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Research Article

Some Identities on Laguerre Polynomials in Connection with Bernoulli and Euler Numbers

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We study some interesting identities and properties of Laguerre polynomials in connection with Bernoulli and Euler numbers. These identities are derived from the orthogonality of Laguerre polynomials with respect to inner product $\langle f, g \rangle = \int_0^\infty e^{-x^2} f(x)g(x)dx$.

1. Introduction/Preliminaries

As is well known, Laguerre polynomials are defined by the generating function as

$$\frac{\exp(-xt/(1-t))}{1-t} = \sum_{n=0}^{\infty} L_n(x)t^n \tag{1.1}$$

(see [1, 2]). By (1.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty} L_n(x)t^n &= \frac{\exp(-xt/(1-t))}{1-t} = \sum_{r=0}^{\infty} \frac{(-1)^r x^r t^r}{r!}(1-t)^{-(r+1)} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^r x^r (r+s)!}{r!r!s!} t^{r+s} = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} \frac{(-1)^r \binom{n}{r}}{r!} x^r\right) t^n.\end{aligned} \tag{1.2}$$

Thus, from (1.2), we have

$$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r \binom{n}{r}}{r!} x^r. \tag{1.3}$$

By (1.3), we see that $L_n(x)$ is a polynomial of degree n with rational coefficients and the leading coefficient $(-1)^n/n!$. It is well known that Rodrigues' formula is given by

$$L_n(x) = \frac{1}{n!} e^x \left(\frac{d^n}{dx^n} e^{-x} x^n \right) \tag{1.4}$$

(see [1–27]). From (1.1), we can derive the following of Laguerre polynomials:

$$\begin{gathered} L_0(x) = 1, \qquad L_1(x) = -x + 1, \\ (n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x), \quad (n \geq 1), \end{gathered} \tag{1.5}$$

$$L'_n(x) = L'_{n-1}(x) - L_{n-1}(x) = 0, \quad (n \geq 1), \tag{1.6}$$

$$xL'_n(x) = nL_n(x) - nL_{n-1}(x) = 0, \quad (n \geq 1). \tag{1.7}$$

By (1.7), we easily see that $u = L_n(x)$ is a solution of the following differential equation of order 2:

$$xu''(x) + (1-x)u'(x) + nu(x) = 0. \tag{1.8}$$

The Bernoulli numbers, B_n, are defined by the generating function as

$$\frac{t}{e^t - 1} = e^{Bt} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \tag{1.9}$$

(see [1–28, 28]), with the usual convention about replacing B^n by B_n.

It is well known that Bernoulli polynomials of degree n are given by

$$B_n(x) = (B + x)^n = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} x^l \tag{1.10}$$

(see [2, 26]). Thus, from (1.10), we have

$$B'_n(x) = \frac{dB_n(x)}{dx} = nB_{n-1}(x) \tag{1.11}$$

(see [3–12]). From (1.9) and (1.10), we can derive the following recurrence relation:

$$B_0 = 1, \qquad (B+1)^n - B_n = \delta_{1,n} \tag{1.12}$$

where $\delta_{n,k}$ is Kronecker's symbol.

The Euler polynomials $E_n(x)$ are also defined by the generating function as

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \tag{1.13}$$

(see [27, 28]), with the usual convention about replacing $E^n(x)$ by $E_n(x)$.

In this special case, $x = 0$, $E_n(0) = E_n$ are called the nth Euler numbers. From (1.13), we note that the recurrence formula of E_n is given by

$$E_0 = 1, \qquad (E+1)^n + E_n = 2\delta_{0,n} \tag{1.14}$$

(see [24]). Finally, we introduce Hermite polynomials, which are defined by

$$e^{2xt-t^2} = e^{H(x)t} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{1.15}$$

(see [29]). In the special case, $x = 0$, $H_n(0) = H_n$ is called the n-th Hermite number. By (1.15), we get

$$H_n(x) = (H+2x)^n = \sum_{l=0}^{n} \binom{n}{l} H_{n-l} 2^l x^l \tag{1.16}$$

(see [29]). It is not difficult to show that

$$\int_0^{\infty} e^{-x} L_m(x) L_n(x)\,dx = \delta_{m,n}, \quad (m, n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}). \tag{1.17}$$

In the present paper, we investigate some interesting identities and properties of Laguerre polynomials in connection with Bernoulli, Euler, and Hermite polynomials. These identities and properties are derived from (1.17).

2. Some Formulae on Laguerre Polynomials in Connection with Bernoulli, Euler, and Hermite Polynomials

Let

$$\mathbf{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\}. \tag{2.1}$$

Then $\mathbf{P}_n$ is an inner product space with the inner product

$$\langle p_1(x), p_2(x) \rangle = \int_0^{\infty} e^{-x} p_1(x) p_2(x)\,dx, \quad (p_1(x), p_2(x) \in \mathbf{P}_n). \tag{2.2}$$

By (1.17), (2.1), and (2.2), we see that $L_0(x), L_1(x), \ldots, L_n(x)$ are orthogonal basis for $\mathbf{P}_n$.

For $p(x) \in \mathbf{P}_n$, it is given by

$$p(x) = \sum_{k=0}^{n} C_k L_k(x), \tag{2.3}$$

where

$$C_k = \langle p(x), L_k(x) \rangle = \int_0^\infty e^{-x} L_k(x) p(x) dx = \frac{1}{k!} \int_0^\infty \left(\frac{d^k}{dx^k} e^{-x} x^k \right) p(x) dx. \tag{2.4}$$

Let us take $p(x) = x^n \in \mathbf{P}_n$. From (2.3) and (2.4), we note that

$$\begin{aligned} C_k &= \frac{1}{k!} \int_0^\infty \left(\frac{d^k}{dx^k} e^{-x} x^k \right) x^n dx = \frac{-n}{k!} \int_0^\infty \left(\frac{d^{k-1}}{dx^{k-1}} e^{-x} x^k \right) x^{n-1} dx \\ &= \frac{(-n)(-(n-1))}{k!} \int_0^\infty \left(\frac{d^{k-2}}{dx^{k-2}} e^{-x} x^k \right) x^{n-2} dx \\ &= \cdots \\ &= (-1)^k \frac{n(n-1)\cdots(n-k+1)}{k!} \int_0^\infty e^{-x} x^n dx = (-1)^k \binom{n}{k} n!. \end{aligned} \tag{2.5}$$

Therefore, by (2.3), (2.4), and (2.5), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$x^n = n! \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_k(x). \tag{2.6}$$

Let us consider $p(x) = B_n(x) \in \mathbf{P}_n$. Then, by (2.3) and (2.4), we get

$$\begin{aligned} C_k &= \frac{1}{k!} \int_0^\infty \left(\frac{d^k}{dx^k} e^{-x} x^k \right) B_n(x) dx = \frac{-n}{k!} \int_0^\infty \left(\frac{d^{k-1}}{dx^{k-1}} e^{-x} x^k \right) B_{n-1}(x) dx \\ &= \frac{(-n)(-(n-1))}{k!} \int_0^\infty \left(\frac{d^{k-2}}{dx^{k-2}} e^{-x} x^k \right) B_{n-2}(x) dx \\ &= \cdots \\ &= (-1)^k \frac{n(n-1)\cdots(n-k+1)}{k!} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} \int_0^\infty e^{-x} x^{k+l} dx \\ &= (-1)^k \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} (k+l)! = n! (-1)^k \sum_{l=0}^{n-k} \binom{k+l}{k} \frac{B_{n-k-l}}{(n-k-l)!}. \end{aligned} \tag{2.7}$$

Therefore, by (2.3), (2.4), and (2.7), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$B_n(x) = n!\sum_{k=0}^{n}\sum_{l=0}^{n-k}(-1)^k\binom{k+l}{k}\frac{B_{n-k-l}}{(n-k-l)!}L_k(x). \tag{2.8}$$

Let us take $p(x) = E_n(x) \in \mathbf{P}_n$. By the same method, we easily see that

$$E_n(x) = n!\sum_{k=0}^{n}\sum_{l=0}^{n-k}(-1)^k\binom{k+l}{k}\frac{E_{n-k-l}}{(n-k-l)!}L_k(x). \tag{2.9}$$

For $p(x) = H_n(x) \in \mathbf{P}_n$, we have

$$H_n(x) = \sum_{k=0}^{n}C_k L_k(x), \tag{2.10}$$

where

$$\begin{aligned}
C_k &= \frac{1}{k!}\int_0^\infty\left(\frac{d^k}{dx^k}e^{-x}x^k\right)H_n(x)dx = \frac{-2n}{k!}\int_0^\infty\left(\frac{d^{k-1}}{dx^{k-1}}e^{-x}x^k\right)H_{n-1}(x)dx \\
&= \frac{(-2n)(-2(n-1))}{k!}\int_0^\infty\left(\frac{d^{k-2}}{dx^{k-2}}e^{-x}x^k\right)H_{n-2}(x)dx \\
&= \cdots \\
&= \frac{(-2n)(-2(n-1))\cdots(-2(n-k+1))}{k!}\int_0^\infty e^{-x}x^k H_{n-k}(x)dx \\
&= \frac{(-1)^k 2^k n(n-1)\cdots(n-k+1)}{k!}\sum_{l=0}^{n-k}\binom{n-k}{l}H_{n-k-l}2^l\int_0^\infty e^{-x}x^{k+l}dx \\
&= (-1)^k\binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}2^{k+l}H_{n-k-l}(k+l)! = n!(-1)^k\sum_{l=0}^{n-k}2^{k+l}\binom{k+l}{k}\frac{H_{n-k-l}}{(n-k-l)!}.
\end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$, one has*

$$H_n(x) = n!\sum_{k=0}^{n}\sum_{l=0}^{n-k}(-1)^k 2^{k+l}\binom{k+l}{k}\frac{H_{n-k-l}}{(n-k-l)!}L_k(x). \tag{2.12}$$

Let $p(x) = \sum_{k=0}^{n} B_k(x)B_{n-k}(x) \in \mathbf{P}_n$. Then we have

$$p(x) = \sum_{k=0}^{n}B_k(x)B_{n-k}(x) = \sum_{k=0}^{n}C_k L_k(x), \tag{2.13}$$

where

$$C_k = \frac{1}{k!}\int_0^\infty \left(\frac{d^k}{dx^k}e^{-x}x^k\right)p(x)dx. \tag{2.14}$$

In [15], it is known that

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(x) = \frac{2}{n+2}\sum_{l=0}^{n-2}\binom{n+2}{l}B_{n-l}B_l(x) + (n+1)B_n(x). \tag{2.15}$$

By (2.14) and (2.15), we get

$$\begin{aligned} C_k = \frac{1}{k!}\Bigg\{ &\frac{2}{n+2}\sum_{l=0}^{n-2}\binom{n+2}{l}B_{n-l}\int_0^\infty\left(\frac{d^k}{dx^k}e^{-x}x^k\right)B_l(x)dx \\ &+(n+1)\int_0^\infty\left(\frac{d^k}{dx^k}e^{-x}x^k\right)B_n(x)dx\Bigg\}. \end{aligned} \tag{2.16}$$

From (2.16), we can derive the following equations ((2.17)-(2.18)):

$$C_n = (-1)^n(n+1)!, \qquad C_{n-1} = n(n+1)!(-1)^{n-1} - \frac{1}{2}(-1)^{n-1}(n+1)!. \tag{2.17}$$

For $0 \le k \le n-2$, we have

$$\begin{aligned} C_k = &\frac{2}{n+2}\sum_{l=k}^{n-2}\sum_{m=0}^{l-k}\binom{n+2}{l}\binom{k+m}{k}l!(-1)^k B_{n-l}\frac{B_{l-k-m}}{(l-k-m)!} \\ &+ (-1)^k(n+1)!\sum_{m=0}^{n-k}\binom{k+m}{k}\frac{B_{n-k-m}}{(n-k-m)!}. \end{aligned} \tag{2.18}$$

Therefore, by (2.13), (2.17), and (2.18), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned} \sum_{k=0}^{n} B_k(x)B_{n-k}(x) = \sum_{k=0}^{n-2}\Bigg\{ &\frac{2}{n+2}\sum_{l=k}^{n-2}\sum_{m=0}^{l-k}(-1)^k l!\binom{n+2}{l}\binom{k+m}{k}B_{n-l}\frac{B_{l-k-m}}{(l-k-m)!} \\ &+(-1)^k(n+1)!\sum_{m=0}^{n-k}\binom{k+m}{k}\frac{B_{n-k-m}}{(n-k-m)!}\Bigg\}L_k(x) \\ &+\left(n(n+1)!(-1)^{n-1} - \frac{1}{2}(-1)^{n-1}(n+1)!\right)L_{n-1}(x) + (-1)^n(n+1)!L_n(x). \end{aligned} \tag{2.19}$$

Let us take $p(x) = \sum_{k=0}^{n} E_k(x)E_{n-k}(x) \in \mathbf{P}_n$. By (2.3) and (2.4), we get

$$p(x) = \sum_{k=0}^{n} E_k(x)E_{n-k}(x) = \sum_{k=0}^{n} C_k L_k(x), \tag{2.20}$$

where

$$C_k = \frac{1}{k!}\int_0^{\infty}\left(\frac{d^k}{dx^k}e^{-x}x^k\right)p(x)dx. \tag{2.21}$$

It is known (see [15]) that

$$\sum_{k=0}^{n} E_k(x)E_{n-k}(x) = \sum_{k=0}^{n-1}\frac{(n+1)\binom{n}{k}}{n-k+1}\left(\sum_{l=k}^{n} E_{l-k}E_{n-l} - 2E_{n-k}\right)E_k(x) + 2(n+1)E_n(x). \tag{2.22}$$

From (2.20), (2.21), and (2.22), we can derive the following equations ((2.23)-(2.24)):

$$C_n = \frac{(-1)^n}{n!}2(n+1)(n!)^2 = 2(-1)^n(n+1)!. \tag{2.23}$$

For $0 \le k \le n-1$, we have

$$\begin{aligned} C_k = {} & \sum_{l=k}^{n-1}\frac{(n+1)\binom{n}{l}}{(n-l+1)!}\left(\sum_{m=l}^{n} E_{m-l}E_{n-m} - 2E_{n-l}\right)\sum_{p=0}^{l-k}(-1)^k l!\binom{k+p}{k}\frac{E_{l-k-p}}{(l-k-p)!} \\ & + 2(n+1)!(-1)^k\sum_{p=0}^{n-k}\binom{k+p}{k}\frac{E_{n-k-p}}{(n-k-p)!}. \end{aligned} \tag{2.24}$$

Therefore, by (2.20) and (2.24), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned} \sum_{k=0}^{n} E_k(x)E_{n-k}(x) = {} & \sum_{k=0}^{n-1}\left\{\sum_{l=k}^{n-1}\frac{(n+1)\binom{n}{l}}{(n-l+1)!}\left(\sum_{m=l}^{n} E_{m-l}E_{n-m} - 2E_{n-l}\right)\sum_{p=0}^{l-k}(-1)^k l!\binom{k+p}{k}\right. \\ & \left. \times\frac{E_{l-k-p}}{(l-k-p)!} + 2(n+1)!(-1)^k\sum_{p=0}^{n-k}\binom{k+p}{k}\frac{E_{n-k-p}}{(n-k-p)!}\right\}L_k(x) \\ & + 2(-1)^n(n+1)!L_n(x). \end{aligned} \tag{2.25}$$

It is known that

$$\sum_{k=0}^{n} E_k(x)E_{n-k}(x) = p(x) = -\frac{4}{n+2}\sum_{k=0}^{n}\binom{n+2}{k}E_{n-k+1}B_k(x) \tag{2.26}$$

(see [15]). From (2.20), (2.21), and (2.23), we have

$$\begin{aligned} C_k &= \frac{1}{k!}\int_0^\infty \left(\frac{d^k}{dx^k}e^{-x}x^k\right)p(x)dx \\ &= -\frac{4}{n+2}\sum_{l=k}^{n}\binom{n+2}{l}E_{n-l+1}\frac{1}{k!}\int_0^\infty\left(\frac{d^k e^{-x}x^k}{dx^k}\right)B_l(x)dx \\ &= -\frac{4}{n+2}\sum_{l=k}^{n}\sum_{m=0}^{l-k}\binom{n+2}{l}(-1)^k E_{n-l+1}\frac{B_{l-k-m}}{(l-k-m)!}l!\binom{m+k}{k}. \end{aligned} \tag{2.27}$$

Therefore, by (2.20) and (2.27), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned} &\sum_{k=0}^{n} E_k(x)E_{n-k}(x) \\ &\quad = -\frac{4}{n+2}\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{m=0}^{l-k}\binom{n+2}{l}(-1)^k E_{n-l+1}\frac{B_{l-k-m}}{(l-k-m)!}l!\binom{m+k}{k}L_k(x). \end{aligned} \tag{2.28}$$

Remark 2.7. Laguerre's differential equation

$$ty'' + (1-t)y' + ny = 0 \tag{2.29}$$

is known to possess polynomial solutions when n is a nonnegative integer. These solutions are naturally called Laguerre polynomials and are denoted by $L_n(t)$. That is, $y = L_n(t)$ are solutions of (2.29) which are given by

$$y = L_n(t) = \sum_{r=0}^{n}\frac{\binom{n}{r}(-1)^r}{r!}t^r, \qquad L_0(1) = 1. \tag{2.30}$$

From (2.30), we note that Laplace transform of $y = L_n(t)$ is given by

$$\mathcal{L}(y) = \mathcal{L}(L_n(t)) = \frac{1}{s}\sum_{r=0}^{n}\binom{n}{r}(-1)^r\left(\frac{1}{s}\right)^r = \frac{(s-1)^n}{s^{n+1}}. \tag{2.31}$$

It is not difficult to show that

$$\mathcal{L}\left(\frac{e^t}{n!}\left(\frac{d^n}{dt^n}e^{-t}t^n\right)\right) = \mathcal{L}(y) = \frac{(s-1)^n}{s^{n+1}}. \tag{2.32}$$

Thus, we conclude that

$$L_n(t) = \frac{e^t}{n!}\left(\frac{d^n}{dt^n}e^{-t}t^n\right), \quad \text{for } n \in \mathbb{Z}_+. \tag{2.33}$$

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