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Generalizations of Laguerre Polynomials

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It is shown that the polynomials $\{L_n^{\alpha, M_0, M_1, \ldots, M_N}(x)\}_{n=0}^{\infty}$ defined by

$$L_n^{\alpha, M_0, M_1, \ldots, M_N}(x) = \sum_{k=0}^{N+1} A_k \cdot D^k L_n^{(\alpha)}(x)$$

for certain real coefficients $\{A_k\}_{k=0}^{N+1}$ are orthogonal with respect to the inner product

$$\langle f, g \rangle = \frac{1}{\Gamma(\alpha+1)} \cdot \int_0^{\infty} x^{\alpha} e^{-x} \cdot f(x)\, g(x)\, dx + \sum_{\nu=0}^{N} M_{\nu} \cdot f^{(\nu)}(0)\, g^{(\nu)}(0),$$

where $\alpha > -1$, $N \in \mathbb{N}$ and $M_{\nu} \geqslant 0$ for all $\nu \in \{0, 1, 2, \ldots, N\}$. For these new polynomials $\{L_n^{\alpha, M_0, M_1, \ldots, M_N}(x)\}_{n=0}^{\infty}$ an orthogonality relation and a second order differential equation are derived. Further we obtain a representation as a ${}_{N+2}F_{N+2}$ hypergeometric series and a $(2N+3)$-terms recurrence relation, which gives rise to a Christoffel–Darboux type formula. © 1990 Academic Press, Inc.

1. INTRODUCTION

In [8, 9] H. L. Krall introduced polynomials which are orthogonal with respect to a weight function consisting of a classical weight function together with a delta function at the endpoint(s) of the interval of orthogonality. These polynomials were described in more detail by A. M. Krall in [7].

In [6] T. H. Koornwinder studied the more general polynomials which are orthogonal on the interval $[-1, 1]$ with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta} + M \cdot \delta(x+1) + N \cdot \delta(x-1)$. These polynomials are generalizations of the classical Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$. In [1] H. Bavinck and H. G. Meijer studied further generalizations of these

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polynomials in the ultraspherical case ($\alpha=\beta$); they computed the polynomials which are orthogonal with respect to the inner product

$$\begin{aligned}\langle f, g\rangle = &\frac{\Gamma(2\alpha+1)}{2^{2\alpha+1}\cdot\Gamma^2(\alpha+1)}\cdot\int_{-1}^{1}(1-x^2)^\alpha\cdot f(x)\,g(x)\,dx\\ &+M\cdot[f(-1)\,g(-1)+f(1)\,g(1)]\\ &+N\cdot[f'(-1)\,g'(-1)+f'(1)\,g'(1)],\end{aligned}$$

where $\alpha>-1$, $M\geqslant 0$, and $N\geqslant 0$.

As a limit case T. H. Koornwinder found the polynomials $\{L_n^{\alpha,N}(x)\}_{n=0}^{\infty}$ which are orthogonal on $[0,\infty)$ with respect to the weight function $x^\alpha e^{-x}+N\cdot\delta(x)$. These polynomials are generalizations of the classical (generalized) Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$. In [5] we listed the most important properties of Koornwinder's generalized Laguerre polynomials. And in [4] R. Koekoek and H. G. Meijer found further generalizations of these polynomials orthogonal with respect to the inner product

$$\langle f, g\rangle=\frac{1}{\Gamma(\alpha+1)}\cdot\int_0^\infty x^\alpha e^{-x}\cdot f(x)\,g(x)\,dx+M\cdot f(0)\,g(0)+N\cdot f'(0)\,g'(0),$$

where $\alpha>-1$, $M\geqslant 0$, and $N\geqslant 0$.

Now it is the aim of the present paper to find the polynomials which are orthogonal with respect to the inner product

$$\langle f, g\rangle=\frac{1}{\Gamma(\alpha+1)}\cdot\int_0^\infty x^\alpha e^{-x}\cdot f(x)(g(x)\,dx+\sum_{\nu=0}^{N}M_\nu\cdot f^{(\nu)}(0)\,g^{(\nu)}(0), \quad (1.1)$$

where $\alpha>-1$, $N\in\mathbb{N}$, and $M_\nu\geqslant 0$ for all $\nu\in\{0,1,2,...,N\}$. We define

$$L_n^{\alpha,M_0,M_1,...,M_N}(x)=\sum_{k=0}^{N+1}A_k\cdot D^kL_n^{(\alpha)}(x). \quad (1.2)$$

We show that the coefficients $\{A_k\}_{k=0}^{N+1}$ can be chosen in such a way that the polynomials $\{L_n^{\alpha,M_0,M_1,...,M_N}(x)\}_{n=0}^{\infty}$ are orthogonal with respect to the inner product (1.1). For $N=1$ these polynomials reduce to the polynomials found in [4] and for $N=0$ we have Koornwinder's generalized Laguerre polynomials.

2. The Classical Laguerre Polynomials

First we state some properties of the classical Laguerre polynomials. For details the reader is referred to [2, 10].

Let $\alpha > -1$. The classical Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ are orthogonal on the interval $[0, \infty)$ with respect to the weight function $x^\alpha e^{-x}$. Their orthogonality relation is

$$\frac{1}{\Gamma(\alpha+1)} \cdot \int_0^\infty x^\alpha e^{-x} \cdot L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\, dx = \binom{n+\alpha}{n} \cdot \delta_{mn}. \tag{2.1}$$

Further we have

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}. \tag{2.2}$$

They can be defined by Rodrigues' formula

$$L_n^{(\alpha)} = \frac{1}{n!} \cdot x^{-\alpha} e^x \cdot D^n[e^{-x} x^{n+\alpha}]. \tag{2.3}$$

Further we have a representation as a hypergeometric series

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} \cdot {}_1F_1(-n; \alpha+1; x) \tag{2.4}$$

and the explicit representation

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} \cdot x^k.$$

Note that

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} \cdot x^n + \text{lower order terms}. \tag{2.5}$$

They satisfy a linear second order differential equation

$$x \cdot y'' + (\alpha + 1 - x) \cdot y' + n \cdot y = 0 \tag{2.6}$$

and a three term recurrence relation

$$(n+1) \cdot L_{n+1}^{(\alpha)}(x) + (x - 2n - \alpha - 1) \cdot L_n^{(\alpha)}(x) + (n+\alpha) \cdot L_{n-1}^{(\alpha)}(x) = 0 \tag{2.7}$$

with $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = \alpha + 1 - x$.

Further we have a Christoffel–Darboux formula

$$(x-y) \cdot \binom{n+\alpha}{n} \cdot \sum_{k=0}^{n} \frac{L_k^{(\alpha)}(x) \cdot L_k^{(\alpha)}(y)}{\binom{k+\alpha}{k}}$$
$$= (n+1) \cdot [L_{n+1}^{(\alpha)}(y) L_n^{(\alpha)}(x) - L_{n+1}^{(\alpha)}(x) L_n^{(\alpha)}(y)]. \tag{2.8}$$

In the so-called confluent form it reads

$$\binom{n+\alpha}{n}\cdot\sum_{k=0}^{n}\frac{\{L_k^{(\alpha)}(x)\}^2}{\binom{k+\alpha}{k}}$$
$$=(n+1)\cdot\left[L_{n+1}^{(\alpha)}(x)\cdot\frac{d}{dx}L_n^{(\alpha)}(x)-L_n^{(\alpha)}(x)\cdot\frac{d}{dx}L_{n+1}^{(\alpha)}(x)\right]. \quad (2.9)$$

Finally we mention the simple differentiation formula $\frac{d}{dx}L_n^{(\alpha)}(x)=-L_{n-1}^{(\alpha+1)}(x)$ or more generally for $k\leqslant n$

$$D^kL_n^{(\alpha)}(x)=(-1)^k\cdot L_{n-k}^{(\alpha+k)}(x). \quad (2.10)$$

This gives us for the definition (1.2)

$$L_n^{\alpha,M_0,M_1,...,M_N}(x)=\sum_{k=0}^{\min(n,N+1)}(-1)^k\cdot A_k\cdot L_{n-k}^{(\alpha+k)}(x). \quad (2.11)$$

3. THE COEFFICIENTS $\{A_k\}_{k=0}^{N+1}$

Now we try to define the coefficients $\{A_k\}_{k=0}^{N+1}$ in such a way that the polynomials $\{L_n^{\alpha,M_0,M_1,...,M_N}(x)\}_{n=0}^{\infty}$ defined by (1.2) or (2.11) are orthogonal with respect to the inner product (1.1).

Let $n\geqslant 1$ and let p denote an arbitrary polynomial of degree $\leqslant n-1$. We want to determine the coefficients $\{A_k\}_{k=0}^{N+1}$, not all zero, such that $\langle p, L_n^{\alpha,M_0,M_1,...,M_N}\rangle=0$. Then $\{L_n^{\alpha,M_0,M_1,...,M_N}(x)\}_{n=0}^{\infty}$ is a set of orthogonal polynomials with respect to the inner product (1.1).

Suppose that the polynomial p can be written as $p(x)=x^{N+1}\cdot q(x)$ for some polynomial q. Then degree $[q]\leqslant n-N-2$ and $n\geqslant N+2$.

In that case we have for $k\leqslant n$

$$\int_0^\infty x^\alpha e^{-x}\cdot p(x)L_{n-k}^{(\alpha+k)}(x)\,dx=\int_0^\infty x^{\alpha+k}e^{-x}\cdot x^{N+1-k}\cdot q(x)L_{n-k}^{(\alpha+k)}(x)\,dx$$

which equals zero in view of the orthogonality property of the classical Laguerre polynomials, since degree $[x^{N+1-k}\cdot q(x)]=N+1-k+\text{degree}[q]\leqslant n-k-1$.

Further we have for $p(x)=x^{N+1}\cdot q(x)$:

$$p^{(\nu)}(0)=0 \qquad \text{for all} \quad \nu\in\{0,1,2,...,N\}.$$

So we have $\langle p, L_n^{\alpha,M_0,M_1,...,M_N}\rangle=0$ if $p(x)=x^{N+1}\cdot q(x)$ for some polynomial q. We conclude: if the coefficients $\{A_k\}_{k=0}^{N+1}$ are chosen in such a way that $\langle p, L_n^{\alpha, M_0, M_1, ..., M_N} \rangle = 0$ for the polynomials $p(x) = x^m$, $m = 0, 1, 2, ..., N$ and $m < n$, then $\langle p, L_n^{\alpha, M_0, M_1, ..., M_N} \rangle = 0$ for every polynomial p with degree $\leqslant n-1$.

Let $p(x) = x^m$ with $m \in \{0, 1, 2, ..., N\}$. Then degree$[p] \leqslant n-1$ implies $n \geqslant m+1$. And for $k \leqslant n$ we have

$$\int_0^\infty x^\alpha e^{-x} \cdot p(x) L_{n-k}^{(\alpha+k)}(x)\, dx = \int_0^\infty x^{\alpha+m} e^{-x} \cdot L_{n-k}^{(\alpha+k)}(x)\, dx.$$

For $m \geqslant k$ we find

$$\int_0^\infty x^{\alpha+m} e^{-x} \cdot L_{n-k}^{(\alpha+k)}(x)\, dx = \int_0^\infty x^{\alpha+k} e^{-x} \cdot x^{m-k} \cdot L_{n-k}^{(\alpha+k)}(x)\, dx = 0$$

since $m-k \leqslant n-k-1$.

Now we use (2.4) and the well-known summation formula ${}_2F_1(-n, b; c; 1) = (c-b)_n/(c)_n$ to find

$$\begin{aligned}
&\int_0^\infty x^{\alpha+m} e^{-x} \cdot L_{n-k}^{(\alpha+k)}(x)\, dx \\
&\quad = \binom{n+\alpha}{n-k} \cdot \sum_{j=0}^{n-k} \frac{(-n+k)_j \cdot \Gamma(\alpha+m+j+1)}{(\alpha+k+1)_j \cdot j!} \\
&\quad = \binom{n+\alpha}{n-k} \cdot \Gamma(m+\alpha+1) \cdot {}_2F_1(-n+k, m+\alpha+1; k+\alpha+1; 1) \\
&\quad = \binom{n-m-1}{n-k} \cdot \Gamma(m+\alpha+1).
\end{aligned} \tag{3.1}$$

For $m < k \leqslant n$ this formula can be found too by using Rodrigues' formula (2.3) for the classical Laguerre polynomials and integration by parts. But later on we use (3.1) for $m = n$.

Further we have

$$p^{(\nu)}(0) = \begin{cases} 0 & \text{for } \nu \neq m \\ m! & \text{for } \nu = m. \end{cases}$$

Hence, $\langle x^m, L_n^{\alpha, M_0, M_1, ..., M_N}(x) \rangle = 0$ for $m = 0, 1, 2, ..., N$ implies, by using (2.2),

$$\begin{aligned}
&\frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)} \cdot \sum_{k=m+1}^{\min(n, N+1)} (-1)^k \cdot \binom{n-m-1}{n-k} \cdot A_k + (-1)^m \cdot m! \cdot M_m \\
&\quad \times \sum_{k=0}^{\min(n, N+1)} (-1)^k \cdot \binom{n+\alpha}{n-k-m} \cdot A_k = 0
\end{aligned}$$

for $m=0, 1, 2, ..., N$. For $n \leqslant N$, m should run to $n-1$. In that case, however, the coefficients $\{A_k\}_{k=n+1}^{N+1}$ in (1.2) are arbitrary. We use this freedom asking for

$$\binom{m+\alpha}{m} \cdot \sum_{k=m+1}^{N+1} (-1)^k \cdot \binom{n-m-1}{n-k} \cdot A_k + (-1)^m \cdot M_m \\ \times \sum_{k=0}^{N+1} (-1)^k \cdot \binom{n+\alpha}{n-m-k} \cdot A_k = 0 \tag{3.2}$$

for $m=0, 1, 2, ..., N$; the number of extra conditions being equal to the number of free parameters. With (3.2) we have found a homogeneous system of $N+1$ equations for the $N+2$ coefficients $\{A_k\}_{k=0}^{N+1}$. So there exists a nontrivial solution.

Note that for $m=N$ in (3.2) we obtain

$$\binom{N+\alpha}{N} \cdot A_{N+1} = M_N \cdot \sum_{k=0}^{N+1} (-1)^k \cdot \binom{n+\alpha}{n-N-k} \cdot A_k \qquad \text{for} \quad n \geqslant N+1.$$

Hence, $A_{N+1}=0$ for $M_N=0$.

We choose the coefficients $\{A_k\}_{k=0}^{N+1}$ in such a way that (3.2) is valid for all n. With this choice we have added some conditions on the coefficients $\{A_k\}_{k=n+1}^{N+1}$ in the case $n \leqslant N$. These conditions imply that $A_k=0$ for $k \in \{n+2, n+3, ..., N+1\}$ and $\binom{n+\alpha}{n} \cdot A_{n+1} = M_n \cdot A_0$ in the case $n \leqslant N$. Thus we find the relation $\binom{n+\alpha}{n} \cdot (A_{n+1} + A_{n+2} + \cdots + A_{N+1}) = M_n \cdot A_0$ for $n \leqslant N$; this implies that the right-hand side of (4.1) has the same form for all n.

From the definition (1.2) it is clear that degree $[L_n^{\alpha, M_0, M_1, ..., M_N}(x)] \leqslant n$, but since $\langle p, L_n^{\alpha, M_0, M_1, ..., M_N} \rangle = 0$ for every polynomial p with degree $\leqslant n-1$ we conclude that degree $[L_n^{\alpha, M_0, M_1, ..., M_N}(x)] = n$.

For the coefficient k_n of x^n in the polynomial $L_n^{\alpha, M_0, M_1, ..., M_N}(x)$ we easily find, by using (2.5),

$$k_n = \frac{(-1)^n}{n!} \cdot A_0, \tag{3.3}$$

from (1.2). Hence $A_0 \neq 0$.

We remark that the coefficients are uniquely determined except for a multiplicative constant. We choose that constant in such a way that $L_n^{\alpha, 0, 0, ..., 0}(x) = L_n^{(\alpha)}(x)$. This proves that the polynomials $\{L_n^{\alpha, M_0, M_1, ..., M_N}(x)\}_{n=0}^{\infty}$ defined by (1.2) with coefficients $\{A_k\}_{k=0}^{N+1}$ satisfying (3.2) are orthogonal with respect to (1.1).

4. The Squared Norm

First of all we prove that

$$\langle L_n^{\alpha, M_0, M_1, \ldots, M_N}, L_n^{\alpha, M_0, M_1, \ldots, M_N} \rangle = \binom{n+\alpha}{n} \cdot A_0 \cdot (A_0 + A_1 + \cdots + A_{N+1}). \tag{4.1}$$

From this we see that

$$A_0 \cdot (A_0 + A_1 + \cdots + A_{N+1}) > 0. \tag{4.2}$$

By using (3.3) we easily see that

$$\langle L_n^{\alpha, M_0, M_1, \ldots, M_N}, L_n^{\alpha, M_0, M_1, \ldots, M_N} \rangle = \frac{(-1)^n}{n!} \cdot A_0 \cdot \langle x^n, L_n^{\alpha, M_0, M_1, \ldots, M_N}(x) \rangle. \tag{4.3}$$

Now we use definition (2.11) to find, with (3.1),

$$\begin{aligned} \langle x^n, L_n^{\alpha, M_0, M_1, \ldots, M_N}(x) \rangle &= \sum_{k=0}^{N+1} \frac{(-1)^k}{\Gamma(\alpha+1)} \cdot A_k \cdot \int_0^\infty x^{\alpha+n} e^{-x} \cdot L_{n-k}^{(\alpha+k)}(x)\, dx \\ &= (-1)^n \cdot \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \cdot \sum_{k=0}^{N+1} A_k, \end{aligned} \tag{4.4}$$

for $n \geqslant N+1$. Hence with (4.3) and (4.4) we have proved (4.1) in the case $n \geqslant N+1$.

In the case $n \leqslant N$ we find

$$\langle x^n, L_n^{\alpha, M_0, M_1, \ldots, M_N}(x) \rangle = (-1)^n \cdot \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \cdot \sum_{k=0}^{n} A_k + (-1)^n \cdot n! \cdot M_n \cdot A_0.$$

Now we apply (3.2) for $m = n$ to see that

$$M_n \cdot A_0 = \binom{n+\alpha}{n} \cdot \sum_{k=n+1}^{N+1} A_k.$$

Hence

$$\begin{aligned} \langle x^n, L_n^{\alpha, M_0, M_1, \ldots, M_N}(x) \rangle &= (-1)^n \cdot \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \cdot \sum_{k=0}^{n} A_k + (-1)^n \cdot n! \\ &\quad \times \binom{n+\alpha}{n} \cdot \sum_{k=n+1}^{N+1} A_k. \end{aligned} \tag{4.5}$$

And with (4.3) and (4.5) we have proved (4.1) and therefore (4.2).

So we have obtained the following orthogonality relation

$$\frac{1}{\Gamma(\alpha+1)}\cdot\int_0^\infty x^\alpha e^{-x}\cdot L_m^{\alpha,M_0,M_1,\ldots,M_N}(x)\,L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\,dx$$
$$+\sum_{\nu=0}^{N} M_\nu\cdot(D^\nu L_m^{\alpha,M_0,M_1,\ldots,M_N})(0)\cdot(D^\nu L_n^{\alpha,M_0,M_1,\ldots,M_N})(0)$$
$$=\binom{n+\alpha}{n}\cdot A_0\cdot(A_0+A_1+\cdots+A_{N+1})\cdot\delta_{mn}.$$

This can be seen as a generalization of (2.1).

5. A Differential Equation

In [4] we found a second order differential equation for our polynomials in the case $N=1$. The same method can be used in the general case, but in [3] J. Koekoek gave a simple proof of the differential equation. We give this proof here.

We prove the following

THEOREM. *The polynomials* $\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\}_{n=0}^{\infty}$ *satisfy a second order differential equation of the form*

$$x\cdot p_2(x)\cdot y''(x)-p_1(x)\cdot y'(x)+n\cdot p_0(x)\cdot y(x)=0, \tag{5.1}$$

where $\{p_k(x)\}_{k=0}^{2}$ are polynomials with

$$\begin{cases} p_2(x)=A_0\cdot(A_0+A_1+\cdots+A_{N+1})\cdot x^{N+1}+\textit{lower order terms}\\ p_1(x)=A_0\cdot(A_0+A_1+\cdots A_{N+1})\cdot x^{N+2}+\textit{lower order terms}\\ p_0(x)=A_0\cdot(A_0+A_1+\cdots+A_{N+1})\cdot x^{N+1}+\textit{lower order terms.}\end{cases} \tag{5.2}$$

Proof. We start with the differential equation (2.6) for the classical Laguerre polynomials

$$x\cdot\frac{d^2}{dx^2}L_n^{(\alpha)}(x)+(\alpha+1-x)\cdot\frac{d}{dx}L_n^{(\alpha)}(x)+n\cdot L_n^{(\alpha)}(x)=0. \tag{5.3}$$

Differentiation of (5.3) leads to

$$x\cdot D^{k+2}L_n^{(\alpha)}(x)+(\alpha+k+1-x)\cdot D^{k+1}L_n^{(\alpha)}(x)+(n-k)\cdot D^kL_n^{(\alpha)}(x)=0 \tag{5.4}$$

for $k \in \mathbb{N}$. By using $k = N-1$ in (5.4) we find

$$x \cdot L_n^{\alpha, M_0, M_1, \ldots, M_N}(x) = \sum_{k=0}^{N} b_k(x) \cdot D^k L_n^{(\alpha)}(x),$$

where

$$\begin{cases} b_k(x) = A_k \cdot x, \qquad k = 0, 1, 2, \ldots, N-2 \\ b_{N-1}(x) = A_{N-1} \cdot x - (n-N+1) \cdot A_{N+1} \\ b_N(x) = A_N \cdot x - (\alpha + N - x) \cdot A_{N+1}. \end{cases}$$

Then we use $k = N-2$ in (5.4) to obtain

$$x^2 \cdot L_n^{\alpha, M_0, M_1, \ldots, M_N}(x) = \sum_{k=0}^{N-1} b_k^*(x) \cdot D^k L_n^{(\alpha)}(x),$$

where

$$\begin{cases} b_k^*(x) = x \cdot b_k(x), \qquad k = 0, 1, 2, \ldots, N-3 \\ b_{N-2}^*(x) = x \cdot b_{N-2}(x) - (n-N+2) \cdot b_N(x) \\ b_{N-1}^*(x) = x \cdot b_{N-1}(x) - (\alpha + N - 1 - x) \cdot b_N(x). \end{cases}$$

Repeating this process we finally obtain, by using $k = 0$ in (5.4),

$$x^N \cdot L_n^{\alpha, M_0, M_1, \ldots, M_N}(x) = q_0(x) \cdot L_n^{(\alpha)}(x) + q_1(x) \cdot \frac{d}{dx} L_n^{(\alpha)}(x) \tag{5.5}$$

for some polynomials q_0 and q_1 with

$$\begin{cases} q_0(x) = A_0 \cdot x^N + \text{lower order terms} \\ q_1(x) = (A_1 + A_2 + \cdots + A_{N+1}) \cdot x^N + \text{lower order terms.} \end{cases} \tag{5.6}$$

Differentiation of (5.5) gives

$$x^N \cdot \frac{d}{dx} L_n^{\alpha, M_0, M_1, \ldots, M_N}(x) + N \cdot x^{N-1} \cdot L_n^{\alpha, M_0, M_1, \ldots, M_N}(x)$$
$$= q_0'(x) \cdot L_n^{(\alpha)}(x) + [q_0(x) + q_1'(x)] \cdot \frac{d}{dx} L_n^{(\alpha)}(x) + q_1(x) \cdot \frac{d^2}{dx^2} L_n^{(\alpha)}(x).$$

Now we multiply by x and use (5.3) and (5.5) to find

$$x^{N+1} \cdot \frac{d}{dx} L_n^{\alpha, M_0, M_1, \ldots, M_N}(x) = r_0(x) \cdot L_n^{(\alpha)}(x) + r_1(x) \cdot \frac{d}{dx} L_n^{(\alpha)}(x), \tag{5.7}$$

where

$$\begin{cases} r_0(x)=x\cdot q_0'(x)-N\cdot q_0(x)-n\cdot q_1(x) \\ r_1(x)=x\cdot q_0(x)+x\cdot q_1'(x)+(x-\alpha-N-1)\cdot q_1(x). \end{cases} \tag{5.8}$$

It follows from (5.6) and (5.8) that

$$\begin{cases} r_0(x)=-n\cdot(A_1+A_2+\cdots+A_{N+1})\cdot x^N+\text{lower order terms} \\ r_1(x)=(A_0+A_1+A_2+\cdots+A_{N+1})\cdot x^{N+1}+\text{lower order terms.} \end{cases} \tag{5.9}$$

In the same way we obtain from (5.7) by using (5.3)

$$x^{N+2}\cdot\frac{d^2}{dx^2}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)=s_0(x)\cdot L_n^{(\alpha)}(x)+s_1(x)\cdot\frac{d}{dx}L_n^{(\alpha)}(x), \tag{5.10}$$

where

$$\begin{cases} s_0(x)=x\cdot r_0'(x)-(N+1)\cdot r_0(x)-n\cdot r_1(x) \\ s_1(x)=x\cdot r_0(x)+x\cdot r_1'(x)+(x-\alpha-N-2)\cdot r_1(x). \end{cases} \tag{5.11}$$

And with (5.9) and (5.11) we have

$$\begin{cases} s_0(x)=-n\cdot(A_0+A_1+A_2+\cdots+A_{N+1})\cdot x^{N+1}+\text{lower order terms} \\ s_1(x)=(A_0+A_1+A_2+\cdots+A_{N+1})\cdot x^{N+2}+\text{lower order terms.} \end{cases} \tag{5.12}$$

Now we eliminate the derivative of the classical Laguerre polynomial from (5.5) and (5.7) to find

$$\begin{aligned}&[q_0(x)r_1(x)-q_1(x)r_0(x)]\cdot L_n^{(\alpha)}(x)\\ &\quad=x^N\cdot\left[r_1(x)\cdot L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)-x\cdot q_1(x)\cdot\frac{d}{dx}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right].\end{aligned}$$

Since $L_n^{(\alpha)}(0)=\binom{n+\alpha}{n}$ we conclude that

$$q_0(x)r_1(x)-q_1(x)r_0(x)=x^N\cdot p_2(x) \tag{5.13}$$

for some polynomial p_2.

In the same way we obtain from (5.5) and (5.10)

$$q_0(x)s_1(x)-q_1(x)s_0(x)=x^N\cdot p_1(x) \tag{5.14}$$

for some polynomial p_1. And from (5.7) and (5.10) it follows that

$$r_0(x)s_1(x)-r_1(x)s_0(x)=n\cdot x^{N+1}\cdot p_0(x) \tag{5.15}$$

for some polynomial p_0. Here we used the fact that for $n=0$ we have

$$q_0(x)=A_0\cdot x^N \qquad \text{and} \qquad r_0(x)=s_0(x)=0$$

which follows from (5.5), (5.8), and (5.11).

In view of (5.5), (5.7), and (5.10) the determinant

$$\begin{vmatrix} x^N\cdot L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & q_0(x) & q_1(x) \\ x^{N+1}\cdot\dfrac{d}{dx}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & r_0(x) & r_1(x) \\ x^{N+2}\cdot\dfrac{d^2}{dx^2}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & s_0(x) & s_1(x) \end{vmatrix}$$

must be zero. The first column can be divided by x^N. Hence, we find by using (5.13), (5.14), and (5.15)

$$0=\begin{vmatrix} L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & q_0(x) & q_1(x) \\ x\cdot\dfrac{d}{dx}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & r_0(x) & r_1(x) \\ x^2\cdot\dfrac{d^2}{dx^2}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & s_0(x) & s_1(x) \end{vmatrix}$$
$$=x^{N+2}\cdot p_2(x)\cdot\frac{d^2}{dx^2}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)-x^{N+1}\cdot p_1(x)\cdot\frac{d}{dx}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)$$
$$+x^{N+1}\cdot n\cdot p_0(x)\cdot L_n^{\alpha,M_0,M_1,\ldots,M_N}(x).$$

This proves (5.1). Now (5.2) follows from (5.13), (5.14), and (5.15) by using (5.6), (5.9), and (5.12). This proves the theorem.

6. Representation as Hypergeometric Series

From (1.2) and (2.4) we obtain

$$L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)=\binom{n+\alpha}{n}\cdot\sum_{k=0}^{N+1}A_k\cdot D^k{}_1F_1(-n;\alpha+1;x)$$
$$=\binom{n+\alpha}{n}\cdot\sum_{m=0}^{n}C_m\cdot\frac{x^m}{m!},$$

where

$$C_m=\sum_{k=0}^{N+1}\frac{(-n)_{m+k}}{(\alpha+1)_{m+k}}\cdot A_k$$
$$=\frac{(-n)_m}{(\alpha+1)_{N+m+1}}\sum_{k=0}^{N+1}(m-n)_k\cdot(m+\alpha+k+1)_{N-k+1}\cdot A_k.$$

From (4.2) it follows that $A_0+A_1+\cdots+A_{N+1}\neq 0$. So we may write

$$C_m=(A_0+A_1+\cdots+A_{N+1})\cdot\frac{(-n)_m}{(\alpha+N+2)_m}\cdot\frac{(m+\beta_0)(m+\beta_1)\cdots(m+\beta_N)}{(\alpha+1)_{N+1}}$$

for certain $\beta_j\in\mathbb{C}$, $j=0, 1, 2, ..., N$. Since $m+\beta_j=\beta_j\cdot(\beta_j+1)_m/(\beta_j)_m$ for $\beta_j\neq 0, -1, -2, ...$ we find in that case

$$L_n^{\alpha, M_0, M_1, ..., M_N}(x)=\frac{\beta_0\beta_1\cdots\beta_N}{(\alpha+1)_{N+1}}\cdot\binom{n+\alpha}{n}\cdot(A_0+A_1+\cdots+A_{N+1})$$
$$\times\,{}_{N+2}F_{N+2}\left(\begin{matrix}-n, \beta_0+1, \beta_1+1, ..., \beta_N+1\\ \alpha+N+2, \beta_0, \beta_1, ..., \beta_N\end{matrix}\,\middle|\, x\right). \quad (6.1)$$

For $-\beta_j\in\mathbb{N}$ we must take the analytic continuation of (6.1).

We remark that (6.1) is a generalization of (2.4).

7. Recurrence Relation

All sets of polynomials which are orthogonal with respect to a positive weight function satisfy a three term recurrence relation. The classical Laguerre polynomials for instance, satisfy (2.7). The polynomials $\{L_n^{\alpha, M_0, M_1, ..., M_N}(x)\}_{n=0}^{\infty}$ in general fail to have this property, but we can prove the following

Theorem. *The polynomials* $\{L_n^{\alpha, M_0, M_1, ..., M_N}(x)\}_{n=0}^{\infty}$ *satisfy a* $(2N+3)$-*terms recurrence relation of the form*

$$x^{N+1}\cdot L_n^{\alpha, M_0, M_1, ..., M_N}(x)=\sum_{k=n-N-1}^{n+N+1} E_k^{(n)}\cdot L_k^{\alpha, M_0, M_1, ..., M_N}(x). \quad (7.1)$$

Proof. Since $x^{N+1}\cdot L_n^{\alpha, M_0, M_1, ..., M_N}(x)$ is a polynomial of degree $n+N+1$ we may write

$$x^{N+1}\cdot L_n^{\alpha, M_0, M_1, ..., M_N}(x)=\sum_{k=0}^{n+N+1} E_k^{(n)}\cdot L_k^{\alpha, M_0, M_1, ..., M_N}(x) \quad (7.2)$$

for some coefficients $E_k^{(n)}\in\mathbb{R}$, $k=0, 1, 2, ..., n+N+1$.

Taking the inner product with $L_m^{\alpha, M_0, M_1, ..., M_N}(x)$ on both sides of (7.2) we find by using (1.1)

$$\begin{aligned}&\langle L_m^{\alpha, M_0, M_1, ..., M_N}, L_m^{\alpha, M_0, M_1, ..., M_N}\rangle\cdot E_m^{(n)}\\ &\quad=\langle x^{N+1}\cdot L_n^{\alpha, M_0, M_1, ..., M_N}(x), L_m^{\alpha, M_0, M_1, ..., M_N}(x)\rangle\\ &\quad=\langle x^{N+1}\cdot L_m^{\alpha, M_0, M_1, ..., M_N}(x), L_n^{\alpha, M_0, M_1, ..., M_N}(x)\rangle. \quad (7.3)\end{aligned}$$

In view of the orthogonality property of the polynomials $\{L_n^{\alpha, M_0, M_1, \ldots, M_N}(x)\}_{n=0}^{\infty}$ we conclude that $E_m^{(n)} = 0$ for $N+1+m \leqslant n-1$ or $m \leqslant n-N-2$. This proves (7.1). Comparing the leading coefficients on both sides of (7.1) we obtain by using (3.3)

$$E_{n+N+1}^{(n)} = \frac{k_n}{k_{n+N+1}} = (-1)^{N+1} \cdot \frac{(n+N+1)!}{n!} \cdot \frac{A_0(n)}{A_0(n+N+1)} \neq 0.$$

Here we wrote $A_0(n)$ instead of A_0, since A_0 depends on n.

If we define

$$\begin{aligned} \Lambda_n &:= \langle L_n^{\alpha, M_0, M_1, \ldots, M_N}, L_n^{\alpha, M_0, M_1, \ldots, M_N} \rangle \\ &= \binom{n+\alpha}{n} \cdot A_0 \cdot (A_0 + A_1 + \cdots + A_{N+1}) \end{aligned}$$

then we find for $E_{n-N-1}^{(n)}$ by using (7.3) and (3.3)

$$E_{n-N-1}^{(n)} = \frac{k_{n-N-1} \cdot \Lambda_n}{\Lambda_{n-N-1} \cdot k_n} \neq 0.$$

The $(2N+3)$ – terms recurrence relation (7.1) clearly is a generalization of (2.7).

Remark. In (7.1) we take $L_k^{\alpha, M_0, M_1, \ldots, M_N}(x) \equiv 0$ for $k < 0$.

8. A Christoffel–Darboux Type Formula

From the recurrence relation (7.1) we easily obtain

$$\begin{aligned} &(x^{N+1} - y^{N+1}) \cdot L_k^{\alpha, M_0, M_1, \ldots, M_N}(x) L_k^{\alpha, M_0, M_1, \ldots, M_N}(y) \\ &= \sum_{m=k-N-1}^{k+N+1} E_m^{(k)} \cdot [L_m^{\alpha, M_0, M_1, \ldots, M_N}(x) L_k^{\alpha, M_0, M_1, \ldots, M_N}(y) \\ &\quad - L_m^{\alpha, M_0, M_1, \ldots, M_N}(y) L_k^{\alpha, M_0, M_1, \ldots, M_N}(x)]. \end{aligned} \tag{8.1}$$

Now we use (7.3) to see that $E_m^{(k)}/\Lambda_k = E_k^{(m)}/\Lambda_m$. So it follows from (8.1) by using

$$\sum_{k=0}^{n} \sum_{m=k-N-1}^{k+N+1} = \sum_{k=0}^{n} \sum_{m=0}^{k+N+1} = \sum_{k=0}^{n} \sum_{m=0}^{n} + \sum_{k=0}^{n} \sum_{m=n+1}^{k+N+1},$$

since the first sum at the right-hand side vanishes, that

$$(x^{N+1}-y^{N+1})\cdot\sum_{k=0}^{n}\Lambda_k^{-1}\cdot L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y)$$
$$=\sum_{k=0}^{n}\sum_{m=n+1}^{k+N+1}\frac{E_m^{(k)}}{\Lambda_k}\cdot[L_m^{\alpha,M_0,M_1,\ldots,M_N}(x)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y)$$
$$-L_m^{\alpha,M_0,M_1,\ldots,M_N}(y)L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)]. \tag{8.2}$$

This can be seen as a Christoffel–Darboux type formula. Note that (8.2) is a generalization of (2.8). We remark that for $n\geqslant N$ we may write

$$\sum_{k=0}^{n}\sum_{m=n+1}^{k+N+1}=\sum_{k=n-N}^{n}\sum_{m=n+1}^{k+N+1}.$$

The right-hand side of (8.2) consists of at most $\frac{1}{2}\cdot(N+1)(N+2)$ summands opposed to the single bracketed "term" in the classical Christoffel–Darboux formula. And if $n<N$, there are fewer terms.

If we divide by $x-y$ and let y tend to x then we obtain the so-called confluent form of the Christoffel–Darboux type formula

$$(N+1)\cdot x^N\cdot\sum_{k=0}^{n}\Lambda_k^{-1}\cdot\{L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)\}^2$$
$$=\sum_{k=0}^{n}\sum_{m=n+1}^{k+N+1}\frac{E_m^{(k)}}{\Lambda_k}\cdot\left[L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)\cdot\frac{d}{dx}L_m^{\alpha,M_0,M_1,\ldots,M_N}(x)\right.$$
$$\left.-L_m^{\alpha,M_0,M_1,\ldots,M_N}(x)\cdot\frac{d}{dx}L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)\right]. \tag{8.3}$$

Note that (8.3) is a generalization of (2.9).

9. Another Definition

Instead of by (1.2) or by (2.11) the polynomials $\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\}_{n=0}^{\infty}$ can be defined by

$$L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)=\sum_{k=0}^{N+1}B_k\cdot x^k\cdot D^kL_n^{(\alpha+k)}(x). \tag{9.1}$$

As before we write by using (2.10)

$$L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)=\sum_{k=0}^{\min(n,N+1)}(-1)^k\cdot B_k\cdot x^k\cdot L_{n-k}^{(\alpha+2k)}(x). \tag{9.2}$$

By comparing (1.2) and (9.2) we see that

$$A_0 = \sum_{k=0}^{N+1} (-n)_k \cdot (-1)^k \cdot B_k$$

and by using (2.2)

$$\binom{n+\alpha}{n} \cdot B_0 = \sum_{k=0}^{N+1} (-1)^k \cdot \binom{n+\alpha}{n-k} \cdot A_k.$$

The definition (9.1) can be proved by using the same method as in Section 3. Now we find

$$\frac{1}{\Gamma(\alpha+1)} \cdot \sum_{k=m+1}^{N+1} (-1)^k \cdot \binom{n-m-1}{n-k} \cdot \Gamma(m+k+\alpha+1) \cdot B_k$$
$$+ (-1)^m \cdot m! \cdot M_m \cdot \sum_{k=0}^{m} k! \cdot \binom{m}{k} \binom{n+\alpha+k}{n-m} \cdot B_k = 0$$

for $m = 0, 1, 2, ..., N$. This is a homogeneous system of $N+1$ equations for the $N+2$ coefficients $\{B_k\}_{k=0}^{N+1}$. Hence there is a nontrivial solution.

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