theorem on power sums. *Discrete Mathematics*. 2009;309:2974-2981.

- [3] Huard JG, Ou ZM, Spearman BK, Williams KS. Elementary Evaluation of Certain Convolution Sums Involving Divisor Functions. *Number theory for the millennium, II*, 2002;229–274.
- [4] Liouville, J. Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres. 1858;3:143–152.
- [5] Liouville J. Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres. 1858;3:241-250.
- [6] Sun ZW. Introduction to Bernoulli and Euler polynomials. In: A lecture given in Taiwan on June 6, 2002.
- [7] Williams KS. *Number Theory in the Spirit of Liouville*: London Mathematical Society, Student Texts 76, Cambridge; 2011 [6mm]

---

©2014 Kim; This is an Open Access article distributed under the terms of the Creative Commons Attribution License <http://creativecommons.org/licenses/by/3.0>, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

**Peer-review history:**

The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)

[www.sciencedomain.org/review-history.php?iid=313&id=6&aid=2446](http://www.sciencedomain.org/review-history.php?iid=313&id=6&aid=2446)