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Some Identities of Frobenius-Type Eulerian Polynomials Arising from Umbral Calculus

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Abstract. In this paper, we study some properties of umbral calculus related with Frobenius-type Eulerian polynomials. From our results of this paper, we can derive many interesting identities with respect to Frobenius-type Eulerian polynomials.

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1. Introduction

Let $\mathbb{C}$ be the complex number field. Throughout this paper, we assume that $\lambda \in \mathbb{C}$ with $\lambda \neq 1$. The *Frobenius-type Eulerian polynomials* of order r are given by

$$\left(\frac{1-\lambda}{e^{(\lambda-1)t}-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} A_n^{(r)}(x|\lambda)\frac{t^n}{n!}, \text{ (see [1,7,8])}. \tag{1.1}$$

In the special case, $x = 0$, $A_n^{(r)}(0|\lambda) = A_n^{(r)}(\lambda)$ are called the *Frobenius-type Eulerian numbers*. By (1.1), we easily get

$$A_n^{(r)}(x|\lambda) = \sum_{k=0}^{\infty} \binom{n}{k} A_k^{(r)}(\lambda)x^{n-k}, \text{ (see [1,3,9,11])}. \tag{1.2}$$

Let $\mathbb{P}$ be the algebra of polynomials in the single variable x over $\mathbb{C}$ and $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. The action of the linear functional on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$. The action $\langle L|p(x)\rangle$ satisfies $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$ and $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant (see [10, 13, 14]).

Let $\mathcal{F}$ denote the algebra of all formal power series in the single variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \;\middle|\; a_k \in \mathbb{C} \right\}. \tag{1.3}$$

For $f(t) \in \mathcal{F}$, we define a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \; (n \geq 0) \text{ (see [10,13,14])}. \tag{1.4}$$

By (1.3) and (1.4), we get

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k} \; (n,k \geq 0), \tag{1.5}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!}t^k$. Then, by (1.5), we easily get $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$ and $f_L(t) = L$, $(n \geq 0)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ is thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra*. The umbral calculus is the study of umbral algebra (see [5, 10, 13, 14]).

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 1$, then $f(t)$ is called a *delta series*. If $o(f(t)) = 0$, then $f(t)$ is called an *invertible series* (see [5, 10, 13, 14]). Let $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\left\langle g(t)f(t)^k|S_n(x)\right\rangle = n!\delta_{n,k}$, where $n, k \geq 0$. The sequence $S_n(x)$ is called *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$ (see [10, 13, 14]). From (1.5), we note that $\langle e^{yt} | \, p(x) \rangle = p(y)$. Let us assume that$f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\left\langle f(t) | x^k \right\rangle}{k!} t^k, \; p(x) = \sum_{k=0}^{\infty} \frac{\left\langle t^k | p(x) \right\rangle}{k!} x^k \text{ (see [5,10,13,14])}. \tag{1.6}$$

From (1.6), we note that

$$p^{(k)}(0) = \left\langle t^k | p(x) \right\rangle, \; \left\langle 1 \left| p^{(k)}(x) \right. \right\rangle = p^{(k)}(0). \tag{1.7}$$

By (1.7), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \; (k \geq 0), \; \text{(see [5,10,13,14])}. \tag{1.8}$$

Let $S_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \text{ for all } y \in \mathbb{C}, \tag{1.9}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [5, 10, 13, 14]).

The purpose of this paper is to study some properties of Frobenius-type Eulerian polynomials arising from umbral calculus. By using our results of this paper, we can obtain many interesting identities of Frobenius-type Eulerian polynomials.

2. Frobenius-type Eulerian polynomails and umbral calculus

In this section, we assume that $r \in \mathbb{Z}$. From (1.1) and (1.9), we note that

$$A_n^{(r)}(x|\lambda) \sim \left(\left(\frac{e^{t(\lambda-1)} - \lambda}{1-\lambda} \right)^r, t \right). \tag{2.1}$$

Let $\mathbb{P}_n = \{p(x) \in \mathbb{C}[x] \mid \deg p(x) \leq n\}$. Then $\mathbb{P}_n$ is the $(n+1)$-dimensional vector space over $\mathbb{C}$. It is easy to show that $\left\{ A_0^{(r)}(x|\lambda), A_1^{(r)}(x|\lambda), \ldots, A_n^{(r)}(x|\lambda) \right\}$ is a good basis for $\mathbb{P}_n$ (see [1-17]).

For $p(x) \in \mathbb{P}_n$, let us assume that

$$p(x) = \sum_{k=0}^{n} c_k A_k^{(r)}(x|\lambda), \; (n \geq 0). \tag{2.2}$$

Then, by (2.1) and (2.2), we get

$$\begin{aligned} \left\langle \left(\frac{e^{t(\lambda-1)} - \lambda}{1-\lambda} \right)^r t^k \middle| p(x) \right\rangle &= \sum_{l=0}^{n} c_l \left\langle \left(\frac{e^{t(\lambda-1)} - \lambda}{1-\lambda} \right)^r t^k \middle| A_l^{(r)}(x|\lambda) \right\rangle \\ &= \sum_{l=0}^{n} c_l l! \delta_{l,k} = k! c_k. \end{aligned} \tag{2.3}$$

Thus, from (2.3), we have

$$\begin{aligned} c_k =& \frac{1}{k!}\left\langle \left(\frac{e^{t(\lambda-1)}-\lambda}{1-\lambda}\right)^r t^k \middle| p(x)\right\rangle = \frac{1}{k!}\left\langle \left(\frac{e^{t(\lambda-1)}-\lambda}{1-\lambda}\right)^r \middle| D^k p(x)\right\rangle \\ =& \frac{1}{k!(1-\lambda)^r}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}\left\langle e^{j(\lambda-1)t}\middle| D^k p(x)\right\rangle \\ =& \frac{1}{k!(1-\lambda)^r}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}\left\langle t^0\middle| D^k p(x+j(\lambda-1))\right\rangle . \end{aligned} \tag{2.4}$$

Therefore, by (2.2) and (2.4), we obtain the following theorem.

Theorem 2.1. *For $r \in \mathbb{Z}_+$, $p(x) \in \mathbb{P}_n$, let*

$$p(x) = \sum_{k=0}^{n} c_k A_k^{(r)}(x|\lambda).$$

Then we have

$$c_k = \frac{1}{k!(1-\lambda)^r}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}D^k p(j(\lambda-1)),$$

where $Dp(x) = \frac{dp(x)}{dx}$.

By Theorem 2.1, we get

$$p(x) = \frac{1}{(1-\lambda)^r}\sum_{k=0}^{n}\left\{\sum_{j=0}^{r}\frac{1}{k!}\binom{r}{j}(-\lambda)^{r-j}D^k p(j(\lambda-1))\right\}A_k^{(r)}(x|\lambda). \tag{2.5}$$

Let us define λ-*difference operator* Δ_λ as follows:

$$\Delta_\lambda f(x) = f(x+\lambda-1) - \lambda f(x), \tag{2.6}$$

and

$$T_\lambda(f) = \frac{1}{1-\lambda}\Delta_\lambda f(x) = \frac{1}{1-\lambda}\left\{f(x+\lambda-1)-\lambda f(x)\right\}. \tag{2.7}$$

From (2.7), we have

$$T_\lambda\left(A_n^{(r)}(x|\lambda)\right) = \frac{1}{1-\lambda}\left\{A_n^{(r)}(x+\lambda-1|\lambda) - \lambda A_n^{(r)}(x|\lambda)\right\}. \tag{2.8}$$

By (1.1), we easily get

$$\begin{aligned} &\sum_{n=0}^{\infty}\left\{A_n^{(r)}(x+\lambda-1|\lambda)-\lambda A_n^{(r)}(x|\lambda)\right\}\frac{t^n}{n!} \\ =& \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^r e^{(x+\lambda-1)t} - \lambda\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^r e^{xt} \\ =& (1-\lambda)\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{r-1} e^{xt} = (1-\lambda)\sum_{n=0}^{\infty}A_n^{(r-1)}(x|\lambda)\frac{t^n}{n!}. \end{aligned} \tag{2.9}$$

Thus, by (2.9), we see that

$$T_\lambda\left(A_n^{(r)}(x|\lambda)\right)=\frac{1}{1-\lambda}\left\{A_n^{(r)}(x+\lambda-1|\lambda)-\lambda A_n^{(r)}(x|\lambda)\right\}=A_n^{(r-1)}(x|\lambda). \tag{2.10}$$

From (2.10), we have

$$T_\lambda^r\left(A_n^{(r)}(x|\lambda)\right)=T_\lambda^{r-1}\left(A_n^{(r-1)}(x|\lambda)\right)=\cdots=A_n^{(0)}(x|\lambda)=x^n. \tag{2.11}$$

By (2.11), we get

$$T_\lambda^r(x^n)=T_\lambda^r\left(A_n^{(0)}(x|\lambda)\right)=A_n^{(-r)}(x|\lambda)=T_\lambda^{2r}\left(A_n^{(r)}(x|\lambda)\right). \tag{2.12}$$

For $s\in\mathbb{Z}_+$, from (2.12), we note that

$$T_\lambda^s\left(A_n^{(r)}(x|\lambda)\right)=A_n^{(r-s)}(x|\lambda). \tag{2.13}$$

On the other hand, by (2.13), we get

$$\begin{aligned}
&T_\lambda^s\left(A_n^{(r)}(x|\lambda)\right)=\left(\frac{e^{t(\lambda-1)}-\lambda}{1-\lambda}\right)^s\left(A_n^{(r)}(x|\lambda)\right)\\
=&\frac{1}{(1-\lambda)^s}\left((1-\lambda)+\sum_{k=1}^\infty\frac{(\lambda-1)^k t^k}{k!}\right)^s A_n^{(r)}(x|\lambda)\\
=&\sum_{m=0}^s\frac{\binom{s}{m}}{(1-\lambda)^m}\sum_{l=m}^\infty\left(\sum_{k_1+\cdots+k_m=l}\frac{1}{k_1!\cdots k_m!}\right)t^l(\lambda-1)^l A_n^{(r)}(x|\lambda)\\
=&\sum_{m=0}^s\frac{\binom{s}{m}}{(1-\lambda)^m}\sum_{l=m}^\infty\frac{(\lambda-1)^l}{l!}\sum_{k_1+\cdots+k_m=l,\ k_i\geq1}\binom{l}{k_1,\ldots,k_m}D^l A_n^{(r)}(x|\lambda)\\
=&\sum_{m=0}^{\min\{s,n\}}\frac{\binom{s}{m}}{(1-\lambda)^m}\sum_{l=m}^n\binom{n}{l}(\lambda-1)^l\sum_{k_1+\cdots+k_m=l,\ k_i\geq1}\binom{l}{k_1,\ldots,k_m}A_{n-l}^{(r)}(x|\lambda)\\
=&\sum_{l=0}^{\min\{s,n\}}\binom{n}{l}\sum_{m=0}^{l}\frac{\binom{s}{m}}{(1-\lambda)^{m-l}}\sum_{k_1+\cdots+k_m=l,\ k_i\geq1}\binom{l}{k_1,\ldots,k_m}A_{n-l}^{(r)}(x|\lambda)\\
&+\sum_{l=\min\{s,n\}+1}^{n}\binom{n}{l}\sum_{m=0}^{\min\{s,n\}}\frac{\binom{s}{m}}{(1-\lambda)^{m-l}}\sum_{k_1+\cdots+k_m=l,\ k_i\geq1}\binom{l}{k_1,\ldots,k_m}A_{n-l}^{(r)}(x|\lambda).
\end{aligned} \tag{2.14}$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.2. *For $r, s \in \mathbb{Z}_+$, we have*

$$A_n^{(r-s)}(x|\lambda) = \sum_{l=0}^{\min\{s,n\}} \sum_{m=0}^{l} \sum_{k_1+\cdots+k_m=l,\ k_i\geq 1} \frac{\binom{n}{l}\binom{s}{m}\binom{l}{k_1,\dots,k_m}}{(1-\lambda)^{m-l}} A_{n-l}^{(r)}(x|\lambda)$$
$$+ \sum_{l=\min\{s,n\}+1}^{n} \sum_{m=0}^{\min\{s,n\}} \sum_{k_1+\cdots+k_m=l,\ k_i\geq 1} \frac{\binom{n}{l}\binom{s}{m}\binom{l}{k_1,\dots,k_m}}{(1-\lambda)^{m-l}} A_{n-l}^{(r)}(x|\lambda).$$

Let us take $r = s$. Then, by Theorem 2.2, we get

$$x^n = \sum_{l=0}^{\min\{r,n\}} \sum_{m=0}^{l} \sum_{k_1+\cdots+k_m=l,\ k_i\geq 1} \frac{\binom{l}{k_1,\dots,k_m}\binom{n}{l}\binom{r}{m}}{(1-\lambda)^{m-l}} A_{n-l}^{(r)}(x|\lambda)$$
$$+ \sum_{l=\min\{r,n\}+1}^{n} \sum_{m=0}^{\min\{r,n\}} \sum_{k_1+\cdots+k_m=l,\ k_i\geq 1} \frac{\binom{l}{k_1,\dots,k_m}\binom{r}{m}\binom{n}{l}}{(1-\lambda)^{m-l}} A_{n-l}^{(r)}(x|\lambda).$$

From (2.6), we can derive the following equation:

$$\Delta_\lambda^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} f((\lambda-1)k). \tag{2.15}$$

Let $s = 2r$. Then, by (2.12) and Theorem 2.2, we get

$$\begin{aligned} T_\lambda^r(x^n) =& A_n^{(-r)}(x|\lambda) = T_\lambda^{2r}\left(A_n^{(r)}(x|\lambda)\right) \\ =& \sum_{l=0}^{\min\{2r,n\}} \sum_{m=0}^{l} \sum_{k_1+\cdots+k_m=l,\ k_i\geq 1} \frac{\binom{l}{k_1,\dots,k_m}\binom{n}{l}\binom{2r}{m}}{(1-\lambda)^{m-l}} A_{n-l}^{(r)}(x|\lambda) \\ &+ \sum_{l=\min\{2r,n\}+1}^{n} \sum_{m=0}^{\min\{2r,n\}} \sum_{k_1+\cdots+k_m=l,\ k_i\geq 1} \frac{\binom{l}{k_1,\dots,k_m}\binom{n}{l}\binom{2r}{m}}{(1-\lambda)^{m-l}} A_{n-l}^{(r)}(x|\lambda). \end{aligned} \tag{2.16}$$

By (2.7), we easily get

$$T_\lambda^r(x^n) = \frac{\Delta_\lambda^r x^n}{(1-\lambda)^r} = \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} (x+(\lambda-1)j)^n. \tag{2.17}$$

For $n, k \geq 0$, let us define λ-*analogue of the Stirling number of the second kind* as follows:

$$S_2(n,k|\lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-\lambda)^{k-j} j^n. \tag{2.18}$$

From (2.18), we note that $S_2(n,k|1) = S_2(n,k)$ where $S_2(n,k)$ is the Striling number of the second kind. Therefore, by (2.16), (2.17) and (2.18), we obtain the following theorem.

Theorem 2.3. *For $n, k \geq 0$, we have*

$$\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}\left(x+(\lambda-1)j\right)^{n}$$
$$=\sum_{l=0}^{\min\{2r,n\}}\sum_{m=0}^{l}\sum_{k_1+\cdots+k_m=l,\ k_i\geq 1}\frac{\binom{2r}{m}\binom{l}{k_1,\ldots,k_m}\binom{n}{l}}{(1-\lambda)^{m-l}}A_{n-l}^{(r)}(x|\lambda)$$
$$+\sum_{l=\min\{2r,n\}+1}^{n}\sum_{m=0}^{\min\{2r,n\}}\sum_{k_1+\cdots+k_m=l,\ k_i\geq 1}\frac{\binom{l}{k_1,\ldots,k_m}\binom{n}{l}\binom{2r}{m}}{(1-\lambda)^{m-l}}A_{n-l}^{(r)}(x|\lambda).$$

Moreover,

$$(\lambda-1)^{n}S_2(n,r|\lambda)$$
$$=\frac{1}{r!}\sum_{l=0}^{\min\{2r,n\}}\sum_{m=0}^{l}\sum_{k_1+\cdots+k_m=l,\ k_i\geq 1}\frac{\binom{l}{k_1,\ldots,k_m}\binom{n}{l}\binom{2r}{m}}{(1-\lambda)^{m-l}}A_{n-l}^{(r)}(\lambda)$$
$$+\frac{1}{r!}\sum_{l=\min\{2r,n\}+1}^{n}\sum_{m=0}^{\min\{2r,n\}}\sum_{k_1+\cdots+k_m=l,\ k_i\geq 1}\frac{\binom{l}{k_1,\ldots,k_m}\binom{n}{l}\binom{2r}{m}}{(1-\lambda)^{m-l}}A_{n-l}^{(r)}(\lambda).$$

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