SOME IDENTITIES FOR BERNOULLI POLYNOMIALS INVOLVING CHEBYSHEV POLYNOMIALS

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ABSTRACT. In this paper we derive some new and interesting identities for Bernoulli, Euler and Hermite polynomials associated with Chebyshev polynomials.

1. INTRODUCTION

The Bernoulli number are defined by the generating function to be

$$\frac{t}{e^t-1}=e^{Bt}=\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n, \quad \text{(see [3,13,14])}, \tag{1}$$

with the usual convention about replacing B^n by B_n.

As is well known, the Bernoulli polynomials are given by

$$B_n(x)=(B+x)^n=\sum_{l=0}^{n}\binom{n}{l}B_{n-l}x^l, \quad \text{(see [1-8])}. \tag{2}$$

From (1), we note that the recurrence relation for the Bernoulli numbers is given by

$$B_0=1, \quad (B+1)^n-B_n=\delta_{1,n}, \quad \text{(see [6-8])},$$

where $\delta_{m,n}$ is the Kronecker symbol.

By (2), we get

$$\frac{dB_n(x)}{dx}=n\sum_{l=0}^{n-1}\binom{n-1}{l}B_{n-1-l}x^l=nB_{n-1}(x). \tag{3}$$

Thus, by (3), we see that

$$\int B_n(x)dx=\frac{B_{n+1}(x)}{n+1}+C, \quad \text{(see [3])}, \tag{4}$$

where C is a some constant.

The Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt}=e^{E(x)t}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \tag{5}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$, (see [1,2,4,10,11]).

In the special case, $x=0$, $E_n(0)=E_n$ are called the n-th Euler numbers.

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It is well known [6, 15] that Hermite polynomials are given by the generating function to be

$$e^{2xt-t^2} = e^{H(x)t} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \tag{6}$$

with the usual convention about replacing $H^n(x)$ by $H_n(x)$.

From (6),we have

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x), \quad H_n(x) = (-1)^n H_n(-x). \tag{7}$$

By (1) and (2), we easily get

$$B_n(x) = \sum_{\substack{k=0\\k\neq 1}}^{n} \binom{n}{k} E_{n-k}(x), \quad \text{(see [1-15])}, \tag{8}$$

$$E_n(x) = -2\sum_{l=0}^{n} \binom{n}{l} \frac{E_{l+1}}{l+1} E_{n-l}(x), \tag{9}$$

and

$$x^n = \frac{1}{n+1}\big(B_{n+1}(x+1) - B_{n+1}(x)\big) = \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l} B_l(x). \tag{10}$$

The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by the relation

$$T_n(x) = \cos n\theta, \quad \text{when } x = \cos\theta, \quad \text{(see [9])}. \tag{11}$$

If the range of the variable x is the interval $[-1, 1]$, then the range of the corresponding variable θ can be taken as $[0, \pi]$. It is known that $\cos n\theta$ is a polynomial of degree n in $\cos\theta$, and indeed we are familiar with elementary formulas $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1, \cdots$.

Thus, by (11), we get

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x,$$
$$T_4(x) = 8x^4 - 8x^2 + 1, \cdots.$$

The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by

$$U_n(x) = \sin(n+1)\theta/\sin\theta, \quad \text{when } x = \cos\theta, \quad \text{(see [9])}. \tag{12}$$

Thus, from (12), we have

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \cdots.$$

By (11), we see that $T_n(x)$ is a polynomial of degree n with integral coefficients and the leading coefficient 2^{n-1} $(n \geq 1)$ and 1 $(n = 0)$. It is not difficult to show that $U_n(x)$ is a polynomial of degree n with integral coefficients and the leading coefficient 2^n $(n \geq 0)$. $T_n(x)$ is a solution of $(1-x^2)y'' - xy' + n^2y = 0$ and $U_n(x)$ is a solution of $(1-x^2)y'' - 3xy' + n(n+2)y = 0$. It is well known [9] that the generating functions of $T_n(x)$ and $U_n(x)$ are given by

$$\frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \tag{13}$$

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and

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \quad \text{for } |x| \le 1,\ |t| < 1. \tag{14}$$

From (11) and (12), we have

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{\pi}{2}, & \text{if } n = m > 0 \\ \pi, & \text{if } n = m = 0 \end{cases}, \tag{15}$$

and

$$\int_{-1}^{1} (1-x^2)^{1/2} U_n(x)U_m(x)dx = \frac{\pi}{2}\delta_{n,m}, \quad \text{(see [9])}. \tag{16}$$

The equations (15) and (16) are used to derive our main result in this paper.

The Rodrigues' formulae for $T_n(x)$ and $U_n(x)$ are known as follows:

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}(1-x^2)^{1/2}\left(\frac{d^n}{dx^n}(1-x^2)^{n-1/2}\right), \tag{17}$$

and

$$U_n(x) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!}(1-x^2)^{-1/2}\left(\frac{d^n}{dx^n}(1-x^2)^{n+1/2}\right). \tag{18}$$

The equations (17) and (18) are also used to derive our result related to orthogonality of Chebyshev polynomials.

From (11) and (12), we can easily derive the following equations (19) and (20):

$$T_n(x) = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2}, \tag{19}$$

and

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}. \tag{20}$$

By the definitions of $T_n(x)$ and $U_n(x)$, we easily get

$$\frac{dT_n(x)}{dx} = nU_{n-1}(x), \quad \frac{dU_n(x)}{dx} = \frac{(n+1)T_{n+1}(x) - xU_n(x)}{x^2-1}. \tag{21}$$

From (21), we have

$$\int U_n(x)dx = \frac{T_{n+1}(x)}{n+1}, \quad \int T_n(x)dx = \frac{nT_{n+1}(x)}{n^2-1} - \frac{xT_n(x)}{n-1}. \tag{22}$$

In this paper we derive some new and interesting identities for Bernoulli, Euler and Hermite polynomials arising from the orthogonality of the Chebyshev polynomials for the inner product space with weighted inner product.

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2. Some identities for Bernoulli, Euler and Hermite polynomials involving Chebyshev polynomials

Let $\mathbf{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\}$. Then $\mathbf{P}_n$ is an inner product space with the weighted inner product

$$\langle p(x),\ q(x)\rangle = \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1-x^2}}dx, \quad \text{where } \ p(x), q(x) \in \mathbf{P}_n.$$

From (15), we note that $\{T_0(x), T_1(x), \cdots, T_n(x)\}$ is an orthogonal basis for $\mathbf{P}_n$. Let us assume $p(x) \in \mathbf{P}_n$. Then $p(x)$ is generated by $\{T_0(x), T_1(x), \cdots, T_n(x)\}$ to be

$$p(x) = \sum_{k=0}^{n} C_k T_k(x). \tag{23}$$

By (15) and (23), we get

$$C_k = \frac{\delta_k}{\pi}\int_{-1}^{1} \frac{T_k(x)p(x)}{\sqrt{1-x^2}}dx = \frac{\delta_k}{\pi}\frac{(-1)^k 2^k k!}{(2k)!}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x^2)^{k-1/2}\right)p(x)dx,$$
$$\text{where } \delta_k = \begin{cases} 1, & \text{if } k=0 \\ 2, & \text{if } k>0. \end{cases} \tag{24}$$

Let us take $p(x) = x^n \in \mathbf{P}_n$. From (24), we have

$$\begin{aligned} C_k &= \frac{(-1)^k 2^k k! \delta_k}{\pi (2k)!}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x^2)^{k-1/2}\right)x^n dx \\ &= \frac{(-1)^k 2^k k!}{\pi(2k)!}\delta_k(-1)^k \frac{n!}{(n-k)!}\int_{-1}^{1}(1-x^2)^{k-1/2}x^{n-k}dx. \end{aligned} \tag{25}$$

It is easy to show that

(26)
$$\begin{aligned} &\int_{-1}^{1}(1-x^2)^{k-1/2}x^{n-k}dx = \frac{(1+(-1)^{n-k})}{2}\int_0^1 (1-y)^{k-1/2}y^{\frac{n-k+1}{2}-1}dy \\ &= \frac{(1+(-1)^{n-k})}{2}\frac{\Gamma(k+1/2)\Gamma(\frac{n-k+1}{2})}{\Gamma(\frac{k+n+2}{2})} = \frac{(1+(-1)^{n-k})}{2}\frac{(n-k)!(2k)!\pi}{2^{n+k}(\frac{n+k}{2})!(\frac{n-k}{2})!k!}. \end{aligned}$$

By (25) and (26), we get

$$C_k = \begin{cases} 0, & \text{if } n-k \equiv 1 \pmod 2 \\ \frac{n!\delta_k}{2^n(\frac{n+k}{2})!(\frac{n-k}{2})!}, & \text{if } n-k \equiv 0 \pmod 2. \end{cases} \tag{27}$$

From (27), we note that

$$x^n = \sum_{k=0}^{n} C_k T_k(x) = \frac{n!}{2^{n-1}}\sum_{\substack{1\le k\le n \\ k\equiv 1 \ (\mathrm{mod}\ 2)}}\frac{T_k(x)}{(\frac{n+k}{2})!(\frac{n-k}{2})!}, \tag{28}$$

where $n \equiv 1 \pmod 2$.

For $n \equiv 0 \pmod 2$, we have

$$x^n = \frac{n!}{2^n}\left\{\frac{T_0(x)}{\left((\frac{n}{2})!\right)^2} + 2\sum_{\substack{2\le k\le n \\ k\equiv 0 \ (\mathrm{mod}\ 2)}}\frac{T_k(x)}{(\frac{n+k}{2})!(\frac{n-k}{2})!}\right\}. \tag{29}$$

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Let us take $p(x) = B_n(x) \in \mathbf{P}_n$. Then

$$\begin{aligned} C_k &= \frac{(-1)^k 2^k k! \delta_k}{\pi(2k)!} \int_{-1}^{1} \left((\frac{d}{dx})^k (1-x^2)^{k-1/2} \right) B_n(x) dx \\ &= \frac{(-1)^k 2^k k! \delta_k}{\pi(2k)!} (-1)^k \frac{n!}{(n-k)!} \int_{-1}^{1} (1-x^2)^{k-1/2} B_{n-k}(x) dx \\ &= \frac{2^k k! \delta_k}{\pi(2k)!} \frac{n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} \int_{-1}^{1} (1-x^2)^{k-1/2} x^l dx. \end{aligned} \tag{30}$$

Now, we compute $\int_{-1}^{1} (1-x^2)^{k-1/2} x^l dx$.

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^{k-1/2} x^l dx &= (1+(-1)^l) \int_{0}^{1} (1-x^2)^{k-1/2} x^l dx \\ &= \begin{cases} 0, & \text{if } l \equiv 1 \pmod 2 \\ \frac{l!(2k)!\pi}{2^{2k+l}(\frac{2k+l}{2})!(\frac{l}{2})!k!}, & \text{if } l \equiv 0 \pmod 2. \end{cases} \end{aligned} \tag{31}$$

By (30) and (31), we get

$$\begin{aligned} C_k &= \frac{2^k k! \delta_k}{\pi(2k)!} \times \frac{n!}{(n-k)!} \times \frac{(2k)!\pi}{2^{2k} k!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \binom{n-k}{l} B_{n-k-l} \frac{l!}{2^l (\frac{2k+l}{2})! (\frac{l}{2})!} \\ &= \frac{n! \delta_k}{2^k (n-k)!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{\binom{n-k}{l} B_{n-k-l} l!}{2^l (\frac{2k+l}{2})! (\frac{l}{2})!}. \end{aligned} \tag{32}$$

Therefore, by (32), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, we have*

$$B_n(x) = n! \sum_{0 \le k \le n} \left(\frac{\delta_k}{2^k (n-k)!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (mod\ 2)}} \frac{\binom{n-k}{l} B_{n-k-l} l!}{2^l (\frac{2k+l}{2})! (\frac{l}{2})!} \right) T_k(x).$$

By the same method, we can derive the following identity:

$$E_n(x) = n! \sum_{0 \le k \le n} \left(\frac{\delta_k}{2^k (n-k)!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{\binom{n-k}{l} E_{n-k-l} l!}{2^l (\frac{2k+l}{2})! (\frac{l}{2})!} \right) T_k(x).$$

Let us take $p(x) = H_n(x) \in \mathbf{P}_n$. From (24), we have

$$\begin{aligned} C_k &= \frac{(-1)^k 2^k k! \delta_k}{\pi(2k)!} \int_{-1}^{1} \left(\frac{d^k}{dx^k} (1-x^2)^{k-1/2} \right) H_n(x) dx \\ &= \frac{(-1)^k 2^k k! \delta_k}{(2k)!\pi} \times (-1)^k 2^k \frac{n!}{(n-k)!} \int_{-1}^{1} (1-x^2)^{k-1/2} H_{n-k}(x) dx \\ &= \frac{2^{2k} k! \delta_k n!}{(2k)!(n-k)!\pi} \sum_{l=0}^{n-k} \binom{n-k}{l} H_{n-k-l} 2^l \int_{-1}^{1} (1-x^2)^{k-1/2} x^l dx, \end{aligned} \tag{33}$$

where H_{n-k-l} is the $(n-k-l)$th Hermite number.

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By (31) and (33), we get

$$C_k = n!\delta_k \sum_{\substack{0\le l\le n-k \\ l\equiv 0 \pmod 2}} \frac{H_{n-k-l}}{(n-k-l)!(\frac{2k+l}{2})!(\frac{l}{2})!}. \tag{34}$$

Therefore, by (34), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, we have*

$$H_n(x) = n! \sum_{0\le k\le n} \left(\delta_k \sum_{\substack{0\le l\le n-k \\ l\equiv 0 \ (mod\ 2)}} \frac{H_{n-k-l}}{(n-k-l)!(\frac{2k+l}{2})!(\frac{l}{2})!} \right) T_k(x).$$

Let $\mathbf{P}_n^* = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\}$. Then $\mathbf{P}_n^*$ is an inner product space with the weighted inner product $\langle p(x),\ q(x)\rangle = \int_{-1}^{1} \sqrt{1-x^2}p(x)q(x)dx$, where $p(x), q(x) \in \mathbf{P}_n$. Then $\{U_0(x), U_1(x), \cdots, U_n(x)\}$ is an orthogonal basis for the inner product space $\mathbf{P}_n^*$.
For $p(x) \in \mathbf{P}_n^*$, let

$$p(x) = \sum_{k=0}^{n} C_k U_k(x), \tag{35}$$

where

$$\begin{aligned} C_k &= \frac{2}{\pi}\langle p(x),\ U_k(x)\rangle = \frac{2}{\pi}\int_{-1}^{1}(1-x^2)^{1/2}U_k(x)p(x)dx \\ &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2}\right)p(x)dx. \end{aligned} \tag{36}$$

Let us assume that $p(x) = x^n \in \mathbf{P}_n^*$. Then, by (36), we get

$$\begin{aligned} C_k &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2}\right)x^n dx \\ &= \frac{(-1)^k 2^{2k+1}(k+1)!}{(2k+1)!\pi} \times \frac{(-1)^k n!}{(n-k)!}\int_{-1}^{1}(1-x^2)^{k+1/2}x^{n-k}dx. \end{aligned} \tag{37}$$

It is easy to show that

(38)

$$\begin{aligned} \int_{-1}^{1}(1-x^2)^{k+1/2}x^{n-k}dx &= (1+(-1)^{n-k})\int_{0}^{1}(1-x^2)^{k+1/2}x^{n-k}dx \\ &= \begin{cases} 0, & \text{if } n-k\equiv 1 \pmod 2 \\ \frac{(n-k)!(2k+2)!\pi}{2^{n+k+2}(\frac{n+k+2}{2})!(\frac{n-k}{2})!(k+1)!}, & \text{if } n-k\equiv 0 \pmod 2). \end{cases} \end{aligned}$$

Therefore, by (37) and (38), we obtain the following proposition.

Proposition 2.3. *For $n \in \mathbb{Z}_+$, we have*

$$x^n = \frac{n!}{2^n} \sum_{\substack{0\le k\le n \\ k\equiv n \ (mod\ 2)}} \frac{k+1}{(\frac{n+k+2}{2})!(\frac{n-k}{2})!} U_k(x).$$

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Let us consider $p(x) = B_n(x) \in \mathbf{P}_n^*$. From (36), we have

$$\begin{aligned} C_k &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2} \right) B_n(x)dx \\ &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \times \frac{(-1)^k n!}{(n-k)!} \int_{-1}^{1} (1-x^2)^{k+1/2} B_{n-k}(x)dx \\ &= \frac{2^{k+1}(k+1)!}{(2k+1)!\pi} \times \frac{n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} \int_{-1}^{1} (1-x^2)^{k+1/2} x^l dx. \end{aligned} \tag{39}$$

It is not difficult to show that

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^{k+1/2} x^l dx &= (1+(-1)^l) \int_{0}^{1} (1-x^2)^{k+1/2} x^l dx \\ &= \begin{cases} 0, & \text{if } l \equiv 1 \pmod 2 \\ \frac{(2k+2)!l!\pi}{2^{2k+2+l}(\frac{2k+2+l}{2})!(k+1)!(\frac{l}{2})!}, & \text{if } l \equiv 0 \pmod 2). \end{cases} \end{aligned} \tag{40}$$

By (39) and (40), we get

$$C_k = \frac{(k+1)n!}{2^k} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{B_{n-k-l}}{(n-k-l)!2^l(\frac{2k+l+2}{2})!(\frac{l}{2})!}. \tag{41}$$

Therefore, by (41), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, we have*

$$B_n(x) = n! \sum_{0 \le k \le n} \left(\frac{k+1}{2^k} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (mod\ 2)}} \frac{B_{n-k-l}}{2^l(n-k-l)!(\frac{2k+l+2}{2})!(\frac{l}{2})!} \right) U_k(x).$$

By the same method, we can derive the following identity:

$$E_n(x) = n! \sum_{0 \le k \le n} \left(\frac{k+1}{2^k} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{E_{n-k-l}}{2^l(n-k-l)!(\frac{2k+l+2}{2})!(\frac{l}{2})!} \right) U_k(x).$$

Let us take $p(x) = H_n(x) \in \mathbf{P}_n^*$. Then $H_n(x) = \sum_{k=0}^{n} C_k U_k(x)$, with

$$\begin{aligned} C_k &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2} \right) H_n(x)dx \\ &= \frac{2^{2k+1}(k+1)!n!}{(2k+1)!\pi(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l H_{n-k-l} \int_{-1}^{1} (1-x^2)^{k+1/2} x^l dx \\ &= n!(k+1) \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{H_{n-k-l}}{(n-k-l)!} \times \frac{1}{(\frac{2k+l+2}{2})!(\frac{l}{2})!}. \end{aligned} \tag{42}$$

Thus, by (42) and (43), we get

$$H_n(x) = n! \sum_{0 \le k \le n} \left((k+1) \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{H_{n-k-l}}{(n-k-l)!(\frac{2k+l+2}{2})!(\frac{l}{2})!} \right) U_k(x).$$

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