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Taekyun Kim, Seog-Hoon Rim, Yilmaz Simsek*, and Daeyeoul Kim

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Abstract. In this paper, by using q-deformed bosonic p-adic integral, we give λ-Bernoulli numbers and polynomials, we prove Witt's type formula of λ-Bernoulli polynomials and Gauss multiplicative formula for λ-Bernoulli polynomials. By using derivative operator to the generating functions of λ-Bernoulli polynomials and generalized λ-Bernoulli numbers, we give Hurwitz type λ-zeta functions and Dirichlet's type λ-L-functions; which are interpolated λ-Bernoulli polynomials and generalized λ-Bernoulli numbers, respectively. We give generating function of λ-Bernoulli numbers with order r. By using Mellin transforms to their function, we prove relations between multiply zeta function and λ-Bernoulli polynomials and ordinary Bernoulli numbers of order r and λ-Bernoulli numbers, respectively. We also study on λ-Bernoulli numbers and polynomials in the space of locally constant. Moreover, we define λ-partial zeta function and interpolation function.

Introduction, definitions and notations

Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will be denoted by the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$, (cf. [2, 3, 4, 5, 6, 7, 8, 9, 16, 17, 20, 26]).

When one talks of q-extension, q considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, as p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. We use the following

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notations:

$$[x] = [x : q] = \frac{1 - q^x}{1 - q} \quad \text{(cf. [3, 4, 5, 6, 8, 9, 24, 26, 28]).}$$

Observe that when $\lim_{q \to 1}[x] = x$, for any x with $|x|_p \leq 1$ in the present p-adic case $[x : a] = \dfrac{1 - a^x}{1 - a}$.

Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$\begin{aligned}
&\mathbb{X} = \varprojlim_{N} \left(\mathbb{Z}/dp^N\mathbb{Z}\right), \\
&\mathbb{X}^* = \cup_{0<a<dp,(a,p)=1}(a + dp\mathbb{Z}_p), \\
&a + dp^N\mathbb{Z}_p = \{x \in \mathbb{X} | x \equiv a \pmod{dp^n}\},
\end{aligned}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. We assume that $u \in \mathbb{C}_p$ with $|1 - u|_p \geq 1$. (cf. [3, 4, 5, 6, 7, 8, 24, 26]).

For $x \in \mathbb{Z}_p$, we say that g is a uniformly differentiable function at point $a \in \mathbb{Z}_p$, and write $g \in UD(\mathbb{Z}_P)$, the set of uniformly differentiable functions, if the difference quotients,

$$F_g(x, y) = \frac{g(y) - g(x)}{y - x},$$

have a limit $l = g'(a)$ as $(x, y) \to (a, a)$. For $f \in UD(\mathbb{Z}_p)$, the *q-deformed bosonic p-adic integral* was defined as

$$\begin{aligned}
I_q(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_q(x) \\
(A) \qquad &= \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x) \mu_q(x + p^N\mathbb{Z}_p) \\
&= \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x) \frac{q^x}{[p^N]}, \text{(cf. [4, 5, 9]).}
\end{aligned}$$

By Eq-(A), we have

$$\lim_{q \to -q} I_q(f) = I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x).$$

This integral, $I_{-q}(f)$, give the q-deformed integral expression of fermioinc. The classical Euler numbers were defined by means of the following generating function:

$$\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!}, \quad |t| < \pi \quad \text{(cf. [6, 7, 20, 21]).}$$

Let u be algebraic in complex number field. Then Frobenius-Euler polynomials [6, 7, 20, 21] were defined by

$$(A1) \qquad \frac{1-u}{e^t-u}e^{xt} = e^{H(u,x)t} = \sum_{m=0}^{\infty} H_m(u,x)\frac{t^m}{m!},$$

where we use technical method's notation by replacing $H^m(u,x)$ by $H_m(u,x)$ symbolically. In case $x=0$, $H_m(u,0)=H_m(u)$, which is called Frobenius-Euler number. The Frobenius-Euler polynomials of order r, denoted by $H_n^{(r)}(u,x)$, were defined by

$$\left(\frac{1-u}{e^t-u}\right)^r e^{tx} = \sum_{n=0}^{\infty} H_n^{(r)}(u,x)\frac{t^n}{n!} \quad \text{(cf. [7, 10, 25, 26])}.$$

The values at $x=0$ are called Frobenius-Euler numbers of order r. When $r=1$, these numbers and polynomials are reduced to ordinary Frobenius-Euler numbers and polynomials. In the usual notation, the n-th Bernoulli polynomial were defined by means of the following generating function:

$$\left(\frac{t}{e^t-1}\right)e^{tx} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

For $x=0$, $B_n(0)=B_n$ are said to be the n-th Bernoulli numbers. The Bernoulli polynomials of order r were defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{tx} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}$$

and $B_n^{(r)}(0)=B_n^{(r)}$ are called the Bernoulli numbers of order r. Let x, w_1, w_2, …, w_r be complex numbers with positive real parts. When the generalized Bernoulli numbers and polynomials were defined by means of the following generating function:

$$\frac{w_1w_2\cdots w_r t^r e^{xt}}{(e^{w_1t}-1)(e^{w_2t}-1)\cdots(e^{w_rt}-1)} = \sum_{n=0}^{\infty} B_n^{(r)}(x \mid w_1,w_2,\ldots,w_r)\frac{t^n}{n!}$$

and $B_n^{(r)}(0 \mid w_1,w_2,\ldots,w_r) = B_n^{(r)}(w_1,w_2,\ldots,w_r)$ (cf. [13, 15]).

The Hurwitz zeta function is defined by

$$\zeta(s,x) = \sum_{n=0}^{\infty}\frac{1}{(x+n)^s},$$

$\zeta(s,1)=\zeta(s)$, which is the Riemann zeta function. The multiple zeta functions [12, 26] were defined by

$$(C) \qquad \zeta_r(s) = \sum_{0<n_1<n_2<\cdots<n_r}\frac{1}{(n_1+\cdots+n_r)^s}.$$

We summarize our paper as follows:

In section 1, by using q-deformed bosonic p-adic integral, the generating functions of λ-Bernoulli numbers and polynomials are given. From these generating functions, we derive many new interesting identities related to these numbers and polynomials and we prove Gauss multiplicative formula for λ-Bernoulli numbers. Witt's type formula of λ-Bernoulli polynomials is given.

In section 2, by using derivative operator $\left(\frac{d}{dt}\right)^k\Big|_{t=0}$ to the generating function of the λ-Bernoulli numbers, we construct Hurwitz' type λ-zeta function, which interpolates λ-Bernoulli polynomials at negative integers.

In section 3, by using same method of section 2, we give Dirichlet type λ-L-function which interpolates generalized λ-Bernoulli numbers.

In section 4, the generating functions of λ-Bernoulli numbers of order r are obtained. From these generating generating functions, we derive some interesting relations between multiple zeta functions and λ-Bernoulli numbers of order r.

In section 5, we give some important identities related to generalized λ-Bernoulli numbers of order r.

In section 6, we study on λ-Bernoulli numbers and polynomials in the space of locally constant. In this section, we also define λ-partial zeta function which interpolates λ-Bernoulli numbers at negative integers.

In section 7, we give p-adic interpolation functions.

1. λ-Bernoulli numbers

In this section, by using Eq-(A), we give integral equation of bosonic p-adic integral. By using this integral equation we define generating function of λ-Bernoulli polynomials. We give fundamental properties of the λ-Bernoulli numbers and polynomials. We also give some new identities related to λ-Bernoulli numbers and polynomials. We prove Gauss multiplicative formula for λ-Bernoulli numbers as well. Witt's type formula of λ-Bernoulli polynomials is given.

To give the expression of bosonic p-adic integral in Eq-(A), we consider the limit

$$I_1(f) = \lim_{q\to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_1(x) \quad \text{(cf. [16, 17, 18, 21])}, \tag{0}$$

in the sense of bosonic p-adic integral on $\mathbb{Z}_p$ (= p-adic invariant integral on $\mathbb{Z}_p$). From this p-adic invariant integral on $\mathbb{Z}_p$, we derive the following integral equation:

$$I_1(f_1) = I_1(f) + f'(0) \quad \text{(cf. [17])}, \tag{1}$$

where $f_1(x) = f(x+1)$. Let C_{p^n} be the space of primitive p^n-th root of unity,

$$C_{p^n} = \{\zeta \mid \zeta^{p^n} = 1\}.$$

Then, we denote

$$T_p = \lim_{n\to\infty} C_{p^n} = \underset{n\geq 0}{\rightarrow} \cup C_{p^n}.$$

For $\lambda \in \mathbb{Z}_p$, we take $f(x) = \lambda^x e^{tx}$, and $f_1(x) = e^t \lambda f(x)$. Thus we have

$$f_1(x) - f(x) = (\lambda e^t - 1) f(x). \tag{2}$$

By substituting (2) into (1), we get

$$(\lambda e^t - 1) I_1(f) = f'(0), (\text{cf. } [4, 21]). \tag{2a}$$

Consequently, we have

$$\frac{\log \lambda + t}{\lambda e^t - 1} := \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}, (\text{cf. } [4]). \tag{3}$$

By using Eq-(3), we obtain

$$\lambda(B(\lambda)+1)^n - B_n(\lambda) = \begin{cases} \log \lambda, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention of replacing $B_n(\lambda)$ by $B^n(\lambda)$, (cf. [4, 17, 18, 21]). From this result, we derive the values of some $B_n(\lambda)$ numbers as follows:

$$B_0(\lambda) = \frac{\log \lambda}{\lambda - 1}, \quad B_1(\lambda) = \frac{\lambda - 1 - \lambda \log \lambda}{(\lambda - 1)^2}, \ldots, \ (\text{cf. } [4, 17, 21]).$$

We note that, if $\lambda \in T_p$, for some $n \in \mathbb{N}$, then Eq-($2a$) is reduced to the following generating function:

$$\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!} \ (\text{cf. } [4]). \tag{3a}$$

If $\lambda = e^{2\pi i/f}$, $f \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, then Eq-(3) is reduced to ($3a$). Eq-($3a$) is obtained by Kim [3]. Let $u \in \mathbb{C}$, then by substituting $x = 0$ into Eq-($A1$), we set

$$\frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!} \ (\text{cf. } [4, 17, 18, 21]). \tag{3b}$$

$H_n(u)$ is denoted Frobenius-Euler numbers. Relation between $H_n(u)$ and $B_n(\lambda)$ is given by the following theorem:

Theorem 1. *Let $\lambda \in \mathbb{Z}_p$. Then*

$$\begin{aligned} B_n(\lambda) =& \frac{\log \lambda}{\lambda - 1} H_n(\lambda^{-1}) + \frac{n H_{n-1}(\lambda^{-1})}{\lambda - 1}, \\ B_0(\lambda) =& \frac{\log \lambda}{\lambda - 1} H_0(\lambda^{-1}). \end{aligned} \tag{4}$$

Proof. By using Eq-(3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(\lambda)\frac{t^n}{n!} &= \frac{\log\lambda + t}{\lambda e^t - 1} = \frac{\log\lambda}{\lambda e^t - 1} + \frac{t}{\lambda e^t - 1}\\ &= \frac{1-\lambda^{-1}}{(1-\lambda^{-1})\lambda}\cdot\left(\frac{\log\lambda}{e^t - \lambda^{-1}}\right) - \frac{(1-\lambda^{-1})}{(e^t-\lambda^{-1})}\cdot\frac{t}{\lambda(1-\lambda^{-1})}\\ &= \frac{\log\lambda}{\lambda-1}\sum_{n=0}^{\infty} H_n(\lambda^{-1})\frac{t^n}{n!} + \frac{t}{\lambda-1}\sum_{n=0}^{\infty} H_n(\lambda^{-1})\frac{t^n}{n!},\end{aligned}$$

the next to the last step being a consequence of Eq-($3b$). After some elementary calculations, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(\lambda)\frac{t^n}{n!} &= \frac{\log\lambda}{\lambda-1}H_0(\lambda^{-1})\\ &\quad + \sum_{n=1}^{\infty}\left(\frac{\log\lambda}{\lambda-1}H_n(\lambda^{-1}) + \frac{n}{\lambda-1}H_{n-1}(\lambda^{-1})\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficient $\frac{t^n}{n!}$ in the above, then we obtain the desired result. $\square$

Observe that, if $\lambda \in T_p$ in Eq-(4), then we have, $B_0(\lambda) = 0$ and $B_n(\lambda) = \frac{nH_{n-1}(\lambda^{-1})}{\lambda-1}$, $n \geq 1$.

By Eq-(3) and Eq-(4), we obtain the following formula:

For $n \geq 0$, $\lambda \in \mathbb{Z}_p$

$$\int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = \begin{cases} \frac{\log\lambda}{\lambda-1}H_0(\lambda^{-1}), & n = 0\\ \frac{\log\lambda}{\lambda-1}H_n(\lambda^{-1}) + \frac{n}{\lambda-1}H_{n-1}(\lambda^{-1}), & n > 0\end{cases} \tag{4a}$$

and

$$\int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = B_n(\lambda), \quad n \geq 0. \tag{4b}$$

Now, we define λ-Bernoulli polynomials, we use these polynomials to give the sums powers of consecutive. The λ-Bernoulli polynomials are defined by means of the following generating function:

$$\frac{\log\lambda + t}{\lambda e^t - 1}e^{tx} = \sum_{n=0}^{\infty} B_n(\lambda;x)\frac{t^n}{n!}. \tag{5}$$

By Eq-(3) and Eq-(5), we have

$$B_n(\lambda;x) = \sum_{k=0}^{n}\binom{n}{k}B_k(\lambda)x^{n-k}.$$

The Witt's formula for $B_n(\lambda;x)$ is given by the following theorem:

Theorem 2. *For $k \in \mathbb{N}$ and $\lambda \in \mathbb{Z}_p$, we have*

$$B_n(\lambda;x) = \int_{\mathbb{Z}_p} (x+y)^n \lambda^y d\mu_1(y). \tag{6}$$

Proof. By substituting $f(y) = e^{t(x+y)}\lambda^y$ into Eq-(1), we have

$$\int_{\mathbb{Z}_p} e^{t(x+y)}\lambda^y d\mu_1(y) = \sum_{n=0}^{\infty} B_n(\lambda;x)\frac{t^n}{n!} = \frac{(\log\lambda + t)e^{tx}}{\lambda e^t - 1}.$$

By using Taylor expansion of e^{tx} in the left side of the above equation, after some elementary calculations, we obtain the desired result. □

We now give the distribution of the λ-Bernoulli polynomials.

Theorem 3. *Let $n \geq 0$, and let $d \in \mathbb{Z}^+$. Then we have*

$$B_n(\lambda;x) = d^{n-1}\sum_{a=0}^{d-1} \lambda^a B_n\left(\lambda^d; \frac{x+a}{d}\right). \tag{7}$$

Proof. By using Eq-(6),

$$\begin{aligned}
B_n(x;\lambda) &= \int_{\mathbb{Z}_p} (x+y)^n \lambda^y d\mu_1(y) \\
&= \lim_{N\to\infty} \frac{1}{dp^N} \sum_{y=0}^{dp^N-1} (x+y)^n \lambda^y \\
&= \lim_{N\to\infty} \frac{1}{dp^N} \sum_{a=0}^{d-1}\sum_{y=0}^{p^N-1} (a+dy+x)^n \lambda^{a+dy} \\
&= d^{n-1} \lim_{N\to\infty} \frac{1}{p^N} \sum_{a=0}^{d-1} \lambda^a \sum_{y=0}^{p^N-1} \left(\frac{a+x}{d} + y\right)^n (\lambda^d)^y \\
&= d^{n-1} \frac{1}{p^N} \sum_{a=0}^{d-1} \lambda^a \int_{\mathbb{Z}_p} \left(\frac{a+x}{d} + y\right)^n (\lambda^d)^y.
\end{aligned}$$

Thus, we have the desired result. □

By substituting $x = 0$ into Eq-(7), we have the following corollary:

Corollary 1. *For $m, n \in \mathbb{N}$, we have*

$$mB_n(\lambda) = \sum_{j=0}^{n} \binom{n}{j} B_j(\lambda^m) m^j \sum_{a=0}^{m-1} \lambda^a a^{n-j}. \tag{8}$$

(*Gauss multiplicative formula for λ-Bernoulli numbers*).

By Eq-(8), we have

Theorem 4. *For $m, n \in \mathbb{N}$ and $\lambda \in \mathbb{Z}_p$, we have*

$$mB_n(\lambda) - m^n[m]_\lambda B_n(\lambda^m) = \sum_{j=0}^{n-1}\binom{n}{j}B_j(\lambda^m)m^j\sum_{k=1}^{m-1}\lambda^k k^{n-j}. \tag{9}$$

Theorem 5. *Let $k \in \mathbb{Z}$, with $k > 1$. Then we have*

$$B_l(\lambda;k) - \lambda^{-k}B_l(\lambda) = \lambda^{-k}l\sum_{n=0}^{k-1}\lambda^n n^{l-1} + (\lambda^{-k}\log\lambda)\sum_{n=0}^{k-1}n^l\lambda^l. \tag{10}$$

Proof. We set

$$\begin{aligned} -\sum_{n=0}^{\infty}e^{(n+k)t}\lambda^n + \sum_{n=0}^{\infty}e^{nt}\lambda^{n-k} &= \sum_{n=0}^{k-1}e^{nt}\lambda^{n-k} \\ &= \sum_{l=0}^{\infty}(\lambda^{-k}\sum_{n=0}^{k-1}n^l\lambda^n)\frac{t^l}{l!} \\ &= \sum_{l=1}^{\infty}(\lambda^{-k}l\sum_{n=0}^{k-1}n^{l-1}\lambda^n)\frac{t^{l-1}}{l!}. \end{aligned} \tag{10a}$$

Multiplying $(t+\log\lambda)$ both side of Eq-($10a$), then by using Eq-(3) and Eq-(5), after some elementary calculations, we have

$$\begin{aligned} &\sum_{l=0}^{\infty}(B_l(\lambda;k) - \lambda^{-k}B_l(\lambda))\frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty}(\lambda^{-k}l\sum_{n=0}^{k-1}\lambda^n n^{l-1} + \lambda^{-k}\log\lambda\sum_{n=0}^{k-1}n^l\lambda^l)\frac{t^l}{l!}. \end{aligned} \tag{10b}$$

By comparing coefficient $\frac{t^l}{l!}$ in both sides of Eq-($10b$). Thus we arrive at the Eq-(10). Thus we complete the proof of theorem. □

Observe that $\lim_{\lambda\to 1}B_l(\lambda) = B_l$. For $\lambda \to 1$, then Eq-(10) reduces the following:

$$B_l(k) - B_l = l\sum_{n=0}^{k-1}n^{l-1}.$$

If $\lambda \in T_p$, then Eq-(10) reduces to the following formula:

$$B_l(\lambda;k) - \lambda^{-k}B_l(\lambda) = \lambda^{-k}l\sum_{n=0}^{k-1}\lambda^n n^{l-1}.$$

Remark. Garrett and Hummel [2] proved combinatorial proof of q-analogue of

$$\sum_{k=1}^{n}k^3 = \binom{n+1}{k}^2$$

as follows:

$$\sum_{k=1}^{n} q^{k-1}[k]_q^2 \left(\begin{bmatrix} k-1 \\ 2 \end{bmatrix}_{q^2} + \begin{bmatrix} k+1 \\ 2 \end{bmatrix}_{q^2} \right) = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2,$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = \prod_{j=1}^{k} \frac{[n+1-j]_q}{[j]_q}$, q-binomial coefficients. In [12], Kim constructed the following formula

$$\begin{aligned} S_{n,q^h}(k) &= \sum_{l=0}^{k-1} q^{h^l}[l]^n \\ &= \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}\beta_{j,q} q^{kj}[k]^{n+1-j} - \frac{(1-q^{(n+1)k})\beta_{n+1,q}}{n+1}, \end{aligned}$$

where $\beta_{j,q}$ are the q-Bernoulli numbers which were defined by

$$e^{\frac{t}{1-q}}\frac{q-1}{\log q} - t\sum_{n=0}^{\infty} q^{n+x}e^{[n+x]t} = \sum_{n=0}^{\infty}\frac{\beta_{n,q}(x)}{n!}t^n, \quad |q|<1, |t|<1,$$

$\beta_{n,q}(0) = \beta_{n,q}$ (cf. [11, 12]).

Schlosser [22] gave for $n = 1, 2, 3, 4, 5$ the value of $S_{n,q^h}[k]$. In [27], the authors also gave another proof of $S_{n,q}(k)$ formula.

2. Hurwitz's type λ-zeta function

In this section, by using generating function of λ-Bernoulli polynomials, we construct Hurwitz's type λ-zeta function, which is interpolate λ-Bernoulli polynomials at negative integers. By Eq-(5), we get

$$\begin{aligned} F_\lambda(t;x) &= \frac{\log\lambda + t}{\lambda e^t - 1}e^{xt} = -(\log\lambda + t)\sum_{n=0}^{\infty}\lambda^n e^{(n+x)t} \\ &= \sum_{n=0}^{\infty} B_n(\lambda)\frac{t^n}{n!}. \end{aligned}$$

By using $\frac{d^k}{dt^k}$ derivative operator to the above, we have

$$\begin{aligned} B_k(\lambda;x) &= \left.\frac{d^k}{dt^k}F_\lambda(t;x)\right|_{t=0}, \\ B_k(\lambda;x) &= -\log\lambda\sum_{n=0}^{\infty}\lambda^n(n+x)^k - k\sum_{n=0}^{\infty}(n+x)^{k-1}\lambda^n. \end{aligned}$$

Thus we arrive at the following theorem:

Theorem 6. *For $k \geq 0$, we have*

$$-\frac{1}{k}B_k(\lambda; x) = \frac{\log \lambda^k}{k}\sum_{n=0}^{\infty}\lambda^n(n+x)^k + \sum_{n=0}^{\infty}\lambda^n(n+x)^{k-1}.$$

Consequently, we define Hurwitz type zeta function as follows:

Definition 1. Let $s \in \mathbb{C}$. Then we define

$$\zeta_\lambda(s,x) = \frac{\log \lambda}{1-s}\sum_{n=0}^{\infty}\frac{\lambda^n}{(n+x)^{s-1}} + \sum_{n=0}^{\infty}\frac{\lambda^n}{(n+x)^s}. \tag{11}$$

Note that $\zeta_\lambda(s,x)$ is analytic continuation, except for $s=1$, in whole complex plane. By Definition 1 and Theorem 6, we have the following:

Theorem 7. *Let $s = 1-k$, $k \in \mathbb{N}$. Then*

$$\zeta_\lambda(1-k, x) = -\frac{B_k(\lambda, x)}{k}. \tag{12}$$

3. Generalized λ-Bernoulli numbers associated with Dirichlet type λ-L-functions

By using Eq-(0), we define

$$I_1(f_d) = I_1(f) + \sum_{j=0}^{d-1} f'(j), \tag{12}$$

where $f_d(x) = f(x+d)$, $\int_{\mathbb{X}} f(x)d\mu(x) = I_1(f)$.

Let χ be a Dirichlet character with conductor $d \in \mathbb{N}^+$, $\lambda \in \mathbb{Z}_p$.

By substituting $f(x) = \lambda^x\chi(x)e^{tx}$ into Eq-(12), then we have

$$\begin{aligned}\int_{\mathbb{X}}\chi(x)\lambda^x e^{tx}d\mu_1(x) &= \sum_{j=0}^{d-1}\frac{\chi(j)\lambda^j e^{tj}(\log\lambda + t)}{\lambda^d e^{dt}-1}\\ &= \sum_{n=0}^{\infty}B_{n,\chi}(\lambda)\frac{t^n}{n!}.\end{aligned} \tag{12a}$$

By Eq-(12a), we easily see that

$$B_{n,\chi}(\lambda) = \int_{\mathbb{X}}\chi(x)x^n\lambda^x d\mu_1(x). \tag{12b}$$

From Eq-(12a), we define generating function of generalized Bernoulli number by

$$F_{\lambda,\chi}(t) = \sum_{j=0}^{d-1}\frac{\chi(j)\lambda^j e^{tj}(\log\lambda + t)}{\lambda^d e^{dt}-1} = \sum_{n=0}^{\infty}B_n(\lambda)\frac{t^n}{n!}. \tag{12c}$$

Observe that if $\lambda \in T_p$, then the above formula reduces to

$$F_{\lambda,\chi}(t) = \sum_{j=0}^{d-1} \frac{\chi(j)\lambda^j e^{tj} t}{\lambda^d e^{dt} - 1} = \sum_{j=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}$$

(for detail see cf. [3, 16, 18, 22, 23, 24]).

From the above, we easily see that

$$F_{\lambda,\chi}(t) = -(\log \lambda + t) \sum_{m=1}^{\infty} \chi(m)\lambda^m e^{tm} = \sum_{n=0}^{\infty} B_{n,\chi}(\lambda) \frac{t^n}{n!}.$$

By applying $\left.\frac{d^k}{dt^k}\right|_{t=0}$ derivative operator both sides of the above equation, we arrive at the following theorem:

Theorem 8. *Let $k \in \mathbb{Z}^+$, $\lambda \in \mathbb{Z}_p$ and let χ be a Derichlet character with conductor d. Then we have*

$$\sum_{m=1}^{\infty} \chi(m)\lambda^m m^{k-1} + \frac{\log \lambda}{k} \sum_{m=1}^{\infty} \lambda^m \chi(m) m^k = -\frac{B_{k,\chi}(\lambda)}{k}. \tag{13}$$

Definition 2 (Dirichlet type λ-L function)**.** For $\lambda, s \in \mathbb{C}$, we define

$$L_\lambda(s,\chi) = \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^s} - \frac{\log \lambda}{s-1} \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^{s-1}}. \tag{14}$$

Relation between $L_\lambda(s,\chi)$ and $\zeta_\lambda(s,y)$ is given by the following theorem :

Theorem 9. *Let $s \in \mathbb{C}$ and $d \in \mathbb{Z}^+$. Then we have*

$$L_\lambda(s,\chi) = d^{-s} \sum_{a=1}^{d} \lambda^a \chi(a) \zeta_{\lambda^d}\left(s, \frac{a}{d}\right).$$

Proof. By substituting $m = a + dk$, $a = 1, 2, \ldots, d$, $k = 0, 1, \ldots, \infty$, into Eq-(14), we have

$$\begin{aligned} L_\lambda(s,\chi) &= \sum_{a=1}^{d} \sum_{k=0}^{\infty} \frac{\lambda^{a+dk}\chi(a+dk)}{(a+dk)^s} - \frac{\log \lambda}{s-1} \sum_{a=1}^{d} \sum_{k=0}^{\infty} \frac{\lambda^{a+dk}\chi(a+dk)}{(a+dk)^{s-1}} \\ &= d^{-s} \sum_{a=1}^{d} (\lambda^a \chi(a)) \left[\sum_{k=0}^{\infty} \frac{(\lambda^d)^k}{(k+\frac{a}{d})^s} - \frac{\log \lambda^d}{s-1} \sum_{k=0}^{\infty} \frac{(\lambda^d)^k}{(k+\frac{a}{d})^{s-1}} \right]. \end{aligned}$$

By using Eq-(11) in the above we obtain the desired result. □

Theorem 10. *For $k \in \mathbb{Z}^+$, we have*

$$L_\lambda(1-k,\chi) = -\frac{1}{k} B_{k,\chi}(\lambda), \quad k > 0.$$

Proof. By substituting $s = 1 - k$ in Definition 2 and using Eq-(13), we easily obtain the desired result. □

Remark. If $\lambda \in T_p$, then from Definition 2, we have

$$L_\lambda(s,\chi) = \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^s}.$$

In [21, 18], Kim studied on the λ-Euler numbers and he gave interesting many relations on λ-Euler numbers and polynomials. λ-Bernoulli numbers and polynomials are corresponding to λ-Euler numbers and polynomials (see [21]). In [17, 18], Kim et al gave λ-(h,q) zeta function and λ-(h,q) L-function. These functions interpolate $\lambda-(h,q)$-Bernoulli numbers at negative integer. Observe that, if we take $s = 1-k$ in Theorem 9, and then using Eq-(12) in Theorem 7, we get another proof of Theorem 10.

4. λ-Bernoulli numbers of order r associated with multiple zeta function

In this section, we define generating function of λ-Bernoulli numbers of order r. By using Mellin transforms and Cauchy residue theorem, we obtain multiple zeta function which is given in Eq-(C). We also gave relations between λ-Bernoulli polynomials of order r and multiple zeta function at negative integers. This relation is important and very interesting. Let $r \in \mathbb{Z}^+$. Generating function of λ-Bernoulli numbers of order r is defined by

$$F_\lambda^{(r)}(t) = \left(\frac{\log\lambda + t}{\lambda e^t - 1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda)\frac{t^n}{n!}. \tag{15}$$

Generating function of λ-Bernoulli polynomials of order r is defined by

$$F_\lambda^{(r)}(t,x) = F_\lambda^{(r)}(t)e^{tx} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda)\frac{t^n}{n!}.$$

Observe that when $r = 1$, Eq-(15) reduces to Eq-(3). By applying Mellin transforms to the Eq-(15) we get

$$\begin{aligned} &\frac{1}{\Gamma(s)}\int_0^{\infty} \lambda^r e^{-tr} F_\lambda^{(r)}(-t)(t-\log\lambda)^{s-r-1}dt \\ &= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{(n_1+n_2+\cdots+n_r+r)^s}. \end{aligned}$$

Thus, we get, by (C)

$$\zeta_r(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} \lambda^r e^{-tr} F_\lambda^{(r)}(-t)(t-\log\lambda)^{s-r-1}dt.$$

By using the above relation, we obtain the following theorem:

Theorem 11. *Let $r, m \in \mathbb{Z}^+$. Then we have*

$$\zeta_r(-m) = (-\lambda)^r m! \sum_{j=0}^{\infty} \binom{-m-r-1}{j} (\log\lambda)^j \frac{B_{m+r+j}(\lambda; r)}{(m+r+j)!}. \tag{D1}$$

Remark. If $\lambda \to 1$, the above theorem reduces to

$$\zeta_r(-m) = (-1)^r m! \frac{B_{m+r}(1;r)}{(m+r)!} \tag{D2}$$

which is given Theorem 6 in [13].

By $(D1)$ and $(D2)$, we obtain relation between λ-Bernoulli polynomials of order r and ordinary Bernoulli polynomials of order r as follows:

$$B_{m+r}(r) = \lambda^r \sum_{j=0}^{\infty} \binom{-m-r-1}{j} (\log \lambda)^j \frac{B_{m+r+j}(\lambda;r)}{(m+r+j)!}(m+r)!,$$

where $m, r \in \mathbb{Z}^+$.

We now give relations between $B_n^{(r)}(\lambda)$ and $H_n^{(r)}(\lambda^{-1})$ as follows:

If $\lambda \in T_p$, then Eq-(15) reduces to the following equation

$$\frac{t^r}{(\lambda e^t - 1)^r} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!}.$$

Thus by the above equation, we easily see that

$$\begin{aligned} t^r &= (\lambda e^t - 1)^r e^{B^{(r)}(\lambda)t} \\ &= \sum_{l=0}^{r} \lambda^l (-1)^{r-l} e^{(B^{(r)}(\lambda)+l)t} \\ &= \sum_{n=0}^{\infty} (\sum_{l=0}^{r} \lambda^l (-1)^{r-l} (B^{(r)}(\lambda)+l)^n) \frac{t^n}{n!}. \end{aligned}$$

Consequently we have

$$\sum_{l=0}^{r} \lambda^l (-1)^{r-l} (B^{(r)}(\lambda)+l)^n = \begin{cases} 0 & \text{if } n \neq r \\ 1 & \text{if } \ n = r. \end{cases}$$

By Eq-(15) we obtain

$$\sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!} = \frac{t^r}{(\lambda-1)^r} \sum_{n=0}^{\infty} H_n^{(r)}(\lambda^{-1}) \frac{t^n}{n!}.$$

By comparing coefficient $\frac{t^n}{n!}$ in the both sides of the above equation, we have

$$B_{n+r}^{(r)}(\lambda) = \frac{\Gamma(n+r+1)}{\Gamma(n+1)} \frac{1}{(\lambda-1)^r} H_n^{(r)}(\lambda^{-1}).$$

Observe that, if we take $r = 1$, then the above identity reduce to Eq-(4), that is

$$B_{n+1}(\lambda) = \frac{(n+1)}{\lambda - 1} H_n(\lambda^{-1}).$$

5. λ-Bernoulli numbers and polynomials associated with multivariate p-adic invariant integral

In this section, we give generalized λ-Bernoulli numbers of order r. Consider the multivariate p-adic invariant integral on $\mathbb{Z}_p$ to define λ-Bernoulli numbers and polynomials.

$$\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-\text{times}} \lambda^{w_1x_1+\cdots+w_rx_r} e^{(w_1x_1+\cdots+w_rx_r)t} d\mu_1(x_1)\cdots d\mu_1(x_r) \\
&= \frac{(w_1\log\lambda + w_1 t)\cdots(w_r\log\lambda + w_r t)}{(\lambda^{w_1}e^{w_1t}-1)\cdots(\lambda^{w_r}e^{w_rt}-1)} \\
&= \sum_{n=0}^{\infty} B_n^{(r)}(\lambda; w_1, w_2, \ldots, w_r)\frac{t^n}{n!},
\end{aligned} \tag{16}$$

where we called $B_n^{(r)}(\lambda; w_1, w_2, \ldots, w_r)$ λ-extension of Bernoulli numbers. Substituting $\lambda = 1$ into Eq-(16), λ-extension of Bernoulli numbers reduce to Barnes Bernoulli numbers as follows :

$$\frac{(w_1t)\cdots(w_rt)}{(e^{w_1t}-1)\cdots(e^{w_rt}-1)} = \sum_{n=0}^{\infty} B_n^{(r)}(w_1, \ldots, w_r)\frac{t^n}{n!},$$

where $B_n^{(r)}(w_1, \ldots, w_r)$ are denoted Barnes Bernoulli umbers and $w_1, \ldots, w_r$ complex numbers with positive real parts [1, 7, 26]. Observe that when $w_1 = w_2 = \cdots = w_r = 1$ in Eq-(16), we obtain the λ-Bernoulli numbers of higher order as follows:

$$\left(\frac{\log\lambda + t}{\lambda e^t - 1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda)\frac{t^n}{n!}.$$

We note that $B_n^{(r)}(\lambda; 1, 1, \ldots, 1) = B_n^{(r)}(\lambda)$.

Consider

$$\left(\frac{\log\lambda + t}{\lambda e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda; x)\frac{t^n}{n!}.$$

Observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n^{(r)}(\lambda; x)\frac{t^n}{n!} &= \left(\frac{\log\lambda + t}{\lambda e^t - 1}\right)^r e^{(\log\lambda + t)x}\lambda^{-x} \\
&= \frac{1}{\lambda^x}\sum_{m=0}^{\infty} B_m^{(r)}(\lambda; x)\frac{(t+\log\lambda)^m}{m!}
\end{aligned}$$

$$=\frac{1}{\lambda^x}\sum_{m=0}^{\infty}\frac{B_m^{(r)}(\lambda;x)}{m!}\sum_{l=0}^{m}\binom{m}{l}(\log\lambda)^m t^{m-l}$$
$$=\sum_{m=0}^{\infty}\left(\frac{1}{\lambda^x}\sum_{l=0}^{\infty}\frac{B_{n+l}^{(r)}(\lambda;x)}{l!}(\log\lambda)^l\right)\frac{t^n}{n!}.$$

Now, comparing coefficient $\frac{t^n}{n!}$ both sides of the above equation, we easily arrive at the following theorem:

Theorem 12. *For $n, r \in \mathbb{N}$ and $\lambda \in \mathbb{Z}_p$, we have*

$$B_n^{(r)}(\lambda;x)=\frac{1}{\lambda^r}\sum_{l=0}^{\infty}B_{n+l}^{(r)}(\lambda;x)\frac{(\log\lambda)^l}{l!},$$

where $0^l=\begin{cases}1 & \text{if } l=0\\ 0 & \text{if } l\neq 0.\end{cases}$

Remark. In Theorem 12, we see that

$$\lim_{\lambda\to 1}B_n^{(r)}(\lambda;x)=\begin{cases}B_n^{(r)}(x) & \text{if } l=0,\\ 0 & \text{if } l\neq 0.\end{cases}$$

6. $\boldsymbol{\lambda}$-Bernoulli numbers and polynomials in the space of locally constant

In this section, we construct partial λ-zeta functions, we need this function in the following section. We need this function in the following section. By Eq-($3b$), Frobenius-Euler polynomials are defined by means of the following generating function:

$$\left(\frac{1-u}{e^t-u}\right)e^{xt}=\sum_{n=0}^{\infty}H_n(u,x)\frac{t^n}{n!}.$$

As well known, we note that the Frobenius-Euler polynomials of order r were defined by

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}(u,x)\frac{t^n}{n!}.$$

The case $x=0$, $H_n^{(r)}(u,0)=H_n^{(r)}(u)$, which are called Frobenius-Euler numbers of order r.

If $\lambda \in T_p$, then λ-Bernoulli polynomials of order r are given by

$$\frac{t^r}{(\lambda e^t-1)^r}e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(\lambda;x)\frac{t^n}{n!}.$$

Hurwitz type λ-zeta function is given by

$$\zeta_\lambda(s,x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(n+x)^s}, \quad \lambda \in T_p. \tag{17}$$

Thus, from Theorem 7, we have

$$\zeta_\lambda(1-k,x) = -\frac{1}{k} B(\lambda; x), \qquad k \in \mathbb{Z}^+. \tag{17a}$$

We now define λ-partial zeta function as follows

$$H_\lambda(s,a|F) = \sum_{m \equiv a \pmod F} \frac{\lambda^m}{m^s}. \tag{17b}$$

From (17), we have

$$H_\lambda(s,a|F) = \frac{\lambda^a}{F^s} \zeta_{\lambda^F}\left(s, \frac{a}{F}\right), \tag{17c}$$

where $\zeta_{\lambda^F}\left(s, \frac{a}{F}\right)$ is given by Eq-(17). By Eq-(17a) we have

$$H_\lambda(1-n,a|F) = -\frac{F^{n-1}\lambda^a B_n(\lambda^F; \frac{a}{F})}{n}, \qquad n \in \mathbb{Z}^+. \tag{18}$$

If $\lambda \in T_p$, then by Eq-(14), we have

$$L_\lambda(s,\chi) = \sum_{n=1}^{\infty} \frac{\lambda^n \chi(n)}{n^s},$$

where $s \in \mathbb{C}$, χ be the primitive Dirichlet character with conductor $f \in \mathbb{Z}^+$. By Theorem 9, Eq-(17c) and Eq-(18) we easily see that

$$L_\lambda(s,\chi) = \sum_{a=1}^{F} \chi(a) H_\lambda\left(s, \frac{a}{F}\right),$$

and

$$L_\lambda(1-k,\chi) = -\frac{B_{k,\chi}(\lambda)}{k}, \quad k \in \mathbb{Z}^+,$$

where $B_{k,\chi}(\lambda)$ is defined by

$$\sum_{a=0}^{F-1} \frac{t\lambda^a \chi(a) e^{at}}{\lambda^F e^{Ft} - 1} = \sum_{a=0}^{\infty} B_{n,\chi}(\lambda) \frac{t^n}{n!}, \quad \lambda \in T_p$$

and F is multiple of f.

Remark.

$$\frac{B_m(\lambda)}{m} = \frac{1}{\lambda - 1} H_{n-1}(\lambda^{-1}), \quad \lambda \in T_p.$$

7. p-adic interpolation function

In this section we give p-adic λ-L function. Let w be the Teichimuller character and let $\langle x\rangle = \dfrac{x}{w(x)}$.

When F is multiple of p and f and $(a,p)=1$, we define

$$H_{p,\lambda}(s,a|F) = \frac{1}{s-1}\lambda^a \langle a\rangle^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} \left(\frac{F}{a}\right)^j B_j(\lambda^F).$$

From this we note that

$$\begin{aligned} H_{p,\lambda}(1-n,a|F) &= -\frac{1}{n}\frac{\lambda^a}{F}\langle a\rangle^n \sum_{j=0}^{n} \binom{n}{j} \left(\frac{F}{a}\right)^j B_j(\lambda^F) \\ &= -\frac{1}{n}F^{n-1}\lambda^a w^{-n}(a) B_n(\lambda^F; \frac{a}{F}) \\ &= w^{-n}(a) H_\lambda(1-n; \frac{a}{F}), \end{aligned}$$

since by Theorem 3 for $\lambda \in T_p$, Eq-(18).

By using this formula, we can consider p-adic λ-L-function for λ-Bernoulli numbers as follows:

$$L_{p,\lambda}(s,\chi) = \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a) H_{p,\lambda}\left(s, \frac{a}{F}\right).$$

By using the above definition, we have

$$\begin{aligned} L_{p,\lambda}(1-n,\chi) &= \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a) H_{p,\lambda}\left(1-n, \frac{a}{F}\right) \\ &= -\frac{1}{n}\left(B_{n,\chi w^{-n}}(\lambda) - p^{n-1}\chi w^{-n}(p) B_{n,\chi w^{-n}}(\lambda^p)\right). \end{aligned}$$

Thus, we define the formula

$$L_{p,\lambda}(s,\chi) = \frac{1}{F}\frac{1}{s-1}\sum_{a=1}^{F}\chi(a)\lambda^a\langle a\rangle^{1-s}\sum_{j=0}^{\infty}\binom{1-s}{j}B_j(\lambda^F)$$

for $s \in \mathbb{Z}_p$.

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Taekyun Kim
Division of General Education-Mathematics
Kwangwoon University
Seoul 139-704, Korea
E-mail address: tkim64@hanmail.net

Seog-Hoon Rim
Department of mathematical Education
Kyungpook National University
Taegu 702-701, Korea
E-mail address: shrim@knu.ac.kr

Yilmaz Simsek
University of Akdeniz
Faculty of Art and Science
Department of Mathematics 07058 Antalya, Turkey
E-mail address: ysimsek@akdeniz.edu.tr

Daeyeoul Kim
National Institute for Mathematical Science
Daejeon 305-340, Korea
E-mail address: daeyeoul@chonbuk.ac.kr