

REVIEW

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Sheffer sequences of polynomials and their applications

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Abstract

In this paper, we investigate some properties of several Sheffer sequences of several polynomials arising from umbral calculus. From our investigation, we can derive many interesting identities of several polynomials.

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1 Introduction

As is well known, the *Bernoulli polynomials* of order a are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^a e^{xt} = \sum_{n=0}^{\infty} B_n^{(a)}(x) \frac{t^n}{n!} \quad (\text{see [1-10]}), \quad (1.1)$$

and the *Narumi polynomials* are also given by

$$\left(\frac{\log(1+t)}{t}\right)^a (1+t)^x = \sum_{n=0}^{\infty} N_n^{(a)}(x) \frac{t^n}{n!} \quad (\text{see [11, 12]}). \quad (1.2)$$

In the special case, $x = 0$, $N_n^{(a)}(0) = N_n^{(a)}$ are called the *Narurni numbers*.

Throughout this paper, we assume that $\lambda \in \mathbb{C}$ with $\lambda \neq 1$. *Frobenius-Euler polynomials* of order a are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t - \lambda}\right)^a e^{xt} = \sum_{n=0}^{\infty} H_n^{(a)}(x|\lambda) \frac{t^n}{n!} \quad (\text{see [10-21]}). \quad (1.3)$$

The *Stirling number of the second kind* is also defined by the generating function to be

$$(e^t - 1)^n = n! \sum_{k=n}^{\infty} S_2(k, n) \frac{t^k}{k!} \quad (\text{see [9-12]}), \quad (1.4)$$

and the *Stirling number of the first kind* is given by

$$(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^n S_1(n, l) x^l \quad (\text{see [9, 11-13]}). \quad (1.5)$$

Let

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \mid a_k \in \mathbb{C} \right\}. \tag{1.6}$$

Let \mathbb{P} be the algebra of polynomials in the variable x over \mathbb{C} and \mathbb{P}^* be the vector space of all linear functionals on \mathbb{P} . The action of the linear functional L on a polynomial $p(x)$ is denoted by $\langle L|p(x) \rangle$. We recall that the vector space structures on \mathbb{P}^* are defined by $\langle L + M|p(x) \rangle = \langle L|p(x) \rangle + \langle M|p(x) \rangle$, $\langle cL|p(x) \rangle = c\langle L|p(x) \rangle$, where c is a complex constant (see [11, 12]).

For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, we define a linear functional $f(t)$ on \mathbb{P} by setting

$$\langle f(t)|x^n \rangle = a_n \quad (n \geq 0). \tag{1.7}$$

By (1.6) and (1.7), we get

$$\langle t^k|x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0), \tag{1.8}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [9–13]).

Suppose that $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k \rangle}{k!} t^k$. Then we have $\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle$ and $f_L(t) = L$. Thus, we note that the map $L \mapsto f_L(t)$ is a vector space isomorphism from \mathbb{P}^* onto \mathcal{F} . Henceforth, \mathcal{F} will be thought of as both a formal power series and a linear functional. We shall call \mathcal{F} the *umbral algebra*. The umbral calculus is the study of umbral algebra (see [9–13]).

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 1$, then $f(t)$ is called a *delta series*. If $o(f(t)) = 0$, then $f(t)$ is called an *invertible series*. Let $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)f(t)^k|S_n(x) \rangle = n! \delta_{n,k}$ ($n, k \geq 0$). The sequence $S_n(x)$ is called *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. By (1.8), we easily get that $\langle e^{yt}|p(x) \rangle = p(y)$. For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k \rangle}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x) \rangle}{k!} x^k, \tag{1.9}$$

and

$$\langle f_1(t) \cdots f_m(t)|x^n \rangle = \sum_{i_1+\dots+i_m=n} \binom{n}{i_1, \dots, i_m} \left(\prod_{j=1}^m \langle f_j(t)|x^{i_j} \rangle \right), \tag{1.10}$$

where $f_1(t), f_2(t), \dots, f_m(t) \in \mathcal{F}$ (see [9–12]). For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, by (1.9), we get

$$p^{(k)}(0) = \langle t^k|p(x) \rangle, \quad \langle 1|p^{(k)}(x) \rangle = p^{(k)}(0). \tag{1.11}$$

Thus, by (1.11), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad (k \geq 0) \text{ (see [10–13])}. \tag{1.12}$$

Let $S_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \quad \text{for all } y \in \mathbb{C}, \tag{1.13}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [11, 12]). By (1.2) and (1.13), we see that $N_n^{(a)}(x) \sim ((\frac{e^t-1}{t})^a, e^t - 1)$.

For $a \neq 0$, the Poisson-Charlier sequences are given by

$$C_n(x; a) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} a^{-k} (x)_k \sim (e^{a(e^t-1)}, a(e^t - 1)). \tag{1.14}$$

In particular, $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, we have

$$\sum_{l=0}^{\infty} C_n(l; a) \frac{t^l}{l!} = e^t \left(\frac{t-a}{a} \right)^n \quad (\text{see [11, 12]}). \tag{1.15}$$

The *Frobenius-type Eulerian polynomials* of order a are given by

$$\left(\frac{1-\lambda}{e^{t(\lambda-1)} - \lambda} \right)^a e^{xt} = \sum_{n=0}^{\infty} A_n^{(a)}(x|\lambda) \tag{1.16}$$

From (1.13) and (1.16), we note that

$$A_n^{(a)}(x|\lambda) \sim \left(\left(\frac{e^{t(1-\lambda)} - \lambda}{1-\lambda} \right)^a, t \right).$$

Let us assume that $p_n(x) \sim (1, f(t))$, $q_n(x) \sim (1, g(t))$. Then we have

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x) \quad (\text{see [11, 12]}). \tag{1.17}$$

Equation (1.17) is important in deriving our results in this paper. The purpose of this paper is to investigate some properties of Sheffer sequences of several polynomials arising from umbral calculus. From our investigation, we can derive many interesting identities of several polynomials.

2 Sheffer sequences of polynomials

Let us assume that $S_n(x) \sim (g(t), f(t))$. Then, by the definition of Sheffer sequence, we see that $g(t)S_n(x) \sim (1, f(t))$. If $g(t)$ is an invertible series, then $\frac{1}{g(t)}$ is also an invertible series. Let us consider the following Sheffer sequences:

$$M_n(x) \sim (1, f(t)), \quad x^n \sim (1, t). \tag{2.1}$$

From (1.17) and (2.1), we note that

$$M_n(x) = x \left(\frac{t}{f(t)} \right)^n x^{-1} x^n = x \left(\frac{t}{f(t)} \right)^n x^{n-1}. \tag{2.2}$$

For $g(t)S_n(x) \sim (1, f(t))$, by (2.2), we get

$$g(t)S_n(x) = x \left(\frac{t}{f(t)} \right)^n x^{n-1}. \tag{2.3}$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.1 For $S_n(x) \sim (g(t), f(t))$ and $n \geq 1$, we have

$$S_n(x) = \frac{1}{g(t)} x \left(\frac{t}{f(t)} \right)^n x^{n-1}.$$

For example, let $S_n(x) = D_n(x) \sim \left(\frac{1-\lambda}{e^t-\lambda}, \frac{e^t-1}{e^{t+1}} \right)$, where $D_n(x)$ is the n th Daehee polynomial (see [1, 8, 9]). Then, by Theorem 2.1, we get

$$\begin{aligned} D_n(x) &= \left(\frac{e^t - \lambda}{1 - \lambda} \right) x \left(\frac{t}{e^t - 1} \right)^n (e^t + 1)^n x^{n-1} = \left(\frac{e^t - \lambda}{1 - \lambda} \right) x \sum_{l=0}^n \binom{n}{l} B_{n-1}^{(n)}(x + l) \\ &= \frac{1}{1 - \lambda} \sum_{l=0}^n \binom{n}{l} \{ (x + 1) B_{n-1}^{(n)}(x + l + 1) - \lambda x B_{n-1}^{(n)}(x + l) \}. \end{aligned}$$

Let us take $S_n(x) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda} \right)^a, \frac{t^2}{e^{bt}-1} \right)$ ($b \neq 0$). Then, by Theorem 2.1, we get

$$\begin{aligned} S_n(x) &= \left(\frac{1 - \lambda}{e^t - \lambda} \right)^a x \left(\frac{e^{bt} - 1}{t} \right)^n x^{n-1} \\ &= \left(\frac{1 - \lambda}{e^t - \lambda} \right)^a x \sum_{k=0}^{n-1} \frac{n! b^{k+n}}{(k+n)!} S_2(k+n, n) x^{n-k-1} (n-1)_k \\ &= \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{\binom{k+n}{n}} S_2(k+n, n) b^{k+n} H_{n-k}^{(a)}(x|\lambda). \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.2 For $n \geq 1$, let $S_n(x) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda} \right)^a, \frac{t^2}{e^{bt}-1} \right)$, $b \neq 0$. Then we have

$$S_n(x) = \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{\binom{k+n}{n}} S_2(k+n, n) b^{k+n} H_{n-k}^{(a)}(x|\lambda).$$

Let

$$S_n(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^a, \frac{t^2 e^{bt}}{e^{ct} - 1} \right), \quad c \neq 0. \tag{2.5}$$

From Theorem 2.1, we can derive

$$\begin{aligned} S_n(x) &= \left(\frac{t}{e^t - 1} \right)^a x \left(\frac{e^{ct} - 1}{t e^{bt}} \right)^n x^{n-1} \\ &= \left(\frac{t}{e^t - 1} \right)^a x e^{-nbt} \sum_{l=0}^{\infty} \frac{n! S_2(l+n, n)}{(l+n)!} c^{l+n} t^l x^{n-1} \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{t}{e^t - 1}\right)^a x \sum_{l=0}^{n-1} \frac{\binom{n-1}{l}}{\binom{l+n}{l}} S_2(l+n, n) c^{n+l} (x-nb)^{n-1-l} \\
 &= \left(\frac{t}{e^t - 1}\right)^a x \sum_{l=0}^{n-1} \sum_{j=0}^{n-1-l} \frac{\binom{n-1}{l}}{\binom{l+n}{l}} \binom{n-1-l}{j} S_2(l+n, n) c^{n+l} (-nb)^j x^{n-1-l-j} \\
 &= \sum_{l=0}^{n-1} \sum_{j=0}^{n-1-l} \frac{\binom{n-1}{l}}{\binom{l+n}{l}} \binom{n-1-l}{j} S_2(l+n, n) c^{n+l} (-nb)^j B_{n-l-j}^{(a)}(x). \tag{2.6}
 \end{aligned}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.3 For $n \geq 1$, let $S_n(x) \sim ((\frac{e^t-1}{t})^a, \frac{t^2 e^{bt}}{e^{ct}-1})$, $c \neq 0$. Then we have

$$S_n(x) = \sum_{l=0}^{n-1} \sum_{j=0}^{n-1-l} \frac{\binom{n-1}{l}}{\binom{l+n}{l}} \binom{n-1-l}{j} S_2(l+n, n) c^{n+l} (-nb)^j B_{n-l-j}^{(a)}(x).$$

Let us take the following Sheffer sequence:

$$S_n(x) \sim \left(\left(\frac{e^t + 1}{2}\right)^\alpha, \frac{t^2}{\log(1+t)} \right). \tag{2.7}$$

By Theorem 2.1 and (2.7), we get

$$\begin{aligned}
 S_n(x) &= \left(\frac{2}{e^t + 1}\right)^\alpha x \left(\frac{\log(1+t)}{t}\right)^n x^{n-1} = \left(\frac{2}{e^t + 1}\right)^\alpha x \sum_{l=0}^{\infty} \frac{N_l^{(n)}}{l!} t^l x^{n-1} \\
 &= \left(\frac{2}{e^t + 1}\right)^\alpha x \sum_{l=0}^{n-1} \binom{n-1}{l} N_l^{(n)} x^{n-l-1} \\
 &= \sum_{l=0}^{n-1} \binom{n-1}{l} N_l^{(n)} E_{n-l}^{(\alpha)}(x), \tag{2.8}
 \end{aligned}$$

where $E_n^{(\alpha)}(x)$ are the n th Euler polynomials of order α which is defined by the generating function to be

$$\left(\frac{2}{e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}.$$

Therefore, by (2.8), we obtain the following theorem.

Theorem 2.4 For $n \geq 1$, let $S_n(x) \sim ((\frac{e^t+1}{2})^\alpha, \frac{t^2}{\log(1+t)})$. Then we have

$$S_n(x) = \sum_{l=0}^{n-1} \binom{n-1}{l} N_l^{(n)} E_{n-l}^{(\alpha)}(x).$$

As is known, we note that

$$\left(\frac{\log(1+t)}{t}\right)^n = n \sum_{l=0}^{\infty} \frac{B_l^{(n+l)}}{n+l} \frac{t^l}{l!}. \tag{2.9}$$

Thus, by Theorem 2.1 and (2.9), we get

$$\begin{aligned}
 S_n(x) &= \left(\frac{2}{e^t + 1}\right)^\alpha x \left(\frac{\log(1+t)}{t}\right)^n x^{n-1} \\
 &= \left(\frac{2}{e^t + 1}\right)^\alpha x n \sum_{l=0}^{n-1} \frac{B_l^{(n+l)}}{n+l} \binom{n-1}{l} x^{n-1-l} \\
 &= n \sum_{l=0}^{n-1} \frac{B_l^{(n+l)}}{n+l} \binom{n-1}{l} E_{n-l}^{(\alpha)}(x).
 \end{aligned} \tag{2.10}$$

Therefore, by Theorem 2.4 and (2.10), we obtain the following corollary.

Corollary 2.5 For $n \geq 1$, and $0 \leq l \leq n - 1$, we have

$$\frac{N_l^{(n)}}{n} = \frac{B_l^{(n+l)}}{n+l}.$$

Remark Let $S_n(x) \sim ((\frac{e^t-1}{t})^\alpha, \log(1+t))$. Then, by Theorem 2.1, we get

$$\begin{aligned}
 S_n(x) &= \left(\frac{t}{e^t - 1}\right)^\alpha x \left(\frac{t}{\log(1+t)}\right)^n x^{n-1} \\
 &= \left(\frac{t}{e^t - 1}\right)^\alpha x \sum_{l=0}^{n-1} \binom{n-1}{l} N_l^{(-n)} x^{n-1-l} \\
 &= \sum_{l=0}^{n-1} \binom{n-1}{l} N_l^{(-n)} B_{n-l}^{(\alpha)}(x).
 \end{aligned} \tag{2.11}$$

Let us assume that

$$S_n(x) \sim \left(\left(\frac{e^t - \lambda}{1 - \lambda}\right)^\alpha, \frac{\log(1+t)}{(1+t)^c}\right) \quad (c \neq 0). \tag{2.12}$$

Then, by Theorem 2.1 and (2.12), we get

$$\begin{aligned}
 S_n(x) &= \left(\frac{1-\lambda}{e^t - \lambda}\right)^\alpha x \left(\frac{t(1+t)^c}{\log(1+t)}\right)^n x^{n-1} \\
 &= \left(\frac{1-\lambda}{e^t - \lambda}\right)^\alpha x \sum_{l=0}^{n-1} B_l^{(l-n+1)}(cn+1) \frac{(n-1)_l}{l!} x^{n-1-l} \\
 &= \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(l-n+1)}(cn+1) \left(\frac{1-\lambda}{e^t - \lambda}\right)^\alpha x^{n-l} \\
 &= \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(l-n+1)}(cn+1) H_{n-l}^{(\alpha)}(x|\lambda).
 \end{aligned} \tag{2.13}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.6 For $n \geq 1$, let $S_n(x) \sim ((\frac{e^t - \lambda}{1 - \lambda})^\alpha, \frac{\log(1+t)}{(1+t)^c})$, $c \neq 0$. Then we have

$$S_n(x) = \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(l-n+1)}(cn+1) H_{n-l}^{(\alpha)}(x|\lambda).$$

As is well known, the *Bernoulli polynomials of the second kind* are defined by the generating function to be

$$\frac{t(1+t)^x}{\log(1+t)} = \sum_{l=0}^{\infty} \frac{b_l(x)}{l!} t^l \quad (\text{see [11, 12]}). \tag{2.14}$$

Thus, by (1.10) and (2.14), we get

$$\left(\frac{t(1+t)^c}{\log(1+t)}\right)^n = \sum_{l=0}^{\infty} \left(\sum_{l_1+\dots+l_n=l} \binom{l}{l_1, \dots, l_n}\right) b_{l_1}(c) \dots b_{l_n}(c) \frac{t^l}{l!}. \tag{2.15}$$

By Theorem 2.1, (2.12) and (2.15), we get

$$\begin{aligned} S_n(x) &= \left(\frac{1-\lambda}{e^t - \lambda}\right)^\alpha x \sum_{l=0}^{n-1} \left(\sum_{l_1+\dots+l_n=l} \binom{l}{l_1, \dots, l_n}\right) \left(\prod_{i=1}^n b_{l_i}(c)\right) \binom{n-1}{l} x^{n-1-l} \\ &= \sum_{l=0}^{n-1} \left(\sum_{l_1+\dots+l_n=l} \binom{l}{l_1, \dots, l_n}\right) \left(\prod_{i=1}^n b_{l_i}(c)\right) \binom{n-1}{l} \left(\frac{1-\lambda}{e^t - \lambda}\right)^\alpha x^{n-l} \\ &= \sum_{l=0}^{n-1} \left(\sum_{l_1+\dots+l_n=l} \binom{l}{l_1, \dots, l_n}\right) \left(\prod_{i=1}^n b_{l_i}(c)\right) \binom{n-1}{l} H_{n-l}^{(\alpha)}(x|\lambda). \end{aligned} \tag{2.16}$$

Therefore, by Theorem 2.6 and (2.16), we obtain the following theorem.

Theorem 2.7 For $n \geq 1$, $0 \leq l \leq n-1$, we have

$$\sum_{l_1+\dots+l_n=l} \binom{l}{l_1, \dots, l_n} \left(\prod_{i=1}^n b_{l_i}(c)\right) = B_l^{(l-n+1)}(cn+1) \quad (c \neq 0).$$

Remark From (1.2), we note that

$$\left(\frac{t(1+t)^c}{\log(1+t)}\right)^n x^{n-1} = \sum_{l=0}^{n-1} \binom{n-1}{l} N_l^{(-n)}(cn) x^{n-1-l}, \tag{2.17}$$

where $c \neq 0$. By Theorem 2.1, (2.12) and (2.17), we get

$$\begin{aligned} S_n(x) &= \left(\frac{1-\lambda}{e^t - \lambda}\right)^\alpha x \left(\frac{t(1+t)^c}{\log(1+t)}\right)^n x^{n-1} \\ &= \sum_{l=0}^{n-1} \binom{n-1}{l} N_l^{(-n)}(cn) H_{n-l}^{(\alpha)}(x|\lambda). \end{aligned} \tag{2.18}$$

From (2.16) and (2.18), we can derive the following identity:

$$N_l^{(-n)}(cn) = \sum_{l_1+\dots+l_n=l} \binom{l}{l_1, \dots, l_n} \left(\prod_{i=1}^n b_{l_i}(c) \right), \tag{2.19}$$

where $n \geq 1$, $0 \leq l \leq n - 1$ and $c \neq 0$. Let

$$S_n(x) \sim \left(\left(\frac{e^{(\lambda-1)t} - \lambda}{1 - \lambda} \right)^\alpha, \frac{t^2(1+t)^c}{\log(1+t)} \right), \quad c \neq 0. \tag{2.20}$$

From Theorem 2.1 and (2.20), we note that

$$\begin{aligned} S_n(x) &= \left(\frac{1 - \lambda}{e^{(\lambda-1)t} - \lambda} \right)^\alpha x \left(\frac{\log(1+t)}{t(1+t)^c} \right)^n x^{n-1} \\ &= \left(\frac{1 - \lambda}{e^{(\lambda-1)t} - \lambda} \right)^\alpha x \sum_{l=0}^{n-1} \binom{n-1}{l} N_l^{(n)}(-cn) x^{n-1-l} \\ &= \sum_{l=0}^{n-1} \binom{n-1}{l} N_l^{(n)}(-cn) A_{n-l}^{(\alpha)}(x|\lambda). \end{aligned} \tag{2.21}$$

Therefore, by (2.21), we obtain the following proposition.

Proposition 2.8 For $n \geq 1$, let $S_n(x) \sim \left(\left(\frac{e^{(\lambda-1)t} - \lambda}{1 - \lambda} \right)^\alpha, \frac{t^2(1+t)^c}{\log(1+t)} \right)$, $c \neq 0$. Then we have

$$S_n(x) = \sum_{l=0}^{n-1} \binom{n-1}{l} N_l^{(n)}(-nc) A_{n-l}^{(\alpha)}(x|\lambda).$$

Now we observe that

$$\begin{aligned} \left(\frac{\log(1+t)}{t(1+t)^c} \right)^n &= (1+t)^{-nc} \left(\frac{\log(1+t)}{t} \right)^n \\ &= (1+t)^{-nc} \left(\sum_{k=0}^{\infty} \frac{n! S_1(k+n, n)}{(k+n)!} t^k \right) \\ &= \left(\sum_{m=0}^{\infty} \binom{-nc}{m} t^m \right) \left(\sum_{k=0}^{\infty} \frac{n! S_1(k+n, n)}{(k+n)!} t^k \right) \\ &= \sum_{l=0}^{\infty} \left\{ \sum_{k=0}^l \frac{n! S_1(k+n, n)}{(k+n)!} \binom{-nc}{l-k} \right\} t^l. \end{aligned} \tag{2.22}$$

By Theorem 2.1, (2.20) and (2.22), we get

$$\begin{aligned} S_n(x) &= \left(\frac{1 - \lambda}{e^{(\lambda-1)t} - \lambda} \right)^\alpha x \left(\frac{\log(1+t)}{t(1+t)^c} \right)^n x^{n-1} \\ &= \sum_{l=0}^{n-1} \binom{n-1}{l} l! \left\{ \sum_{k=0}^l \frac{n!}{(k+n)!} S_1(n+k, n) \binom{-nc}{l-k} \right\} A_{n-l}^{(\alpha)}(x|\lambda). \end{aligned} \tag{2.23}$$

Therefore, by Proposition 2.8 and (2.23), we obtain the following theorem.

Theorem 2.9 For $n \geq 1$, $0 \leq l \leq n - 1$ and $c \neq 0$, we have

$$N_l^{(n)}(-cn) = l! \sum_{k=0}^l \frac{n!}{(n+k)!} S_1(k+n, n) \binom{-nc}{l-k}.$$

Remark It is easy to show that

$$(\log(1+t))^n = \sum_{l=0}^{\infty} \frac{n!}{(l+n)!} S_1(l+n, n) t^{l+n}. \tag{2.24}$$

By Theorem 2.1, (2.7) and (2.24), we get

$$\begin{aligned} S_n(x) &= \left(\frac{2}{e^t+1}\right)^\alpha x \left(\frac{\log(1+t)}{t}\right)^n x^{n-1} \\ &= \left(\frac{2}{e^t+1}\right)^\alpha x \sum_{l=0}^{n-1} \frac{n!l!}{(l+n)!} \binom{n-1}{l} S_1(l+n, n) x^{n-1-l} \\ &= \sum_{l=0}^{n-1} \frac{\binom{n-1}{l}}{\binom{l+n}{n}} S_2(l+n, n) E_{n-l}^{(\alpha)}(x). \end{aligned} \tag{2.25}$$

From Theorem 2.4 and (2.25), we can derive the following identity:

$$N_l^{(n)} = \frac{S_2(l+n, n)}{\binom{l+n}{n}}, \quad \text{where } n \geq 1, 0 \leq l \leq n - 1. \tag{2.26}$$

Let us consider the following Sheffer sequence:

$$S_n(x) \sim \left(\left(\frac{e^{(\lambda-1)t} - \lambda}{1 - \lambda} \right)^\alpha, \frac{t}{e^{ct}(1+bt)^m} \right), \quad b, c \neq 0, m \in \mathbb{Z}_+. \tag{2.27}$$

By Theorem 2.1 and (2.27), we get

$$\begin{aligned} S_n(x) &= \left(\frac{1-\lambda}{e^{(\lambda-1)t} - \lambda}\right)^\alpha x (e^{ct}(1+bt)^m)^n x^{n-1} \\ &= \left(\frac{1-\lambda}{e^{(\lambda-1)t} - \lambda}\right)^\alpha x e^{nct} (1+bt)^{mn} x^{n-1}. \end{aligned} \tag{2.28}$$

From (1.15) and (2.28), we can derive

$$\begin{aligned} S_n(x) &= \left(\frac{1-\lambda}{e^{(\lambda-1)t} - \lambda}\right)^\alpha x (-1)^{mn} \sum_{l=0}^{n-1} C_{mn} \left(l; -\frac{nc}{b} \right) (nc)^l \binom{n-1}{l} x^{n-1-l} \\ &= (-1)^{mn} \sum_{l=0}^{n-1} C_{mn} \left(l; -\frac{nc}{b} \right) (nc)^l \binom{n-1}{l} A_{n-l}^{(\alpha)}(x|\lambda). \end{aligned} \tag{2.29}$$

Therefore, by (2.29), we obtain the following theorem.

Theorem 2.10 For $n \geq 1$, let $S_n(x) \sim ((\frac{e^{(\lambda-1)t}-\lambda}{1-\lambda})^\alpha, \frac{t}{e^{ct}(1+bt)^m})$, where $m \in \mathbb{Z}_+$, $b \neq 0$ and $c \neq 0$. Then we have

$$S_n(x) = (-1)^{mn} \sum_{l=0}^{n-1} C_{mn} \left(l; -\frac{nc}{b} \right) (nc)^l \binom{n-1}{l} A_{n-l}^{(\alpha)}(x|\lambda).$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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