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Abstract

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Umbral calculus and Sheffer sequences of polynomials

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I. INTRODUCTION

The main goal of this paper is to use umbral calculus to obtain several new and interesting identities of Sheffer polynomials. Umbral calculus has been used in numerous problems of applied mathematics, theoretical physics, approximation theory, and several diverse areas of mathematics, like analysis, combinatorics, statistics and topology. Di Bucchianico and Loeb[9] present more than five hundred old and new findings related to the study of Sheffer polynomial sequences. For example, applications of umbral calculus to the physics of gases can be found in[35] and umbral techniques have been used in group theory and quantum mechanics by Biedenharn et al.[5,6]. Other instances of the relation between umbral calculus and physics can be found in[13] (see also the references therein), where Gzyl linked umbral calculus to the Hamiltonian approach in physics and quantum mechanics, and in[26,27], where Morikawa presented application for umbral calculus in statistical physics. Umbral calculus, in particular Sheffer sequences, has also been applied to the normal ordering of expressions involving bosonic creation and annihilation operators[7,8].

In this paper, umbral calculus is considered for some special Sheffer polynomials (to be defined in Section II) such as *Frobenius-Euler polynomials*, *Bernoulli polynomials*, *Changhee polynomials*, *Daehee polynomials* and *Bessel polynomials*.

Throughout this paper, we assume that $\lambda \in \mathbb{C}$ with $\lambda \neq 1$. For $\alpha \in \mathbb{R}$, the *Frobenius-Euler polynomials* are defined by the generating function

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^\alpha e^{xt} = \sum_{n\geq 0} H_n^{(\alpha)}(x|\lambda)\frac{t^n}{n!}, \tag{1}$$

(see[1,10,11,22]). In the special case $x=0$, $H_n^{(\alpha)}(0|\lambda) = H_n^{(\alpha)}(\lambda)$ are called the *n-th Frobenius-Euler numbers* of order α. Writing the left-hand side of (1) as a product of a series in t (involving $H_n^{(\alpha)}(\lambda)$), that represents the left factor, and a power series of the exponential function (that represents the right factor), and comparing with the right-hand side of (1), we get

$$H_n^{(\alpha)}(x|\lambda) = \sum_{k=0}^{n}\binom{n}{k} H_{n-k}^{(\alpha)}(\lambda)x^k. \tag{2}$$

The *Bernoulli polynomials* of order α are defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt} = \sum_{n\geq 0} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \tag{3}$$

(see[3,12,19]), and the *Euler polynomials* of order α are given by

$$\left(\frac{2}{e^t+1}\right)^\alpha e^{xt} = \sum_{n\geq 0} E_n^{(\alpha)}(x)\frac{t^n}{n!}, \tag{4}$$

(see[4]). The *Changhee polynomials of the second kind* are defined by the generating function

$$\frac{(1+t)^x}{1+\lambda(1+t)} = \sum_{k\geq 0} Ch_k(x|\lambda)\frac{t^k}{k!}, \tag{5}$$

where $|\lambda|<1$ (see[2,18]). The *Dachee polynomials* are given by

$$\frac{1-\lambda+t(1+\lambda)}{(1-\lambda)(1-t)}\left(\frac{1+t}{1-t}\right)^x = \sum_{k\geq 0} D_k(x|\lambda)\frac{t^k}{k!} \tag{6}$$

(see[2,17]). The *Poisson-Charlier sequence* is given by

$$C_n(x;a) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}a^{-k}(x)_k \sim \left(e^{a(e^t-1)}, a(e^t-1)\right), \tag{7}$$

(for the notation $\sim$ see the next section) where $a\neq 0$, $n\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}$ and $(x)_k = x(x-1)\cdots(x-k+1)$. It follows that

$$\sum_{k\geq 0} C_n(k;a)\frac{t^k}{k!} = \left(\frac{t-a}{a}\right)^n e^t, \tag{8}$$

(see[20,21,30,31]). From (7) and (8), we derive the generating function of *Poisson-Charlier polynomials*:

$$\sum_{k\geq 0} C_k(x;a)\frac{t^k}{k!} = e^{-t}\left(\frac{t+a}{a}\right)^x, \tag{9}$$

(see[20,21,30,31]). We shall also need a solution of the Bessel differential equation

$$x^2y''+2(x+1)y'+n(n+1)y=0$$

given by

$$y_n(x) = \sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!k!}\left(\frac{x}{2}\right)^k. \tag{10}$$

The aim of the present paper is to deduce some new identities between some particular classes of Sheffer polynomials by using umbral calculus techniques. More precisely, for two

given sequences of Sheffer polynomials $\{R_n(x)\}_{n\geq 0}$ and $\{S_n(x)\}_{n\geq 0}$, we are interested in finding constants $c_{n,k}$ such that

$$S_n(x) = \sum_{k=0}^{n} c_{n,k}R_k(x) \tag{11}$$

holds for all $n\in\mathbb{N}_0$. In this context, such an identity will be called the $(S_n;R_n)$*-identity* and the $c_{n,k}$ will be called *connection constants*. The most famous example of such an $(S_n;R_n)$-identity is connected to $S_2(n,k)$, the Stirling numbers of the second kind. The description given by James Stirling on how to compute these values makes it clear that he did not use the recurrence relation of the Stirling numbers of the second kind which Saka obtained in 1782. To read more about how James Stirling used the falling polynomials for accelerating the convergence of series, see the English translation of *Methodus Differentialis* with annotations by Tweedle[34]. Despite Stirling's earlier discovery of the values of Stirling numbers of the second kind, Saka deserves credit for being the first one to associate a combinatorial meaning to these values, which are now named after James Stirling. James Stirling showed

$$x^n = \sum_{k=1}^{n} S_2(n,k)(x)_k, \tag{12}$$

which is the $(x^n;(x)_n)$-identity. Identity (12) has been considered, extended and generalized by many researchers (see[32]). For instance, in[7] (see also[23]), Identity (12) has been presented as normal ordering of the operator $(XD)^n$ (here X and D are defined as $(Xp)(x)=xp(x)$ and $(Dp)(x)=\frac{d}{dx}p(x)$, for any formal power series $p(x)$):

$$(XD)^n = \sum_{k=1}^{n} S_2(n,k)X^kD^k, \text{ (see[7] (Eq. (25))).}$$

Note that to intepret (12) in a direct fashion as a relation for normal ordering operators, one sets $x=XD$ and uses the relation $(XD)_k = XD(XD-1)(XD-2)\cdots(XD-k+1) = X^kD^k$ already known to George Boole in the 1860s. In the present paper, we briefly consider a few related consequences of (11) for normal ordering, see, e.g., Theorem IV.1.

The aim of the present paper is to present several new $(S_n;R_n)$-identities by the use of umbral calculus. At first, in the next section, we present the necessary definitions, notation and results from umbral algebra and umbral calculus. Then, in Section 3, we present several $(S_n;R_n)$-identities where S_n and R_n are related to the above polynomials. Finally, in Section

4, we establish a connection between our $(S_n; R_n)$-identities and the problem of normal ordering.

II. UMBRAL CALCULUS

Let Π be the algebra of polynomials in a single variable x over $\mathbb{C}$ and let Π^* be the vector space of all linear functionals on Π. The action of a linear functional L on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$, and the vector space structure on Π^* is defined by

$$\langle cL + c'L'|p(x)\rangle = c\langle L|p(x)\rangle + c'\langle L'|p(x)\rangle,$$

where c, c' are any two complex constants (see[20,21,30,31]). Let $\mathcal{H}$ denote the algebra of formal power series in a single variable t:

$$\mathcal{H} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}. \tag{13}$$

The formal power series in the variable t define a linear functional on Π by setting

$$\langle f(t)|x^n\rangle = a_n, \text{ for all } n \geq 0, \text{ (see}^{20,21,30,31}). \tag{14}$$

By (13) and (14), we easily get

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \text{ for all } n, k \geq 0, \text{ (see}^{20,21,30,31}), \tag{15}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let $f_L(t) = \sum_{k\geq 0}\langle L|x^k\rangle\frac{t^k}{k!}$. From (15), we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from Π^* onto $\mathcal{H}$. Henceforth, $\mathcal{H}$ is thought of as set of both formal power series and linear functionals. We call $\mathcal{H}$ the *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

The *order* $O(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish (see[20,21,30,31]). If $O(f(t)) = 1$ (respectively, $O(f(t)) = 0$), then $f(t)$ is called a *delta* (respectively, an *invertable*) series.

Suppose that $O(f(t)) = 1$ and $O(g(t)) = 0$, then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)(f(t))^k|S_n(x)\rangle = n!\delta_{n,k}$, where $n, k \geq 0$. The sequence $S_n(x)$ is called the *Sheffer* sequence for $(g(t), f(t))$ which is denoted by $S_n(x) \sim (g(t), f(t))$ (see[20,21,30,31]).

For $f(t) \in \mathcal{H}$ and $p(x) \in \Pi$, we have

$$\langle e^{yt}|p(x)\rangle = p(y), \quad \langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle, \tag{16}$$

and

$$f(t) = \sum_{k\geq 0}\langle f(t)|x^k\rangle\frac{t^k}{k!}, \quad p(x) = \sum_{k\geq 0}\langle t^k|p(x)\rangle\frac{x^k}{k!}, \tag{17}$$

(see[20,21,30,31]). From (17), we derive

$$\langle t^k|p(x)\rangle = p^{(k)}(0), \quad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0), \tag{18}$$

where $p^{(k)}(0)$ denotes the k-th derivative of $p(x)$ with respect to x at $x = 0$. Thus, by (18), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k}p(x), \text{ for all } k \geq 0, \text{ (see}^{20-22,29-31}). \tag{19}$$

Let $S_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))}e^{y\bar{f}(t)} = \sum_{k\geq 0} S_k(y)\frac{t^k}{k!}, \tag{20}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see[20-22,29-31]).

For $S_n(x) \sim (g(t), f(t))$ and $R_n(x) \sim (h(t), \ell(t))$, let

$$S_n(x) = \sum_{k=0}^{n} c_{n,k}R_k(x), \tag{21}$$

then we have

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}(\ell(\bar{f}(t)))^k \Big| x^n \right\rangle, \tag{22}$$

(see[20,21,30,31]). The equations (21) and (22) are important to derive our main results in this paper.

III. IDENTITIES AND SHEFFER SEQUENCES OF POLYNOMIALS

In this section, we present several $(S_n; R_n)$-identities based on (22), where $\{R_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ are two sequences of Sheffer polynomials.

A. Bessel function

The solution of the Bessel differential equation $y_n(x)$ is called *Bessel function*. It is well known that

$$x^n y_{n-1}(1/x) \sim (1, t - t^2/2), \quad (\text{see}^{30,31}). \tag{23}$$

By (20) and (23), we get $\sum_{k\geq 0} x^k y_{k-1}(1/x)\frac{t^k}{k!} = e^{x(1-\sqrt{1-2t})}$. For $x^n \sim (1,t)$ and $x^n y_{n-1}(1/x) \sim (1, t-t^2/2)$, let

$$x^n = \sum_{k=0}^{n} c_{n,k} x^k y_{k-1}(1/x). \tag{24}$$

Then, by (22), we have

$$\begin{aligned}
c_{n,k} &= \frac{1}{k!}\langle (t-t^2/2)^k | x^n \rangle = \frac{1}{k!}\langle t^k (1-t/2)^k | x^n \rangle \\
&= (-1)^k \binom{n}{k} \langle (t-2)^k/2^k | x^{n-k} \rangle = (-1)^k \binom{n}{k} \sum_{\ell=0}^{n-k} \frac{C_k(\ell;2)}{\ell!} \langle t^\ell e^{-t} | x^{n-k} \rangle \\
&= (-1)^k \binom{n}{k} \sum_{\ell=0}^{n-k} C_k(\ell;2) \binom{n-k}{\ell} \langle e^{-t} | x^{n-k-\ell} \rangle \\
&= \binom{n}{k} \sum_{\ell=0}^{n-k} C_k(\ell;2) \binom{n-k}{\ell} (-1)^{n-\ell},
\end{aligned}$$

where $C_k(\ell;2)$ is the Poisson-Charlier polynomial. Therefore, by (24), we obtain the following result.

Theorem III.1 *For $n \geq 0$,*

$$x^n = \sum_{k=0}^{n} \sum_{\ell=0}^{n-k} (-1)^{n-\ell} C_k(\ell;2) \binom{n}{k} \binom{n-k}{\ell} x^k y_{k-1}(1/x).$$

For an arbitrary $b \in \mathbb{R}$, the *Abel polynomials* form a polynomial sequence, the n-th term of which is given by $A_n(x;b) = x(x-bn)^{n-1} \sim (1, te^{bt})$, (see[20,21,30,31]). Let

$$x^n = \sum_{k=0}^{n} c_{n,k} A_k(x;b). \tag{25}$$

Then, by using that $x^n \sim (1,t)$, (16) and (22), we obtain

$$c_{n,k} = \frac{1}{k!}\langle (te^{bt})^k | x^n \rangle = \frac{1}{k!}\langle t^k e^{kbt} | x^n \rangle = \binom{n}{k} \langle e^{kbt} | x^{n-k} \rangle = \binom{n}{k} (bk)^{n-k}.$$

Therefore, by (25), we can state the following result.

Theorem III.2 *For $n \geq 1$,*

$$x^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k^{n-k} b^{n-k} (x-bk)^{k-1}.$$

B. Laguerre polynomial

Laguerre polynomials are defined by

$$L_n(x) = \sum_{\ell=1}^{n} \frac{n!}{\ell!} \binom{n-1}{\ell-1} (-x)^\ell \sim \left(1, \frac{t}{t-1}\right), \quad (\text{see}^{20,21,30,31}).$$

Let

$$x^n = \sum_{k=0}^{n} c_{n,k} L_k(x). \tag{26}$$

Then, by using that $x^n \sim (1,t)$, (16) and (22), we obtain

$$\begin{aligned}
c_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{t}{t-1}\right)^k \Big| x^n \right\rangle = \frac{(-1)^k}{k!} \sum_{\ell \geq 0} \binom{k+\ell-1}{\ell} \langle t^{\ell+k} | x^n \rangle \\
&= \frac{(-1)^k}{k!} \sum_{\ell=0}^{n-k} \binom{k+\ell-1}{\ell} n! \delta_{\ell,n-k} = (-1)^k (n-k)! \binom{n}{k} \binom{n-1}{k-1}.
\end{aligned}$$

Therefore, by (26), we can state the following result.

Theorem III.3 *For $n \geq 0$,*

$$x^n = \sum_{k=0}^{n} (-1)^k (n-k)! \binom{n}{k} \binom{n-1}{k-1} L_k(x).$$

C. Bernoulli polynomial

Consider $P_n(x) \sim \left(\left(\frac{e^t-1}{t}\right)^r, t\right)$, $r \in \mathbb{R}$, then by (3) and (20) we get

$$\sum_{k\geq 0} P_k(x) \frac{t^k}{k!} = \frac{t^r}{(e^t-1)^r} e^{xt} = \sum_{k \geq 0} B_k^{(r)}(x) \frac{t^k}{k!}.$$

Hence, $P_n(x) = B_n^{(r)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^r, t\right)$. Let

$$B_n^{(r)}(x) = \sum_{k=0}^{n} c_{n,k} x^k y_{k-1}(1/x). \tag{27}$$

Then, by (16), (22) and (23), we obtain

$$\begin{aligned} c_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{t}{e^t-1}\right)^r (t-t^2/2)^k \middle| x^n \right\rangle = \frac{(-1)^k}{k!}\left\langle t^k\frac{(t-2)^k}{2^k} \middle| B_n^{(r)}(x)\right\rangle \\ &= \frac{(-1)^k (n)_k}{k!}\left\langle \frac{(t-2)^k}{2^k} \middle| B_{n-k}^{(r)}(x)\right\rangle \\ &= (-1)^k\binom{n}{k}\sum_{\ell=0}^{n-k}\frac{C_k(\ell;2)}{\ell!}\langle t^\ell | B_{n-k}^{(r)}(x-1)\rangle \\ &= (-1)^k\binom{n}{k}\sum_{\ell=0}^{n-k}C_k(\ell;2)\binom{n-k}{\ell}\langle 1 | B_{n-k-\ell}^{(r)}(x-1)\rangle \\ &= (-1)^k\binom{n}{k}\sum_{\ell=0}^{n-k}C_k(\ell;2)\binom{n-k}{\ell}B_{n-k-\ell}^{(r)}(-1). \end{aligned}$$

Therefore, by (27), we can state the following result.

Theorem III.4 *For $n \geq 0$,*

$$B_n^{(r)}(x) = \sum_{k=0}^{n}\sum_{\ell=0}^{n-k}(-1)^k\binom{n}{k}\binom{n-k}{\ell}C_k(\ell;2)B_{n-k-\ell}^{(r)}(-1)x^k y_{k-1}(1/x).$$

D. Changhee polynomial

Consider $Q_n(x) \sim (1+\lambda e^t, e^t-1)$, then by (5) and (20) we get

$$\sum_{k\geq 0} Q_k(x)\frac{t^k}{k!} = \frac{(1+t)^x}{1+\lambda(1+t)} = \sum_{k\geq 0} Ch_k(x|\lambda)\frac{t^k}{k!}.$$

Hence, $Q_n(x) = Ch_n(x|\lambda) \sim (1+\lambda e^t, e^t-1)$. Let

$$x^n = \sum_{k=0}^{n} c_{n,k}Ch_k(x|\lambda). \tag{28}$$

Then, by using the fact that $x^n \sim (1,t)$, (16) and (22), we obtain

$$c_{n,k} = \frac{1}{k!}\langle (1+\lambda e^t)(e^t-1)^k | x^n\rangle = \frac{1+\lambda}{k!}\langle (e^t-1)^k|x^n\rangle + \frac{\lambda}{k!}\langle (e^t-1)^{k+1}|x^n\rangle.$$

Using the fact that $\frac{(e^t-1)^k}{k!} = \sum_{\ell\geq 0}\frac{S_2(\ell+k,k)}{(\ell+k)!}t^{\ell+k}$, where $S_2(m,k)$ is the Stirling number of the second kind, we obtain

$$\begin{aligned} c_{n,k} &= (1+\lambda)\sum_{\ell\geq 0}\frac{S_2(\ell+k,k)}{(\ell+k)!}\langle t^{\ell+k}|x^n\rangle + \lambda(k+1)\sum_{\ell\geq 0}\frac{S_2(\ell+k+1,k+1)}{(\ell+k+1)!}\langle t^{\ell+k+1}|x^n\rangle \\ &= (1+\lambda)\sum_{\ell=0}^{n-k}\frac{S_2(\ell+k,k)}{(\ell+k)!}\langle t^{\ell+k}|x^n\rangle + \lambda(k+1)\sum_{\ell=0}^{n-k-1}\frac{S_2(\ell+k+1,k+1)}{(\ell+k+1)!}\langle t^{\ell+k+1}|x^n\rangle \\ &= (1+\lambda)S_2(n,k) + \lambda(k+1)S_2(n,k+1). \end{aligned}$$

Therefore, by (28) we can state the following result.

Theorem III.5 *For $n \geq 0$,*

$$x^n = (1+\lambda)\sum_{k=0}^{n}S_2(n,k)Ch_k(x|\lambda) + \lambda\sum_{k=0}^{n}(k+1)S_2(n,k+1)Ch_k(x|\lambda).$$

Let

$$B_n^{(r)}(x) = \sum_{k=0}^{n} c_{n,k}Ch_k(x|\lambda). \tag{29}$$

Since $Ch_n(x|\lambda) \sim (1+\lambda e^t, e^t-1)$ and $B_n^{(r)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^r, t\right)$, we obtain, by (16) and (22), that

$$c_{n,k} = \frac{1}{k!}\left\langle \left(\frac{t}{e^t-1}\right)^r (1+\lambda e^t)(e^t-1)^k \middle| x^n\right\rangle.$$

Using the fact that $\frac{(e^t-1)^k}{k!} = \sum_{\ell\geq 0}\frac{S_2(\ell+k,k)}{(\ell+k)!}t^{\ell+k}$, where $S_2(m,k)$ is the Stirling number of the second kind, we obtain

$$\begin{aligned} c_{n,k} &= \sum_{\ell\geq 0}\frac{S_2(\ell+k,k)}{(\ell+k)!}\left\langle \left(\frac{t}{e^t-1}\right)^r (1+\lambda e^t)t^{\ell+k} \middle| x^n\right\rangle \\ &= \sum_{\ell\geq 0}\frac{S_2(\ell+k,k)}{(\ell+k)!}\langle (1+\lambda e^t)t^{\ell+k}|B_n^{(r)}(x)\rangle \\ &= \sum_{\ell\geq 0}\frac{S_2(\ell+k,k)}{(\ell+k)!}\langle t^{\ell+k}|B_n^{(r)}(x)\rangle + \lambda\sum_{\ell\geq 0}\frac{S_2(\ell+k,k)}{(\ell+k)!}\langle t^{\ell+k}|B_n^{(r)}(x+1)\rangle \\ &= \sum_{\ell=0}^{n-k}\frac{S_2(\ell+k,k)}{(\ell+k)!}\langle t^{\ell+k}|B_n^{(r)}(x)\rangle + \lambda\sum_{\ell=0}^{n-k}\frac{S_2(\ell+k,k)}{(\ell+k)!}\langle t^{\ell+k}|B_n^{(r)}(x+1)\rangle \\ &= \sum_{\ell=0}^{n-k}S_2(\ell+k,k)\binom{n}{\ell+k}(B_{n-k-\ell}^{(r)} + \lambda B_{n-k-\ell}^{(r)}(1)). \end{aligned}$$

Therefore, by (29), we can state the following result.

Theorem III.6 *For $n \geq 0$,*

$$B_n^{(r)}(x) = \sum_{k=0}^{n}\sum_{\ell=0}^{n-k}S_2(\ell+k,k)\binom{n}{\ell+k}(B_{n-k-\ell}^{(r)} + \lambda B_{n-k-\ell}^{(r)}(1))Ch_k(x|\lambda).$$

E. Daehee polynomial

Consider $T_n(x) \sim \left(\frac{1-\lambda}{e^t-\lambda}, \frac{e^t-1}{e^t+1}\right)$, then by (6) and (20) we get

$$\sum_{k\geq 0} T_k(x)\frac{t^k}{k!} = \frac{1-\lambda+t(1+\lambda)}{(1-\lambda)(1-t)}\left(\frac{1+t}{1-t}\right)^x = \sum_{k\geq 0} D_k(x|\lambda)\frac{t^k}{k!}.$$

Hence, $T_n(x) = D_n(x|\lambda) \sim \left(\frac{1-\lambda}{e^t-\lambda}, \frac{e^t-1}{e^t+1}\right)$. Let

$$H_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} c_{n,k} D_k(x|\lambda). \tag{30}$$

Then, by using the fact that $H_n^{(r)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^r, t\right)$, (16) and (22), we obtain

$$\begin{aligned} c_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^{r+1}\left(\frac{e^t-1}{e^t+1}\right)^k |x^n\right\rangle \\ &= \frac{1}{k!}\left\langle \left(\frac{e^t-1}{e^t+1}\right)^k |H_n^{(r+1)}(x|\lambda)\right\rangle. \end{aligned}$$

Using the fact that $H_n^{(r)}(x|\lambda) = \sum_{k=0}^{n}\binom{n}{k}H_{n-k}^{(r)}(\lambda)x^k$, we get

$$\begin{aligned} c_{n,k} &= \frac{1}{k!}\sum_{\ell=0}^{n}\binom{n}{\ell}H_{n-\ell}^{(r+1)}(\lambda)\left\langle \left(\frac{e^t-1}{e^t+1}\right)^k |x^\ell\right\rangle \\ &= \frac{1}{k!}\sum_{\ell=0}^{n}\binom{n}{\ell}H_{n-\ell}^{(r+1)}(\lambda)\left\langle \left(\frac{e^t-1}{2}\frac{2}{e^t+1}\right)^k |x^\ell\right\rangle. \end{aligned}$$

Since $\sum_{k\geq 0} E_k^{(r)}(x)\frac{t^k}{k!} = \frac{2^r e^{xt}}{(e^t+1)^r} = \sum_{k\geq 0}\frac{2^r x^k}{(e^t+1)^r}\frac{t^k}{k!}$, we have $E_n^{(r)}(x) \sim \left(\left(\frac{e^r+1}{2}\right)^r, t\right)$. Thus, by the fact that $(e^t-1)^k = \sum_{j\geq 0}\binom{k}{j}(-1)^{k-j}e^{jt}$, we get

$$c_{n,k} = \frac{1}{k!2^k}\sum_{\ell=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{n}{\ell}\binom{k}{j}H_{n-\ell}^{(r+1)}(\lambda)E_\ell^{(k)}(j).$$

Therefore, by (30), we can state the following result.

Theorem III.7 *For $n \geq 0$,*

$$H_n^{(r)}(x|\lambda) = \sum_{k=0}^{n}\sum_{\ell=0}^{n}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{k!2^k}\binom{n}{\ell}\binom{k}{j}H_{n-\ell}^{(r+1)}(\lambda)E_\ell^{(k)}(j)D_k(x|\lambda).$$

Using the fact that $\frac{(e^t-1)^k}{k!} = \sum_{\ell\geq 0}\frac{S_2(\ell+k,k)}{(\ell+k)!}t^{\ell+k}$, where $S_2(m,k)$ is the Stirling number of the second kind, we obtain

$$\begin{aligned} c_{n,k} &= \frac{1}{2^k}\sum_{\ell=0}^{n}\sum_{j\geq 0}\frac{S_2(j+k,k)}{(j+k)!}\binom{n}{\ell}H_{n-\ell}^{(r+1)}(\lambda)\langle t^{j+k}|E_\ell^{(k)}(x)\rangle \\ &= \frac{1}{2^k}\sum_{\ell=0}^{n}\sum_{j=0}^{\ell-k}S_2(j+k,k)\binom{n}{\ell}\binom{\ell}{j+k}H_{n-\ell}^{(r+1)}(\lambda)E_{\ell-k-j}^{(k)}. \end{aligned}$$

Therefore, by (30), we can state the following result.

Theorem III.8 *For $n \geq 0$.*

$$H_n^{(r)}(x|\lambda) = \sum_{k=0}^{n}\sum_{\ell=0}^{n}\sum_{j=0}^{\ell-k}2^{-k}\binom{n}{\ell}\binom{\ell}{j+k}S_2(j+k,k)H_{n-\ell}^{(r+1)}(\lambda)E_{\ell-k-j}^{(k)}D_k(x|\lambda).$$

F. Hermite polynomial

The Hermite polynomials $H_n(x)$ are given by $H_n(x) \sim (e^{t^2/4}, t/2)$. Let

$$H_n(x) = \sum_{k=0}^{n} c_{n,k}x^k y_{k-1}(1/x). \tag{31}$$

Then, by (22), (16) and (23), we obtain

$$\begin{aligned} c_{n,k} &= \frac{1}{k!}\langle e^{-t^2}(2t-2t^2)^k|x^n\rangle \\ &= \frac{(-1)^k 2^k}{k!}\langle e^{-t^2}t^k(t-1)^k|x^n\rangle \\ &= (-1)^k 2^k\binom{n}{k}\langle e^{-t^2}(t-1)^k|x^{n-k}\rangle. \end{aligned}$$

Using the fact that

$$\begin{aligned} e^{-t^2}(t-1)^k &= \sum_{j\geq 0}\sum_{\ell\geq 0}\frac{(-1)^j C_k(\ell;1)}{j!\ell!}t^{2j+\ell}e^{-t} \\ &= \sum_{m\geq 0}\sum_{j=0}^{[m/2]}\frac{(-1)^j C_k(m-2j;1)}{j!(m-2j)!}t^m e^{-t}, \end{aligned}$$

we get

$$
\begin{aligned}
c_{n,k} &= (-1)^k 2^k \binom{n}{k} \sum_{m\ge 0}\sum_{j=0}^{[m/2]} \frac{(-1)^j C_k(m-2j;1)}{j!(m-2j)!}\langle t^m e^{-t} | x^{n-k}\rangle \\
&= (-1)^k 2^k \binom{n}{k} \sum_{m\ge 0}\sum_{j=0}^{[m/2]} \frac{(-1)^j C_k(m-2j;1)}{j!(m-2j)!}\langle t^m | (x-1)^{n-k}\rangle \\
&= 2^k \binom{n}{k} \sum_{m=0}^{n-k}\sum_{j=0}^{[m/2]} \frac{(-1)^{n-m-j} C_k(m-2j;1)m!}{j!(m-2j)!}\binom{n-k}{m}.
\end{aligned}
$$

Therefore, by (31), we can state the following result.

Theorem III.9 *For $n \ge 0$,*

$$
H_n(x) = \sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{j=0}^{[m/2]} \frac{(-1)^{n-m-j} C_k(m-2j;1)m!2^k}{j!(m-2j)!}\binom{n}{k}\binom{n-k}{m} x^k y_{k-1}(1/x).
$$

G. Chebyshev polynomials of the second kind

The *Chebyshev polynomials of the second kind* are defined by $U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$ for $r \ge 0$. Evidently, $U_r(x)$ is a polynomial of degree r in x with integer coefficients. For example, $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, and in general, $U_r(x) = 2xU_{r-1}(x) - U_{r-2}(x)$. Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory (see[28]).

Let $\alpha = x + \sqrt{x^2-1}$, $\beta = x - \sqrt{x^2-1}$ and $\gamma = \sqrt{x^2-1}$. From the above recurrence relation it is not hard to see that

$$
\sum_{n\ge 0} U_n(x)\frac{t^n}{n!} = \frac{\alpha e^{\alpha t} - \beta e^{\beta t}}{\alpha - \beta} = \frac{\alpha e^{\gamma t} - \beta e^{-\gamma t}}{2\gamma} e^{xt},
$$

which implies that $U_n(x) \sim \left(\frac{2\gamma}{\alpha e^{\gamma t} - \beta e^{-\gamma t}}, t\right)$.

By (23) we have that $x^n y_{n-1}(1/x) \sim (1, t - t^2/2)$. Assume

$$
U_n(x) = \sum_{k=0}^{n} c_{n,k} x^k y_{k-1}(1/x). \tag{32}
$$

Then, by (22), we have

$$
\begin{aligned}
c_{n,k} &= \frac{1}{k!}\left\langle \frac{\alpha e^{\gamma t} - \beta e^{-\gamma t}}{2\gamma}(t - t^2/2)^k | x^n \right\rangle \\
&= (-1)^k \binom{n}{k} \sum_{\ell=0}^{n-k} \frac{C_k(\ell;2)}{\ell!} \left\langle \frac{\alpha e^{\gamma t} - \beta e^{-\gamma t}}{2\gamma} t^\ell e^{-t} | x^{n-k} \right\rangle \\
&= \frac{(-1)^k}{2\gamma}\binom{n}{k} \sum_{\ell=0}^{n-k} C_k(\ell;2)\binom{n-k}{\ell} \langle \alpha e^{(\gamma-1)t} - \beta e^{-(\gamma+1)t} | x^{n-k-\ell}\rangle,
\end{aligned}
$$

which, by (16), implies

$$
c_{n,k} = \binom{n}{k} \sum_{\ell=0}^{n-k} (-1)^{n-\ell} C_k(\ell;2) \binom{n-k}{\ell} \frac{\alpha(1-\gamma)^{n-k-\ell} - \beta(1+\gamma)^{n-k-\ell}}{2\gamma}. \tag{33}
$$

By a simple induction, we have

$$
\begin{aligned}
\frac{(1-\gamma)^m - (1+\gamma)^m}{2\gamma} &= -\sqrt{2-x^2}^{m-1} U_{m-1}\left(\frac{1}{\sqrt{2-x^2}}\right), \\
\frac{(1-\gamma)^m + (1+\gamma)^m}{2} &= \sqrt{2-x^2}^{m} U_{m}\left(\frac{1}{\sqrt{2-x^2}}\right) - \sqrt{2-x^2}^{m-1} U_{m-1}\left(\frac{1}{\sqrt{2-x^2}}\right),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\frac{\alpha(1-\gamma)^m - \beta(1+\gamma)^m}{2\gamma} \\
&= -\sqrt{2-x^2}^{m-1}\left(\frac{x}{\sqrt{2-x^2}} U_{m-1}\left(\frac{1}{\sqrt{2-x^2}}\right) + U_{m-2}\left(\frac{1}{\sqrt{2-x^2}}\right)\right).
\end{aligned} \tag{34}
$$

Theorem III.10 *For all $n \ge 0$,*

$$
U_n(x) = \sum_{k=0}^{n}\left[\sum_{\ell=0}^{n-k} (-1)^{n-1-\ell} C_k(\ell;2)\binom{n-k}{\ell}\binom{n}{k} V_{n-k-\ell}(x)\right] x^k y_{k-1}(1/x),
$$

where $V_m(x) = \sqrt{2-x^2}^{m-1}\left(\frac{x}{\sqrt{2-x^2}}U_{m-1}\left(\frac{1}{\sqrt{2-x^2}}\right) + U_{m-2}\left(\frac{1}{\sqrt{2-x^2}}\right)\right)$.

IV. NORMAL ORDERING

Since the seminal work of Katriel[14] the combinatorial aspects of normal ordering arbitrary words in the creation and annihilation operators $a^\dagger$ and a of a single-mode boson having the usual commutation relations

$$
[a, a^\dagger] = aa^\dagger - a^\dagger a = 1, \quad [a, a] = 0, \quad [a^\dagger, a^\dagger] = 0 \tag{35}
$$

have been studied intensively, see[7,8,14,16,23–25,33] and references therein. Recall that *normal ordering* $\mathcal{N}(F(a,a^\dagger))$ is a functional representation of an operator function $F(a,a^\dagger)$ in which all the creation operators stand to the left of the annihilation operators. For instance, Katriel[14] showed that normal ordering of $(a^\dagger a)^n$ is given by

$$\mathcal{N}[(a^\dagger a)^n] = \sum_{k=0}^{n} S_2(n,k)(a^\dagger)^k a^k.$$

Using the properties of coherent states (for example, see[7] (Appendix A)), we obtain

$$\langle z|e^{ta^\dagger a}|z\rangle = \sum_{n\geq 0}\langle z|(a^\dagger a)^n|z\rangle \frac{t^n}{n!} = e^{|z|^2(e^t-1)}, \text{ (see}^7 \text{ (Eq. 35)).}$$

Choosing z such that $|z|^2 = 1$, the right-hand side is the exponential generating function of the *Bell numbers* B_n. Thus, one directly obtains $\langle z|\hat{N}^n|z\rangle = B_n$, where we have introduced the number operator $\hat{N} = a^\dagger a$. This connection between Bell numbers and expectation values of the number operator with respect to coherent states was also observed by Katriel[15].

In[7] (see also[8]), it is shown that if $S_n(x) \sim (g(t), f(t))$, and if the action of M and P on $S_n(x)$ satisfy $(MS_n)(x) = S_{n+1}(x)$ and $(PS_n)(x) = nS_{n-1}(x)$, then $P = f(D)$ and $M = \left(X - \frac{g'(D)}{g(D)}\right)\frac{1}{f'(D)}$, where D is the derivative operator, X is the multiplication operator, namely $(Dp)(x) = \frac{dp}{dx}(x)$ and $(Xp)(x)) = xp(x)$, and $f'(t)$ denotes the derivative of $f(t)$ respect to t. The operator M is also called *Sheffer shift* and the above formula is a well known result of umbral calculus[31] (Theorem 3.7.1). Note that we can formally write $S_n(x) = M^n \cdot 1$ (where $S_0(x) = 1$), i.e., the sequence $S_n(x)$ is generated successively by application of M. For instance, if $M = 2X - D$ and $P = \frac{1}{2}D$, then $S_n(x) = H_n(x)$ (Hermite polynomials). As another example, let $M = -XD^2 + (2X-1)D - X + 1$ and $P = -\frac{D}{1-D} = -\sum_{j\geq 1} D^j$, then $S_n(x) = n!L_n(x)$ (Laguerre polynomials).

The operators M and P define an action on the index n of the *quasi-monomial* $S_n(x)$. This resembles the property of the operators $a^\dagger$ and a in Fock space given by $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$. This suggests the following correspondance $P \leftrightarrow a$, $M \leftrightarrow a^\dagger$ and $S_n(x) \leftrightarrow |n\rangle$ for all $n \geq 0$ (for more details, we refer the reader to[7] (Section 5.2.3) or[8]).

Let $M_{g,f}(x,y) = \left(y - \frac{g'(x)}{g(x)}\right)\frac{1}{f'(x)}$. Then the above correspondence with $P = D$, $M = X$ and $S_n(x) = x^n$ shows that in general $S_n(x) = [M_{g,f}(D,X)]^n \cdot 1$, hence,

$$[M_{g,f}(a,a^\dagger)]^n|0\rangle = S_n(a^\dagger)|0\rangle.$$

As a relation for coherent states one obtains, therefore,

$$\langle z|[M_{g,f}(a,a^\dagger)]^n|0\rangle = S_n(z^*)\langle z|0\rangle. \tag{36}$$

Exponentiating $M_{g,f}(a,a^\dagger)$ and using (20) for the generating function of the $S_n(z^*)$, one obtains

$$\langle z|e^{\lambda M_{g,f}(a,a^\dagger)}|0\rangle = \frac{1}{g(\bar{f}(\lambda))}e^{z^*\bar{f}(\lambda)}\langle z|0\rangle. \tag{37}$$

This result can be extended to a general matrix element $\langle z|e^{\lambda M_{g,f}(a,a^\dagger)}|z'\rangle$ by using $|z'\rangle = e^{-|z'|^2/2}e^{z'a^\dagger}|0\rangle$, giving after some tedious algebra ([7] (Section 5.3))

$$\langle z|e^{\lambda M_{g,f}(a,a^\dagger)}|z'\rangle = \frac{g(z')}{g(\bar{f}(\lambda + f(z')))}e^{z^*[\bar{f}(\lambda+f(z'))-z']}\langle z|z'\rangle, \tag{38}$$

where $\langle z|z'\rangle = e^{z^*z' - \frac{1}{2}|z'|^2 - \frac{1}{2}|z|^2}$ is the coherent states overlapping factor. Let us return to (36). If $S_n(x) \sim (g(t), f(t))$ and $R_n(x) \sim (h(t), \ell(t))$, then the identity $S_n(x) = \sum_{k=0}^n c_{n,k}R_k(x)$ with $c_{n,k}$ given in (22) can be written as

$$\langle z|[M_{g,f}(a,a^\dagger)]^n|0\rangle = S_n(z^*)\langle z|0\rangle = \sum_{k=0}^n c_{n,k}R_k(z^*)\langle z|0\rangle = \sum_{k=0}^n c_{n,k}\langle z|[M_{h,\ell}(a,a^\dagger)]^k|0\rangle,$$

which shows that our identities can be interpreted as a transformation between $\langle z|[M_{g,f}(a,a^\dagger)]^n|0\rangle$ and $\langle z|[M_{h,\ell}(a,a^\dagger)]^k|0\rangle$, or, between the operator functions $[M_{g,f}(a,a^\dagger)]^n$ and $[M_{h,\ell}(a,a^\dagger)]^k$ themselves. Let us state this relation explicitly in the following theorem.

Theorem IV.1 *Let the two Sheffer sequences $S_n(x) \sim (g(t), f(t))$ and $R_n(x) \sim (h(t), \ell(t))$ be related by $S_n(x) = \sum_{k=0}^n c_{n,k}R_k(x)$. Then one has*

$$[M_{g,f}(a,a^\dagger)]^n = \sum_{k=0}^n \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}(\ell(\bar{f}(t)))^k \Big| x^n\right\rangle [M_{h,\ell}(a,a^\dagger)]^k + \mathcal{O}, \tag{39}$$

where $\bar{f}$ denotes the compositional inverse of f and $\mathcal{O}$ denotes an operator satisfying $\mathcal{O}|0\rangle = 0$. Consequently, when evaluated between the coherent states $\langle z|$ and $|0\rangle$, this implies

$$\langle z|[M_{g,f}(a,a^\dagger)]^n|0\rangle = \sum_{k=0}^n \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}(\ell(\bar{f}(t)))^k \Big| x^n\right\rangle \langle z|[M_{h,\ell}(a,a^\dagger)]^k|0\rangle. \tag{40}$$

Let us consider the particular case where $R_k(x) = x^k$, i.e., $S_n(x) = \sum_{k=0}^n c_{n,k}x^k$. For $R_k(x) = x^k$, one has $x^n \sim (1,t)$, i.e., $h(t) = 1$ and $\ell(t) = t$. It follows that $M_{h,\ell}(a,a^\dagger) = a^\dagger$, thus

$$\langle z|[M_{h,\ell}(a,a^\dagger)]^k|0\rangle = (z^*)^k\langle z|0\rangle.$$

The coefficients $c_{n,k}$ are given in this case by $\frac{1}{k!}\left\langle\frac{h(\bar{f}(t))}{g(\bar{f}(t))}(\ell(\bar{f}(t)))^k|x^n\right\rangle=\frac{1}{k!}\left\langle\frac{1}{g(\bar{f}(t))}(\bar{f}(t))^k|x^n\right\rangle$, so that

$$\frac{\langle z|[M_{g,f}(a,a^\dagger)]^n|0\rangle}{\langle z|0\rangle}=\sum_{k=0}^{n}\frac{(z^*)^k}{k!}\left\langle\frac{1}{g(\bar{f}(t))}(\bar{f}(t))^k|x^n\right\rangle.$$

Let us make contact with the normal ordering procedure. Above, we have denoted for a general operator function $F(a,a^\dagger)$ its normally ordered form $\mathcal{N}(F(a,a^\dagger))\equiv F(a,a^\dagger)$, which is obtained by moving all annihilation operators a to the right, using the commutation relation. We may additionally define the operation $:G(a,a^\dagger):$ which means normally order $G(a,a^\dagger)$ without taking into account the commutation relation (i.e., treat a and $a^\dagger$ as commuting objects). The normal ordering problem for $F(a,a^\dagger)$ is solved if we can find an operator $G(a,a^\dagger)$ for which $F(a,a^\dagger)=:G(a,a^\dagger):$ is satisfied. Now, it is a well known result that if $\langle z|F(a,a^\dagger)|z'\rangle=G(z^*,z')\langle z|z'\rangle$, then $\mathcal{N}(F(a,a^\dagger))=:G(a^\dagger,a):$. Thus, we can write (38) as (see[7] (Equation (5.23)) or[8])

$$\mathcal{N}(e^{\lambda M_{g,f}(a,a^\dagger)})=:\frac{g(a)}{g(\bar{f}(\lambda+f(a)))}e^{a^\dagger[\bar{f}(\lambda+f(a))-a]}:. \qquad (41)$$

By (39) and the results in the previous sections, we can obtain several nice normal ordering identities. In the following, we present several examples.

Example IV.2 *Let* $(g,f)=(1,t)$ *and* $(h,\ell)=(1,t-t^2/2)$, *so* $M_{g,f}(a,a^\dagger)=a^\dagger$ *and* $M_{h,\ell}(a,a^\dagger)=a^\dagger(1-a)^{-1}=a^\dagger\sum_{j\geq0}a^j$. *Then, by Section 3.1 and (39), we obtain*

$$(a^\dagger)^n=\sum_{k=0}^{n}\binom{n}{k}\sum_{\ell=0}^{n-k}C_k(\ell;2)\binom{n-k}{\ell}(-1)^{n-\ell}\left(a^\dagger(1-a)^{-1}\right)^k+\mathcal{O}.$$

The expression on the right-hand side looks rather complicated. When evaluated between the states $\langle z|$ *and* $|0\rangle$, *the experssion* $\langle z|\left(a^\dagger(1-a)^{-1}\right)^k|0\rangle$ *reduces to* $p_k(z^*)\langle z|0\rangle$, *where* p_k *is a polynomial of degree* k. *For example, if* $k=1$, *then* $\langle z|a^\dagger(1-a)^{-1}|0\rangle=\langle z|a^\dagger|0\rangle=z^*\langle z|0\rangle$ *due to* $a^r|0\rangle=0$ *for* $r\geq1$. *Similarly, if* $k=2$, *then* $\langle z|\left(a^\dagger(1-a)^{-1}\right)^2|0\rangle=\langle z|a^\dagger(1-a)^{-1}a^\dagger|0\rangle=\langle z|a^\dagger(1+a)a^\dagger|0\rangle$, *where we have used that* $a^ra^\dagger|0\rangle=0$ *for* $r\geq2$. *It follows that* $\langle z|\left(a^\dagger(1-a)^{-1}\right)^2|0\rangle=\left((z^*)^2+z^*\right)\langle z|0\rangle$. *In the same fashion one determines for* $k=3$ *that* $\langle z|\left(a^\dagger(1-a)^{-1}\right)^3|0\rangle=\left((z^*)^3+3(z^*)^2+3z^*\right)\langle z|0\rangle$. *In general, the above equation gives*

$$(z^*)^n=\sum_{k=0}^{n}\binom{n}{k}\sum_{\ell=0}^{n-k}C_k(\ell;2)\binom{n-k}{\ell}(-1)^{n-\ell}p_k(z^*).$$

Comparing this with Theorem III.1, we see that $p_k(z^*)=(z^*)^ky_{k-1}(1/z^*)$ *where* y_{k-1} *denotes the Bessel function defined in* (10).

Example IV.3 *Let* $(g,f)=(1,t)$ *and* $(h,\ell)=(1,t/(1-t))$, *so* $M_{g,f}(a,a^\dagger)=a^\dagger$ *and* $M_{h,\ell}(a,a^\dagger)=a^\dagger(1-a)^{-2}=a^\dagger\sum_{j\geq0}(j+1)a^j$. *Then, by Section 3.2 and (39), we obtain*

$$(a^\dagger)^n=\sum_{k=0}^{n}(-1)^k(n-k)!\binom{n}{k}\binom{n-1}{k-1}\left(a^\dagger(1-a)^{-2}\right)^k+\mathcal{O}.$$

Example IV.4 *Let* $(g,f)=(1,t)$ *and* $(h,\ell)=(1+\lambda e^t,e^t-1)$, *so* $M_{g,f}(a,a^\dagger)=a^\dagger$ *and* $M_{h,\ell}(a,a^\dagger)=a^\dagger e^{-a}-\frac{\lambda}{1+\lambda e^a}$. *Then, by Section 3.4 and (39), we obtain*

$$(a^\dagger)^n=\sum_{k=0}^{n}((1+\lambda)S_2(n,k)+\lambda(k+1)S_2(n,k+1))\left(a^\dagger e^{-a}-\frac{\lambda}{1+\lambda e^a}\right)^k+\mathcal{O}.$$

Example IV.5 *Let* $(g,f)=(e^{t^2/4},t/2)$ *and* $(h,\ell)=(1,t-t^2/2)$, *so* $M_{g,f}(a,a^\dagger)=2a^\dagger-a$ *and* $M_{h,\ell}(a,a^\dagger)=a^\dagger(1-a)^{-1}=a^\dagger\sum_{j\geq0}a^j$. *Then, by Section 3.6 and (39), we obtain*

$$(2a^\dagger-a)^n=\sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{j=0}^{[m/2]}\frac{(-1)^{n-m-j}C_k(m-2j;1)m!2^k}{j!(m-2j)!}\binom{n}{k}\binom{n-k}{m}\left(a^\dagger(1-a)^{-1}\right)^k+\mathcal{O}.$$

When evaluated between the states $\langle z|$ *and* $|0\rangle$, *the left-hand side becomes* $\langle z|(2a^\dagger-a)^n|0\rangle$. *According to*[7] *(Section 5.4.1), this can be simplified to* $H_n(z^*)\langle z|0\rangle$, *where* H_n *denotes the* n*-th Hermite polynomial. Using from the first example that* $\langle z|\left(a^\dagger(1-a)^{-1}\right)^k|0\rangle=(z^*)^ky_{k-1}(1/z^*)\langle z|0\rangle$, *this yields*

$$H_n(z^*)=\sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{j=0}^{[m/2]}\frac{(-1)^{n-m-j}C_k(m-2j;1)m!2^k}{j!(m-2j)!}\binom{n}{k}\binom{n-k}{m}(z^*)^ky_{k-1}(1/z^*),$$

which is the contents of Theorem III.9.

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