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New Changhee q-Euler numbers and polynomials associated with p-adic q-integrals

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Abstract

Using non-archimedean q-integrals on $\mathbb{Z}_p$ defined in [T. Kim, On a q-analogue of the p-adic log gamma functions and related integrals, J. Number Theory 76 (1999) 320–329; T. Kim, q-Volkenborn integration, Russ. J. Math. Phys. 9 (2002) 288–299], we define new Changhee q-Euler polynomials and numbers which are different from those of Kim [T. Kim, p-adic q-integrals associated with the Changhee–Barnes' q-Bernoulli polynomials, Integral Transforms Spec. Funct. 15 (2004) 415–420] and Carlitz [L. Carlitz, q-Bernoulli and Eulerian numbers, Trans. Amer. Math. Soc. 76 (1954) 332–350]. We define generating functions of multiple q-Euler numbers and polynomials. Furthermore we construct a multivariate Hurwitz type zeta function which interpolates the multivariate q-Euler numbers or polynomials at negative integers.

Keywords: p-Adic q-Integral; q-Euler numbers and polynomials

1. Introduction

Let p be a fixed odd prime in this paper. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ denote the ring of rational integers, the ring of p-adic integers, the field of p-adic numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\nu_p(p)$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one speaks of a q-extension, q can be regarded as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$; it is always clear from the context. If $q \in \mathbb{C}$, then one usually assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then one usually assumes that $|q-1|_p < p^{-\frac{1}{p-1}}$, and hence $q^x = \exp(x \log q)$ for $x \in \mathbb{Z}_p$. In this paper, we use the notation

$$[x]_q = \frac{1-q^x}{1-q}, \qquad (a:q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad \text{cf. [3–8]}.$$

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Note that $\lim_{q\to 1}[x]_q = x$ for any x with $|x|_p \le 1$ in the p-adic case. For a fixed positive integer d with $(p, d) = 1$, set

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z},$$
$$X_1 = \mathbb{Z}_p, \qquad X^* = \bigcup_{\substack{0<a<dp\\(a,p)=1}} a + dp\mathbb{Z}_p,$$
$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a (\bmod\, dp^N)\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$ (cf. [10,11]). We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x, y) = \frac{f(x)-f(y)}{x-y}$ have a limit $f'(a)$ as $(x, y) \to (a, a)$. For $f \in UD(\mathbb{Z}_p)$, let us begin with the expression

$$\frac{1}{[p^N]_q}\sum_{0\le j<p^N} q^j f(j) = \sum_{0\le j<p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p), \quad \text{cf. [4–6,8]},$$

which represents a q-analogue of Riemann sums for f. The integral of f on $\mathbb{Z}_p$ is defined as the limit of those sums (as $n \to \infty$) if this limit exists. The p-adic q-integral of a function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_X f(x)\mathrm{d}\mu_q(x) = \int_{X_d} f(x)\mathrm{d}\mu_q(x) = \lim_{N\to\infty}\frac{1}{[dp^N]_q}\sum_{x=0}^{dp^N-1} f(x)q^x.$$

Recently, many mathematicians studied Bernoulli and Euler numbers (see [1–24]). Using non-archimedean q-integrals on $\mathbb{Z}_p$ defined in [15,16], we define new Changhee q-Euler polynomials and numbers which are different from those of Kim [7] and Carlitz [2]. We define generating functions of multiple q-Euler numbers and polynomials. Furthermore we construct a multivariate Hurwitz type zeta function which interpolates the multivariate q-Euler numbers or polynomials at negative integers.

2. Multivariate q-Euler numbers and polynomials

Using p-adic q-integrals on $\mathbb{Z}_p$, we now define new q-Euler polynomials as follows:

$$\int_{\mathbb{Z}_p} q^y (x + y)^n \mathrm{d}\mu_{-1}(y) = E_{n,q}(x).$$

Note that $\lim_{q\to 1} E_{n,q}(x) = E_n(x)$, where $E_n(x)$ are Euler polynomial which are defined by $\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$. In the case $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called new q-Euler numbers. And note that $\lim_{q\to 1} E_{n,q} = E_n$, where E_n are classical Euler numbers. Let $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r$ be positive integers. Then we consider a multivariate integral as follows:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{b_1x_1+\cdots+b_rx_r}(x + a_1x_1 + \cdots + a_rx_r)^n \mathrm{d}\mu_{-1}(x_1)\cdots\mathrm{d}\mu_{-1}(x_r)$$
$$= E_{n,q}{}^{(r)}(x|a_1, a_2, \ldots, a_r;\ b_1, b_2, \ldots, b_r). \tag{1}$$

Here $E_{n,q}{}^{(r)}(x|a_1, a_2, \ldots, a_r;\ b_1, b_2, \ldots, b_r)$ are called multivariate q-Euler polynomials of order r.

In the case $x = 0$ in (1), $E_{n,q}(0|a_1, a_2, \ldots, a_r;\ b_1, b_2, \ldots, b_r) = E_{n,q}(a_1, a_2, \ldots, a_r;\ b_1, b_2, \ldots, b_r)$ will be called multivariate q-Euler numbers of order r.

From (1), we derive the following generating function for multivariate q-Euler polynomials:

$$F_q^{(r)}(t, x|a_1, a_2, \ldots, a_r;\ b_1, b_2, \ldots, b_r)$$
$$= \sum_{n=0}^{\infty} E_{n,q}{}^{(r)}(x|a_1, a_2, \ldots, a_r;\ b_1, b_2, \ldots, b_r)\frac{t^n}{n!}$$

$$= \frac{2^r}{(q^{b_1}e^{a_1 t}+1)(q^{b_2}e^{a_2 t}+1)\cdots(q^{b_r}e^{a_r t}+1)}e^{xt}. \tag{2}$$

Note that ${E_{0,q}}^{(r)}(x|a_1,a_2,\ldots,a_r;\ b_1,b_2,\ldots,b_r) = \frac{2^r}{[2]_{q^{b_1}}[2]_{q^{b_2}}\cdots[2]_{q^{b_r}}}$.

Let χ be the Dirichlet character with conductor $f(=\text{odd}) \in \mathbb{N}$. Then we define the multivariate generalized q-Euler numbers attached to χ as follows:

$$\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)q^{b_1x_1+\cdots+b_rx_r}(a_1x_1+\cdots+a_rx_r)^n \mathrm{d}\mu_{-1}(x_1)\cdots\mathrm{d}\mu_{-1}(x_r)$$
$$= E_{n,\chi,q}(a_1,\ldots,a_r;\ b_1,\ldots,b_r). \tag{3}$$

From (3), we can derive generating functions for the multivariate generalized q-Euler numbers attached to χ,

$$F^{(r)}_{\chi,q}(t|a_1,\ldots,a_r;\ b_1,\ldots,b_r) = \sum_{n=0}^{\infty} E_{n,\chi,q}(a_1,\ldots,a_r;\ b_1,\ldots,b_r)\frac{t^n}{n!}$$
$$= \sum_{n_1,\ldots,n_r=0}^{f-1} \frac{2^r(-1)^{n_1+\cdots+n_r}q^{b_1n_1+\cdots+b_rn_r}e^{(a_1n_1+\cdots+a_rn_r)t}\chi(n_1)\cdots\chi(n_r)}{(q^{fb_1}e^{fa_1t}+1)\cdots(q^{fb_r}e^{fa_rt}+1)}, \tag{4}$$

where n is odd.

Let n be an odd positive integer. From the definition of ${E_{n,\chi,q}}^{(r)}(a_1,\ldots,a_r;\ b_1,\ldots,b_r)$ in (3), we have the following:

$$
\begin{aligned}
&{E_{n,\chi,q}}^{(r)}(a_1,\ldots,a_r;\ b_1,\ldots,b_r)\\
&= \lim_{N\to\infty} \sum_{n_1,\ldots,n_r=0}^{fp^N-1} \chi(n_1)\cdots\chi(n_r)(-1)^{n_1+\cdots+n_r}q^{b_1n_1+\cdots+b_rn_r}(a_1x_1+\cdots+a_rx_r)^n\\
&= \lim_{N\to\infty} \sum_{n_1,\ldots,n_r=0}^{f-1}\ \sum_{x_1,\ldots,x_r=0}^{p^N-1} \prod_{j=1}^{r}\chi(n_j+fn_j)(-1)^{\sum_{j=1}^{r}(n_j+fx_j)} q^{\sum_{j=1}^{r}b_j(n_j+x_jf_1)} \sum_{j=1}^{r}a_j(n_j+fx_j)^n\\
&= f^n \sum_{n_1,\ldots,n_r=0}^{f-1}(-1)^{\sum_{j=1}^{r}n_j}\prod_{j=1}^{r}\chi(n_j)q^{\sum_{j=1}^{r}b_jn_j}\\
&\quad\times \lim_{N\to\infty}\sum_{x_1,\ldots,x_r=0}^{p^N-1}(-1)^{\sum_{j=1}^{r}(x_j)} q^{f\sum_{j=1}^{r}b_jx_j}\left(\frac{\sum_{j=1}^{r}a_jn_j}{f}+\sum_{j=1}^{r}a_jx_j\right)^n\\
&= f^n \sum_{n_1,\ldots,n_r=0}^{f-1}(-1)^{\sum_{j=1}^{r}n_j}\prod_{j=1}^{r}\chi(n_j)q^{\sum_{j=1}^{r}b_jn_j}\\
&\quad\times \underbrace{\int_{\mathbb{Z}_r}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}} q^{f\sum_{j=1}^{r}b_jx_j}\left(\frac{\sum_{j=1}^{n}a_jn_j}{f}+\sum_{j=1}^{n}a_jx_j\right)^n \mathrm{d}\mu_{-1}(x_1)\cdots\mathrm{d}\mu_{-1}(x_r)\\
&= f^n \sum_{n_1,\ldots,n_r=0}^{f-1}(-1)^{\sum_{j=1}^{r}n_j}\prod_{j=1}^{r}\chi(n_j)q^{\sum_{j=1}^{r}b_jn_j}E_{n,q^f}\left(\frac{\sum_{j=1}^{n}a_jn_j}{f}\,\middle|\,a_1,\ldots,a_r;\ b_1,\ldots,b_r\right).
\end{aligned}
$$

Therefore we obtain:

Theorem 1. *Let $a_1, \dots, a_r, b_1, \dots, b_r$ be positive integers; then we have*

$$E_{n,\chi,q}{}^{(r)}(a_1, \dots, a_r;\ b_1, \dots, b_r) = \sum_{n_1,\dots,n_r=0}^{f-1} (-1)^{\sum_{j=1}^{r} n_j} \prod_{j=1}^{r} \chi(j) q^{\sum_{j=1}^{r} b_j n_j} f^n E_{n,q^f}\left(\left. \frac{\sum_{j=1}^{n} a_j n_j}{f} \right| a_1, \dots, a_r; b_1, \dots, b_r \right), \tag{5}$$

where f, n are odd positive integers.

3. Multivariate q-zeta functions in $\mathbb{C}$

In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let us assume that $a_1, \dots, a_r, b_1, \dots, b_r$ are positive integers.

The purpose of this section is to study a multivariate Hurwitz type zeta function which interpolates multivariate q-Euler polynomials of order r at negative integers. By (2), we easily see that

$$F_q^{(r)}(t, x \mid a_1, \dots, a_r;\ b_1, \dots, b_r) = 2^r \sum_{n_1,\dots,n_r=0}^{\infty} (-1)^{\sum_{j=1}^{r} n_j} q^{\sum_{j=1}^{r} b_j n_j} e^{(\sum_{j=1}^{r} a_j n_j + x)t}. \tag{6}$$

By taking derivatives of order k, on both sides of (6) we obtain the following:

Theorem 2. *Let k be positive odd integer and let $a_1, \dots, a_r, b_1, \dots, b_r$ be positive integers. Then we have*

$$E_{k,q}{}^{(r)}(x \mid a_1, \dots, a_r;\ b_1, \dots, b_r) = 2^r \sum_{n_1,\dots,n_r=0}^{\infty} (-1)^{\sum_{j=1}^{r} n_j} q^{\sum_{j=1}^{r} b_j n_j} \left(\sum_{j=1}^{r} a_j n_j + x \right)^k. \tag{7}$$

By the above theorem, we may now construct the complex multivariate q-zeta functions as follows:

Definition 1. For $s \in \mathbb{C}$, we define

$$\zeta_r(s, x \mid a_1, \dots, a_r;\ b_1, \dots, b_r) = \sum_{n_1,\dots,n_r=0}^{\infty} \frac{2^r (-1)^{\sum_{j=1}^{r} n_j} q^{\sum_{j=1}^{r} b_j n_j}}{\left(\sum_{j=1}^{r} a_j n_j + x \right)^s}. \tag{8}$$

Thus we note that this function in (8) is an analytic continuation in the whole complex plane. And we see that this multivariate q-zeta function interpolates q-Euler polynomials at negative integers.

Theorem 3. *Let n be an odd positive integer. Then we have*

$$\zeta_r(-n, x \mid a_1, \dots, a_r;\ b_1, \dots, b_r) = E_{n,q}{}^{(r)}(x \mid a_1, \dots, a_r;\ b_1, \dots, b_r). \tag{9}$$

We now give the complex integral representation of $\zeta_r(s, x \mid a_1, \dots, a_r;\ b_1, \dots, b_r)$. Using (6), we have the following:

$$\begin{aligned}
&\frac{1}{\Gamma(s)} \oint_{\mathbb{C}} F_q^{(r)}(-t, x \mid a_1, \dots, a_r;\ b_1, \dots, b_r) t^{s-1} \mathrm{d}t \\
&= 2^r \sum_{n_1,\dots,n_r=0}^{\infty} (-1)^{\sum_{j=1}^{r} n_j} q^{\sum_{j=1}^{r} b_j n_j} \frac{1}{\Gamma(s)} \oint_{\mathbb{C}} e^{-(\sum_{j=1}^{r} n_j a_j + x)t} t^{s-1} \mathrm{d}t \\
&= 2^r \sum_{n_1,\dots,n_r=0}^{\infty} (-1)^{\sum_{j=1}^{r} n_j} \frac{q^{\sum_{j=1}^{r} b_j n_j}}{\left(\sum_{j=1}^{r} n_j a_j + x \right)^s} \frac{1}{\Gamma(s)} \oint_{\mathbb{C}} e^{-y} y^{s-1} \mathrm{d}y
\end{aligned}$$

$$= \zeta_r(s, x \mid a_1, \ldots, a_r;\ b_1, \ldots, b_r). \tag{10}$$

On the other hand,

$$\frac{1}{\Gamma(s)} \oint_{\mathbb{C}} F_q^{(r)}(-t, x \mid a_1, \ldots, a_r;\ b_1, \ldots, b_r) t^{s-1} \mathrm{d}t$$
$$= \sum_{m=0}^{\infty} (-1)^m \frac{{E_{m,q}}^{(r)}(x \mid a_1, \ldots, a_r;\ b_1, \ldots, b_r)}{m!} \frac{1}{\Gamma(s)} \oint_{\mathbb{C}} t^{m+s-1} \mathrm{d}t. \tag{11}$$

Thus by (10) and (11), we have the following:

$$\zeta_r(s, x \mid a_1, \ldots, a_r;\ b_1, \ldots, b_r) = \sum_{m=0}^{\infty} (-1)^m \frac{E_m^{(r)}(x \mid a_1, \ldots, a_r;\ b_1, \ldots, b_r)}{m!} \frac{1}{\Gamma(s)} \oint_{\mathbb{C}} t^{m+s-1} \mathrm{d}t.$$

Thus we have

$$\zeta_r(-n, x \mid a_1, \ldots, a_r;\ b_1, \ldots, b_r) = E_n^{(r)}(x \mid a_1, \ldots, a_r;\ b_1, \ldots, b_r). \tag{12}$$

To construct the multivariate Dirichlet L-function we investigate the generating function of generalized multivariate q-Euler numbers attached to χ, which is derived in (4).

$$2^r \sum_{n_1,\ldots,n_r=0}^{f-1} \frac{(-1)^{\sum_{j=1}^r n_j} q^{\sum_{j=1}^r b_j n_j} \mathrm{e}^{\sum_{j=1}^r a_j n_j t} \prod_{j=1}^r \chi(n_j)}{\prod_{j=1}^r (q^{fbj} \mathrm{e}^{f a_j t} + 1)}$$
$$= 2^r \sum_{n_1,\ldots,n_r=0}^{f-1} (-1)^{\sum_{j=1}^r n_j} q^{\sum_{j=1}^r b_j n_j} \prod_{j=1}^r \chi(n_j) \sum_{x_1,\ldots,x_r=0}^{\infty} (-1)^{\sum_{j=1}^r n_j} q^{\sum_{j=1}^r f b_j x_j} \mathrm{e}^{\sum_{j=1}^r f a_j x_j t}$$
$$= 2^r \sum_{x_1,\ldots,x_r=0}^{\infty} \sum_{n_1,\ldots,n_r=0}^{f-1} (-1)^{\sum_{j=1}^r (n_j + f x_j)} q^{\sum_{j=1}^r b_j (n_j + f x_j)} \prod_{j=1}^r \chi(n_j + f x_j) \mathrm{e}^{t \sum_{j=1}^r a_j (n_j + f x_j)}$$
$$= 2^r \sum_{n_1,\ldots,n_r=0}^{\infty} (-1)^{\sum_{j=1}^r n_j} q^{\sum_{j=1}^r b_j n_j} \prod_{j=1}^r \chi(n_j) \mathrm{e}^{t \sum_{j=1}^r a_j n_j}.$$

Thus we can write

$${F_{\chi,q}}^{(r)}(t \mid a_1, \ldots, a_r;\ b_1, \ldots, b_r) = 2^r \sum_{n_1,\ldots,n_r=1}^{\infty} (-1)^{\sum_{j=1}^r n_j} q^{\sum_{j=1}^r b_j n_j} \prod_{j=1}^r \chi(n_j) \mathrm{e}^{t \sum_{j=1}^r a_j n_j}$$

(since $\chi(0) = 0$, we can start at $n_1 = 1, n_2 = 1, \ldots, n_r = 1$)

$$= \sum_{n=1}^{\infty} E_{n,\chi,q}(a_1, \ldots, a_r;\ b_1, \ldots, b_r) \frac{t^n}{n!}. \tag{13}$$

From (13), we can derive the following:

$$2^r \sum_{n_1,\ldots,n_r=1}^{\infty} (-1)^{\sum_{j=1}^r n_j} q^{\sum_{j=1}^r b_j n_j} \left(\prod_{j=1}^r \chi(n_j) \right) (a_1 n_1 + \cdots + a_r n_r)^k = E_{k,\chi,q}(a_1, \ldots, a_r;\ b_1, \ldots, b_r). \tag{14}$$

Therefore we have the following:

Definition 2. For $s \in \mathbb{C}$, define multivariate Dirichlet L-function as follows:

$$L_r(s, \chi \mid a_1, \ldots, a_r;\ b_1, \ldots, b_r) = 2^r \sum_{n_1, \ldots, n_r = 1}^{\infty} \frac{(-1)^{n_1 + \cdots + n_r} q^{b_1 n_1 + \cdots + b_r n_r} \chi(n_1) \cdots \chi(n_r)}{(a_1 n_1 + \cdots + a_r n_r)^s}. \tag{15}$$

Note that $L_r(s, \chi \mid a_1, \ldots, a_r;\ b_1, \ldots, b_r)$ is also an analytic function in the whole complex plane. By (12)–(15), we see that the q-analogue multivariate Dirichlet L-function interpolates multivariate generalized q-Euler numbers attached to χ at negative integers as follows:

Theorem 4. *Let k be a positive integer. Then we have*

$$L_r(-k, \chi \mid a_1, \ldots, a_r;\ b_1, \ldots, b_r) = E_{k,\chi,q}(a_1, \ldots, a_r;\ b_1, \ldots, b_r).$$

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References

[1] L. Carlitz, q-Bernoulli numbers and polynomials, Duke Math. J. 15 (1948) 987–1000.
[2] L. Carlitz, q-Bernoulli and Eulerian numbers, Trans. Amer. Math. Soc. 76 (1954) 332–350.
[3] M. Cenkci, M. Can, Some results on q-analogue of the Lerch zeta function, Adv. Stud. Contemp. Math. 12 (2006) 213–223.
[4] M. Cenkci, M. Can, V. Kurt, p-adic interpolation functions and Kummer-type congruences for q-twisted and q-generalized twisted Euler numbers, Adv. Stud. Contemp. Math. 9 (2004) 203–216.
[5] H.S. Cho, E.S. Kim, Translation-invariant p-adic integral on $\mathbb{Z}_p$, Notes Number Theory Discrete Math. 7 (2001) 70–77.
[6] A.S. Hegazi, M. Mansour, A note on q-Bernoulli numbers and polynomials, J. Nonlinear Math. Phys. 13 (2006) 9–18.
[7] T. Kim, p-adic q-integrals associated with the Changhee–Barnes' q-Bernoulli polynomials, Integral Transforms Spec. Funct. 15 (2004) 415–420.
[8] T. Kim, S.H. Rim, Y. Simsek, A note on the alternating sums of powers of consecutive q-integers, Adv. Stud. Contemp. Math. 13 (2006) 159–164.
[9] T. Kim, On the analogs of Euler numbers and polynomials associated with p-adic q-integrals on $\mathbb{Z}_p$ at $q = -1$, J. Math. Anal. Appl. (2006). doi:10.1016/j.jmaa.2006.09.027.
[10] T. Kim, On the q-extension of Euler and Genocchi numbers, J. Math. Anal. Appl. 326 (2007) 1458–1465.
[11] T. Kim, L.C. Jang, H.K. Park, A note on q-Euler and Genocchi numbers, Proc. Japan Acad. 77 (2001) 139–141.
[12] T. Kim, Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function, Russ. J. Math. Phys. 12 (2005) 189–196.
[13] T. Kim, Non-Archimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials, Russ. J. Math. Phys. 10 (2003) 91–98.
[14] T. Kim, An invariant p-adic integral associated with Daehee numbers, Integral Transforms Spec. Funct. 13 (2002) 65–69.
[15] T. Kim, On a q-analogue of the p-adic log gamma functions and related integrals, J. Number Theory 76 (1999) 320–329.
[16] T. Kim, q-Volkenborn integration, Russ. J. Math. Phys. 9 (2002) 288–299.
[17] T. Kim, q-Euler numbers and polynomials associated with p-adic q-integrals, J. Nonlinear Math. Phys. 14 (2007) 15–27.
[18] T. Kim, q-Euler numbers and polynomials associated with p-adic q-integrals and basic q-zeta function, Trends Math. (Information Center for Mathematical Sciences) 9 (2006) 7–12.
[19] B.A. Kupershmidt, Reflection symmetries of q-Bernoulli polynomials, J. Nonlinear Math. Phys. 12 (2005) 412–422.
[20] J. Satoh, q-analogue of Riemann's ζ-function and q-Euler numbers, J. Number Theory 31 (1989) 346–362.
[21] K. Shiratani, S. Yamamoto, On a p-adic interpolation function for the Euler numbers and its derivatives, Mem. Fac. Sci. Kyushu Univ. Ser. A 39 (1985) 113–125.
[22] Y. Simsek, Theorems on twisted L-function and twisted Bernoulli numbers, Adv. Stud. Contemp. Math. 11 (2005) 205–218.
[23] Y. Simsek, On p-adic twisted q-L-functions related to generalized twisted Bernoulli numbers, Russ. J. Math. Phys. 13 (2006) 327–339.
[24] Q.-M. Luo, H.M. Srivastava, Some relationships between the Apostol–Bernoulli and Apostol–Euler polynomials, Comput. Math. Appl. 10 (2005) 631–642.