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A Note on Poly-Bernoulli Polynomials Arising from Umbral Calculus

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Abstract

In this paper, we give some recurrence formula and new and interesting

identities for the poly-Bernoulli numbers and polynomials which are derived from umbral calculus.

1 Introduction

The classical polylogarithmic function $Li_s(x)$ are

$$Li_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s}, \quad s \in \mathbf{Z}, \quad (\text{see [3, 5]}). \quad (1)$$

In [5], poly-Bernoulli polynomials are defined by the generating function to be

$$\frac{Li_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} = e^{B^{(k)}(x)t} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad (\text{see [3, 5]}), \quad (2)$$

with the usual convention about replacing $(B^{(k)}(x))^n$ by $B_n^{(k)}(x)$.

As is well known, the Bernoulli polynomials of order r are defined by the generating function to be

$$\left(\frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [7, 9]}). \quad (3)$$

In the special case, $r = 1$, $\mathbb{B}_n^{(r)}(x) = B_n(x)$ is called the n -th ordinary Bernoulli polynomial. Here we denote higher-order Bernoulli polynomials as $\mathbb{B}_n^{(r)}$ to avoid conflict of notations.

If $x = 0$, then $B_n^{(k)}(0) = B_n^{(k)}$ is called the n -th poly-Bernoulli number. From (2), we note that

$$B_n^{(k)}(x) = \sum_{l=0}^n \binom{n}{l} B_{n-l}^{(k)} x^l = \sum_{l=0}^n \binom{n}{l} B_l^{(k)} x^{n-l}. \quad (4)$$

Let \mathcal{F} be the set of all formal power series in the variable t over \mathbf{C} as follows:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbf{C} \right\}, \quad (5)$$

and let $\mathbb{P} = \mathbf{C}[x]$ and \mathbb{P}^* denote the vector space of all linear functionals on \mathbb{P} . $\langle L|p(x) \rangle$ denotes the action of linear functional L on the polynomial $p(x)$, and it is well known that the vector space operations on \mathbb{P}^* are defined

by $\langle L + M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle$, $\langle cL | p(x) \rangle = c \langle L | p(x) \rangle$, where c is a complex constant (see [6, 9]).

For $f(t) \in \mathcal{F}$, let $\langle f(t) | x^n \rangle = a_n$. Then, by (5), we easily get

$$\langle t^k | x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \quad (\text{see [1, 4, 6, 9, 10]}), \quad (6)$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let us assume that $f_L(t) = \sum_{k=0}^{\infty} \langle L | x^k \rangle \frac{t^k}{k!}$. Then, by (6), we see that $\langle f_L(t) | x^n \rangle = \langle L | x^n \rangle$. That is, $f_L(t) = L$. Additionally, the map $L \mapsto f_L(t)$ is a vector space isomorphism from \mathbb{P}^* onto \mathcal{F} . Henceforth, \mathcal{F} denotes both the algebra of the formal power series in t and the vector space of all linear functionals on \mathbb{P} , and so an element $f(t)$ of \mathcal{F} will be thought as a formal power series and a linear functional. \mathcal{F} is called the umbral algebra. The umbral calculus is the study of umbral algebra. The order $O(f(t))$ of the power series $f(t) \neq 0$ is the smallest integer for which a_k does not vanish. If $O(f(t)) = 0$, then $f(t)$ is called an invertible series. If $O(f(t)) = 1$, then $f(t)$ is called a delta series. For $f(t), g(t) \in \mathcal{F}$, we have

$$\langle f(t)g(t) | p(x) \rangle = \langle f(t) | g(t)p(x) \rangle = \langle g(t) | f(t)p(x) \rangle. \quad (7)$$

Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we have

$$f(t) = \sum_{k=0}^{\infty} \langle f(t) | x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k | p(x) \rangle \frac{x^k}{k!}, \quad (\text{see [6, 9]}). \quad (8)$$

From (8), we can easily derive

$$p^{(k)}(x) = \frac{d^k p(x)}{dx^k} = \sum_{l=k}^{\infty} \frac{\langle t^l | p(x) \rangle}{(l-k)!} x^{l-k}. \quad (9)$$

Thus, by (8) and (9), we get

$$p^{(k)}(0) = \langle t^k | p(x) \rangle = \langle 1 | p^{(k)}(x) \rangle. \quad (10)$$

Hence, from (10), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad (\text{see [1, 6, 9]}). \quad (11)$$

It is easy to show that

$$e^{yt} p(x) = p(x+y), \quad \langle e^{yt} | p(x) \rangle = p(y). \quad (12)$$

Let $O(f(t)) = 1$ and $O(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k | s_n(x) \rangle = n! \delta_{n,k}$, for $n, k \geq 0$. The sequence $s_n(x)$ is called a Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$. The Sheffer sequence $s_n(x)$ for $(g(t), t)$ is called the Appell sequence for $g(t)$. For $p(x) \in \mathbb{P}$, $f(t) \in \mathcal{F}$, we have

$$\langle f(t) | xp(x) \rangle = \langle \partial_t f(t) | p(x) \rangle = \langle f'(t) | p(x) \rangle, \quad (\text{see [9, 10]}). \quad (13)$$

Let $s_n(x) \sim (g(t), f(t))$. Then the following equations are known:

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | s_k(x) \rangle}{k!} g(t) f(t)^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t) f(t)^k | p(x) \rangle}{k!} s_k(x), \quad (14)$$

where $h(t) \in \mathcal{F}$, $p(x) \in \mathbb{P}$,

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} s_k(y) \frac{t^k}{k!}, \quad \text{for all } y \in \mathbf{C}, \quad (15)$$

where $\bar{f}(t)$ is the compositional inverse for $f(t)$ with $\bar{f}(f(t)) = t$, and

$$f(t)s_n(x) = ns_{n-1}(x). \quad (16)$$

As is well known, the Stirling numbers of the second kind are also defined by the generating function to be

$$(e^t - 1)^m = m! \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!} = \sum_{l=0}^{\infty} \frac{m!}{(l+m)!} S_2(l+m, m) t^{l+m}. \quad (17)$$

Let $s_n(x) \sim (g(t), t)$. Then the Appell identity is given by

$$s_n(x+y) = \sum_{k=0}^n \binom{n}{k} s_k(y) x^{n-k} = \sum_{k=0}^n \binom{n}{k} s_{n-k}(y) x^k, \quad (\text{see [6, 9]}), \quad (18)$$

and

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) s_n(x), \quad (\text{see [6.9]}). \quad (19)$$

For $s_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$, we have

$$s_n(x) = \sum_{m=0}^n r_m(x) c_{n,m}, \quad (20)$$

where

$$c_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \middle| x^n \right\rangle, \quad (\text{see [9]}). \quad (21)$$

The equations (20) and (21) are important in deriving our main results of this paper.

In this paper, we give some recurrence formula and new and interesting identities for the poly-Bernoulli numbers and polynomials which are derive from umbral calculus.

2 Poly-Bernoulli numbers and polynomials

Let $g_k(t) = \frac{1-e^{-t}}{Li_k(1-e^{-t})}$. Then, by (2) and (15), we get

$$B_n^{(k)}(x) \sim (g_k(t), t). \quad (22)$$

That is, poly-Bernoulli polynomial $B_n^{(k)}(x)$ is an Appell sequence.

By (1), we easily get

$$\frac{d}{dx} Li_k(x) = \frac{1}{x} Li_{k-1}(x), \quad tB_n^{(k)}(x) = \frac{d}{dx} B_n^{(k)}(x) = nB_{n-1}^{(k)}(x). \quad (23)$$

From (2) and (15), we have

$$B_n^{(k)}(x) = \frac{1}{g_k(t)} x^n = \frac{Li_k(1-e^{-t})}{1-e^{-t}} x^n. \quad (24)$$

Let $k \in \mathbf{Z}$ and $n \geq 0$. Then we have

$$\begin{aligned} B_n^{(k)}(x) &= \frac{Li_k(1-e^{-t})}{1-e^{-t}} x^n = \sum_{m=1}^{\infty} \frac{(1-e^{-t})^{m-1}}{m^k} x^n \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} (1-e^{-t})^m x^n = \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^m (-1)^j \binom{m}{j} e^{-jt} x^n \\ &= \sum_{m=0}^n \frac{1}{(m+1)^k} \sum_{j=0}^m (-1)^j \binom{m}{j} (x-j)^n. \end{aligned} \quad (25)$$

By (17) and (25), we get

$$\begin{aligned}
 B_n^{(k)}(x) &= \sum_{m=0}^n \frac{1}{(m+1)^k} \sum_{a=0}^{\infty} (-1)^a \frac{m!}{(a+m)!} S_2(a+m, m) t^{a+m} x^n \quad (26) \\
 &= \sum_{m=0}^n \frac{1}{(m+1)^k} \sum_{a=0}^{n-m} (-1)^a \frac{m!}{(a+m)!} S_2(a+m, m) (n)_{a+m} x^{n-a-m} \\
 &= \sum_{l=0}^n \left\{ \sum_{m=0}^{n-l} \frac{(-1)^{n-m-l}}{(m+1)^k} \binom{n}{l} m! S_2(n-l, m) \right\} x^l,
 \end{aligned}$$

where $(a)_n = a(a-1)(a-2)\cdots(a-n+1)$.

Now, we use the well-known transfer formula for Appell sequences (see equation (19)).

By (19) and (22), we get

$$B_{n+1}^{(k)}(x) = \left(x - \frac{g'_k(t)}{g_k(t)} \right) B_n^{(k)}(x), \quad (27)$$

where

$$\begin{aligned}
 \frac{g'_k(t)}{g_k(t)} &= (\log g_k(t))' = (\log(1-e^{-t}) - \log Li_k(1-e^{-t}))' \quad (28) \\
 &= \frac{e^{-t}}{1-e^{-t}} \left\{ 1 - \frac{Li_{k-1}(1-e^{-t})}{Li_k(1-e^{-t})} \right\} \\
 &= \frac{1}{e^t - 1} \left(\frac{Li_k(1-e^{-t}) - Li_{k-1}(1-e^{-t})}{Li_k(1-e^{-t})} \right).
 \end{aligned}$$

From (27) and (28), we have

$$\begin{aligned}
 B_{n+1}^{(k)}(x) &= x B_n^{(k)}(x) - \frac{g'_k(t)}{g_k(t)} B_n^{(k)}(x) \quad (29) \\
 &= x B_n^{(k)}(x) - \left(\frac{t}{e^t - 1} \right) \left(\frac{Li_k(1-e^{-t}) - Li_{k-1}(1-e^{-t})}{t(1-e^{-t})} \right) x^n.
 \end{aligned}$$

Here, we note that

$$\begin{aligned}
 \frac{Li_k(1-e^{-t}) - Li_{k-1}(1-e^{-t})}{1-e^{-t}} &= \sum_{m=2}^{\infty} \left(\frac{1}{m^k} - \frac{1}{m^{k-1}} \right) (1-e^{-t})^{m-1} \quad (30) \\
 &= \left(\frac{1}{2^k} - \frac{1}{2^{k-1}} \right) t + \cdots
 \end{aligned}$$

is a delta series.

For any delta series $f(t)$, we observe that

$$\frac{f(t)}{t}x^n = f(t)\frac{x^{n+1}}{n+1}. \quad (31)$$

By (29), (30) and (31), we get

$$\begin{aligned} B_{n+1}^{(k)}(x) &= xB_n^{(k)}(x) - \left(\frac{t}{e^t-1}\right) \left(\frac{1}{n+1} \frac{Li_k(1-e^{-t}) - Li_{k-1}(1-e^{-t})}{1-e^{-t}} x^{n+1}\right) \\ &= xB_n^{(k)}(x) - \frac{1}{n+1} \sum_{l=0}^{\infty} \frac{B_l}{l!} t^l \{B_{n+1}^{(k)}(x) - B_{n+1}^{(k-1)}(x)\} \\ &= xB_n^{(k)}(x) - \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_l \{B_{n+1-l}^{(k)}(x) - B_{n+1-l}^{(k-1)}(x)\}. \end{aligned} \quad (32)$$

Therefore, by (32), we obtain the following theorem.

Theorem 1. For $k \in \mathbf{Z}$, $n \geq 0$, we have

$$B_{n+1}^{(k)}(x) = xB_n^{(k)}(x) - \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_l \{B_{n+1-l}^{(k)}(x) - B_{n+1-l}^{(k-1)}(x)\},$$

where B_n is the n -th ordinary Bernoulli number.

It is easy to show that

$$\begin{aligned} txB_n^{(k)}(x) &= t \sum_{l=0}^n \binom{n}{l} B_{n-l}^{(k)} x^{l+1} = \sum_{l=0}^n \binom{n}{l} B_{n-l}^{(k)} (l+1)x^l \\ &= nx \sum_{l=0}^{n-1} \binom{n-1}{l} B_{n-1-l}^{(k)} x^l + \sum_{l=0}^n \binom{n}{l} B_{n-l}^{(k)} x^n \\ &= nx B_{n-1}^{(k)}(x) + B_n^{(k)}(x). \end{aligned} \quad (33)$$

Applying to t on both sides of Theorem 1, by (33), we get

$$(n+1)B_n^{(k)}(x) = nx B_{n-1}^{(k)}(x) + B_n^{(k)}(x) - \sum_{l=0}^n \binom{n}{l} B_{n-l} \{B_l^{(k)}(x) - B_l^{(k-1)}(x)\}. \quad (34)$$

Therefore, by (34), we obtain the following corollary.

Corollary 2. For $k \in \mathbf{Z}$ and $n \geq 1$, we have

$$\begin{aligned} (n+1)B_n^{(k)}(x) - n\left(x + \frac{1}{2}\right)B_{n-1}^{(k)}(x) + \sum_{l=0}^{n-2} \binom{n}{l} B_{n-l} B_l^{(k)}(x) \\ = \sum_{l=0}^n \binom{n}{l} B_{n-l} B_l^{(k-1)}(x). \end{aligned}$$

From (2) and (6), we note that

$$\begin{aligned} B_n^{(k)}(y) &= \left\langle \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^n \right\rangle = \left\langle \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| xx^{n-1} \right\rangle \quad (35) \\ &= \left\langle \partial_t \left(\frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{yt} \right) \middle| x^{n-1} \right\rangle \\ &= \left\langle \partial_t \left(\frac{Li_k(1-e^{-t})}{1-e^{-t}} \right) e^{yt} \middle| x^{n-1} \right\rangle + y \left\langle \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^{n-1} \right\rangle \\ &= \left\langle \frac{Li_{k-1}(1-e^{-t}) - Li_k(1-e^{-t})}{(1-e^{-t})^2} e^{(y-1)t} \middle| x^{n-1} \right\rangle + y B_{n-1}^{(k)}(y). \end{aligned}$$

Now, we observe that

$$\begin{aligned} \frac{Li_{k-1}(1-e^{-t}) - Li_k(1-e^{-t})}{(1-e^{-t})^2} &= \frac{1}{(1-e^{-t})^2} \sum_{m=1}^{\infty} \left\{ \frac{(1-e^{-t})^m}{m^{k-1}} - \frac{(1-e^{-t})^m}{m^k} \right\} \quad (36) \\ &= \sum_{m=2}^{\infty} \left\{ \frac{1}{m^{k-1}} - \frac{1}{m^k} \right\} (1-e^{-t})^{m-2} \\ &= \sum_{m=0}^{\infty} \left\{ \frac{1}{(m+2)^{k-1}} - \frac{1}{(m+2)^k} \right\} (1-e^{-t})^m. \end{aligned}$$

Thus, by (36), we get

$$\begin{aligned} \left\langle \frac{Li_{k-1}(1-e^{-t}) - Li_k(1-e^{-t})}{(1-e^{-t})^2} e^{(y-1)t} \middle| x^{n-1} \right\rangle \quad (37) \\ = \sum_{m=0}^{\infty} \left\{ \frac{1}{(m+2)^{k-1}} - \frac{1}{(m+2)^k} \right\} \langle (1-e^{-t})^m e^{(y-1)t} \middle| x^{n-1} \rangle \\ = \sum_{m=0}^{n-1} \left(\frac{1}{(m+2)^{k-1}} - \frac{1}{(m+2)^k} \right) \langle (1-e^{-t})^m \middle| (x+y-1)^{n-1} \rangle \\ = \sum_{m=0}^{n-1} \left(\frac{1}{(m+2)^{k-1}} - \frac{1}{(m+2)^k} \right) \sum_{a=0}^{n-1} \binom{n-1}{a} (y-1)^{n-1-a} \langle (1-e^{-t})^m \middle| x^a \rangle. \end{aligned}$$

From (6) and (7), we have

$$\langle (1 - e^{-t})^m | x^a \rangle = (-1)^{a+m} m! S_2(a, m). \tag{38}$$

From (37) and (38), we have

$$\begin{aligned} & \left\langle \frac{Li_{k-1}(1 - e^{-t}) - Li_{k-1}(1 - e^{-t})e^{(y-1)t}}{(1 - e^{-t})^2} \middle| x^{n-1} \right\rangle \\ &= \sum_{m=0}^{n-1} \sum_{a=0}^{n-1} (-1)^{a+m} \binom{n-1}{a} m! \left(\frac{1}{(m+2)^{k-1}} - \frac{1}{(m+2)^k} \right) \\ & \quad \times S_2(a, m)(y-1)^{n-1-a} \\ &= \sum_{m=0}^{n-1} \sum_{l=0}^{n-1} (-1)^{n-1-l+m} \binom{n-1}{l} m! \left(\frac{1}{(m+2)^{k-1}} - \frac{1}{(m+2)^k} \right) \\ & \quad \times S_2(n-1-l, m)(y-1)^l. \end{aligned} \tag{39}$$

Therefore, by (35) and (39), we obtain the following theorem.

Theorem 3. For $k \in \mathbf{Z}$ and $n \geq 1$, we have

$$\begin{aligned} B_n^{(k)}(x) &= xB_{n-1}^{(k)}(x) + \sum_{l=0}^{n-1} (-1)^{n-1-l} \binom{n-1}{l} \\ & \quad \times \left\{ \sum_{m=0}^{n-1} (-1)^m \frac{(m+1)!}{(m+2)^k} S_2(n-1-l, m) \right\} (x-1)^l. \end{aligned}$$

Now, we try to compute $\langle Li_k(1 - e^{-t}) | x^{n+1} \rangle$ in two ways. On the one hand,

$$\begin{aligned} \langle Li_k(1 - e^{-t}) | x^{n+1} \rangle &= \left\langle (1 - e^{-t}) \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \middle| x^{n+1} \right\rangle \\ &= \left\langle \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \middle| (1 - e^{-t}) x^{n+1} \right\rangle \\ &= \left\langle \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \middle| x^{n+1} - (x-1)^{n+1} \right\rangle \\ &= \sum_{m=0}^n \binom{n+1}{m} (-1)^{n-m} \left\langle 1 \middle| \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} x^m \right\rangle \\ &= \sum_{m=0}^n \binom{n+1}{m} (-1)^{n-m} B_m^{(k)}. \end{aligned} \tag{40}$$

On the other hand,

$$\begin{aligned}
 \langle Li_k(1 - e^{-t}) | x^{n+1} \rangle &= \left\langle \int_0^t (Li_k(1 - e^{-s}))' ds \middle| x^{n+1} \right\rangle \tag{41} \\
 &= \left\langle \int_0^t e^{-s} \frac{Li_{k-1}(1 - e^{-s})}{1 - e^{-s}} ds \middle| x^{n+1} \right\rangle \\
 &= \left\langle \int_0^t \left(\sum_{a=0}^{\infty} \frac{(-s)^a}{a!} \right) \left(\sum_{m=0}^{\infty} \frac{B_m^{(k-1)}}{m!} s^m \right) ds \middle| x^{n+1} \right\rangle \\
 &= \left\langle \sum_{l=0}^{\infty} \left(\sum_{m=0}^l \binom{l}{m} (-1)^{l-m} B_m^{(k-1)} \right) \frac{1}{l!} \int_0^t s^l ds \middle| x^{n+1} \right\rangle \\
 &= \sum_{l=0}^n \sum_{m=0}^l \binom{l}{m} (-1)^{l-m} \frac{B_m^{(k-1)}}{(l+1)!} (n+1)! \delta_{n+1, l+1} = \sum_{m=0}^n \binom{n}{m} (-1)^{n-m} B_m^{(k-1)}.
 \end{aligned}$$

By (40) and (41), we get

$$\sum_{m=0}^n (-1)^{n-m} \binom{n}{m} B_m^{(k-1)} = \sum_{m=0}^n \binom{n+1}{m} (-1)^{n-m} B_m^{(k)}. \tag{42}$$

By (2) and (3), we see that

$$B_n^{(k)}(x) \sim \left(\frac{1 - e^{-t}}{Li_k(1 - e^{-t})}, t \right), \quad \mathbb{B}_n^{(r)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^r, t \right), \quad r \geq 0. \tag{43}$$

From (20), (21) and (43), we have

$$B_n^{(k)}(x) = \sum_{m=0}^n C_{n,m} \mathbb{B}_m^{(r)}(x), \tag{44}$$

where

$$\begin{aligned}
 C_{n,m} &= \frac{1}{m!} \left\langle \frac{\left(\frac{e^t - 1}{t} \right)^r}{\frac{1 - e^{-t}}{Li_k(1 - e^{-t})}} t^m \middle| x^n \right\rangle \tag{45} \\
 &= \frac{1}{m!} \left\langle \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \left(\frac{e^t - 1}{t} \right)^r \middle| t^m x^n \right\rangle \\
 &= \binom{n}{m} \left\langle \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \middle| \left(\frac{e^t - 1}{t} \right)^r x^{n-m} \right\rangle.
 \end{aligned}$$

By (17), we easily get

$$\left(\frac{e^t - 1}{t} \right)^r = \sum_{l=0}^{\infty} \frac{r!}{(l+r)!} S_2(l+r, r) t^l. \tag{46}$$

Thus, from (46), we have

$$\left(\frac{e^t - 1}{t}\right)^r x^{n-m} = \sum_{l=0}^{n-m} \frac{r!}{(l+r)!} S_2(l+r, r)(n-m)_l x^{n-m-l}. \tag{47}$$

By (45) and (47), we get

$$\begin{aligned} C_{n,m} &= \binom{n}{m} \sum_{l=0}^{n-m} \frac{r!}{(l+r)!} S_2(l+r, r)(n-m)_l \left\langle \frac{Li_k(1-e^{-t})}{1-e^{-t}} \middle| x^{n-m-l} \right\rangle \tag{48} \\ &= \binom{n}{m} \sum_{l=0}^{n-m} \frac{r!}{(l+r)!} S_2(l+r, r)(n-m)_l \left\langle t^0 \middle| \frac{Li_k(1-e^{-t})}{1-e^{-t}} x^{n-m-l} \right\rangle \\ &= \binom{n}{m} \sum_{l=0}^{n-m} \frac{r!}{(l+r)!} S_2(l+r, r)(n-m)_l B_{n-m-l}^{(k)}. \end{aligned}$$

Therefore, by (44) and (48), we obtain the following theorem.

Theorem 4. For $k \in \mathbf{Z}$ and $r \in \mathbf{Z}_{\geq 0}$, we have

$$B_n^{(k)}(x) = \sum_{m=0}^n \left\{ \binom{n}{m} \sum_{l=0}^{n-m} \frac{r!(n-m)_l}{(l+r)!} S_2(l+r, r) B_{n-l-m}^{(k)} \right\} \mathbb{B}_n^{(r)}(x).$$

For $r \in \mathbf{Z}_{\geq 0}$, the Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [2, 6, 8]}). \tag{49}$$

By (2) and (49), we see that

$$B_n^{(k)}(x) \sim \left(\frac{1-e^{-t}}{Li_k(1-e^{-t})}, t\right), \quad E_n^{(r)}(x) \sim \left(\left(\frac{e^t + 1}{2}\right)^r, t\right). \tag{50}$$

From (20), (21) and (50), we have

$$B_n^{(k)}(x) = \sum_{m=0}^n C_{n,m} E_m^{(r)}(x), \tag{51}$$

where

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{Li_k(1-e^{-t})}{1-e^{-t}} \left(\frac{e^t + 1}{2}\right)^r \middle| t^m x^n \right\rangle \tag{52} \\ &= \frac{\binom{n}{m}}{2^r} \left\langle \frac{Li_k(1-e^{-t})}{1-e^{-t}} \middle| (e^t + 1)^r x^{n-m} \right\rangle \\ &= \frac{\binom{n}{m}}{2^r} \sum_{j=0}^r \binom{r}{j} \left\langle \frac{Li_k(1-e^{-t})}{1-e^{-t}} \middle| e^{jt} x^{n-m} \right\rangle \end{aligned}$$

$$\begin{aligned}
 &= \frac{\binom{n}{m}}{2^r} \sum_{j=0}^r \binom{r}{j} \left\langle t^0 \left| \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} (x + j)^{n-m} \right. \right\rangle \\
 &= \frac{\binom{n}{m}}{2^r} \sum_{j=0}^r \binom{r}{j} B_{n-m}^{(k)}(j).
 \end{aligned}$$

Therefore, by (51) and (52), we obtain the following theorem.

Theorem 5. For $k \in \mathbf{Z}$ and $r \in \mathbf{Z}_{\geq 0}$, we have

$$B_n^{(k)}(x) = \frac{1}{2^r} \sum_{m=0}^n \left\{ \binom{n}{m} \sum_{j=0}^r \binom{r}{j} B_{n-m}^{(k)}(j) \right\} E_n^{(r)}(x).$$

Let $\lambda \in \mathbf{C}$ with $\lambda \neq 1$. For $r \in \mathbf{Z}_{\geq 0}$, the Frobenius-Euler polynomials are also defined by the generating function to be

$$\left(\frac{1 - \lambda}{e^t - \lambda} \right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!}, \quad (\text{see [2, 6, 7]}). \tag{53}$$

From (2), (15) and (53), we note that

$$B_n^{(k)}(x) \sim \left(\frac{1 - e^{-t}}{Li_k(1 - e^{-t})}, t \right), \quad H_n^{(r)}(x|\lambda) \sim \left(\left(\frac{e^t - \lambda}{1 - \lambda} \right)^r, t \right). \tag{54}$$

By (20), (21) and (54), we get

$$B_n^{(k)}(x) = \sum_{m=0}^n C_{n,m} H_m^{(r)}(x|\lambda), \tag{55}$$

where

$$\begin{aligned}
 C_{n,m} &= \frac{1}{m!} \left\langle \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \left(\frac{e^t - \lambda}{1 - \lambda} \right)^r \left| t^m x^n \right. \right\rangle \\
 &= \frac{\binom{n}{m}}{(1 - \lambda)^r} \left\langle \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \left| (e^t - \lambda)^r x^{n-m} \right. \right\rangle \\
 &= \frac{\binom{n}{m}}{(1 - \lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} \left\langle \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \left| e^{jt} x^{n-m} \right. \right\rangle \\
 &= \frac{\binom{n}{m}}{(1 - \lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} \left\langle t^0 \left| \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} (x + j)^{n-m} \right. \right\rangle \\
 &= \frac{\binom{n}{m}}{(1 - \lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} B_{n-m}^{(k)}(j).
 \end{aligned} \tag{56}$$

Therefore, by (55) and (56), we obtain the following theorem.

Theorem 6. For $k \in \mathbf{Z}$ and $r \in \mathbf{Z}_{\geq 0}$, we have

$$B_n^{(k)}(x) = \frac{1}{(1-\lambda)^r} \sum_{m=0}^n \left\{ \binom{n}{m} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} B_{n-m}^{(k)}(j) \right\} H_m^{(r)}(x|\lambda).$$

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