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Generalizations of a q-Analogue of Laguerre Polynomials

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We study polynomials $\{L_n^{\alpha, M_0, M_1, ..., M_N}(x; q)\}_{n=0}^{\infty}$ orthogonal with respect to the inner product

$$\langle f, g\rangle = \frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\,\Gamma(\alpha+1)} \int_0^{\infty} \frac{x^{\alpha}}{(-(1-q)x; q)_{\infty}} f(x)\, g(x)\, dx + \sum_{\nu=0}^{N} M_{\nu} (D_q^{\nu} f)(0)(D_q^{\nu} g)(0),$$

where $\alpha > -1$, N is an integer, and $M_{\nu} \geqslant 0$ for all $\nu \in \{0, 1, 2, ..., N\}$. These polynomials are q-analogues of the polynomials $\{L_n^{\alpha, M_0, M_1, ..., M_N}(x)\}_{n=0}^{\infty}$ orthogonal with respect to the (Sobolev) inner product

$$\langle f, g\rangle = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} x^{\alpha} e^{-x} f(x)\, g(x)\, dx + \sum_{\nu=0}^{N} M_{\nu} f^{(\nu)}(0)\, g^{(\nu)}(0).$$

We prove the orthogonality relation for which we give a discrete form (q-integral) too. We give a representation as a basic hypergeometric series, a recurrence relation is derived, a Christoffel–Darboux type formula and a second order q-difference equation satisfied by these new basic orthogonal polynomials.

1. INTRODUCTION

In [12] we studied the polynomials $\{L_n^{\alpha, M_0, M_1, ..., M_N}(x)\}_{n=0}^{\infty}$ which are orthogonal with respect to the (Sobolev) inner product

$$\langle f, g\rangle = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} x^{\alpha} e^{-x} f(x)\, g(x)\, dx + \sum_{\nu=0}^{N} M_{\nu} f^{(\nu)}(0)\, g^{(\nu)}(0), \tag{1.1}$$

where $\alpha > -1$, N is an integer, and $M_{\nu} \geqslant 0$ for all $\nu \in \{0, 1, 2, ..., N\}$. These

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polynomials are generalizations of the classical Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ and can be defined by

$$L_n^{\alpha, M_0, M_1, \ldots, M_N}(x) = \sum_{k=0}^{N+1} A_k D^k L_n^{(\alpha)}(x)$$

for certain coefficients $\{A_k\}_{k=0}^{N+1}$. The special case $N=1$ was treated in [15].

Note that for $N>0$ the inner product defined by (1.1) cannot be obtained from a weight function. That is why the polynomials $\{L_n^{\alpha, M_0, M_1, \ldots, M_N}(x)\}_{n=0}^{\infty}$ have some properties which differ from the well-known properties of the classical orthogonal polynomials (see for instance [3, 18]). For $N=0$ these polynomials reduce to the polynomials $\{L_n^{\alpha, M}(x)\}_{n=0}^{\infty}$ found by Koornwinder in [16]. The most important properties of Koornwinder's generalized Laguerre polynomials can be found in [10]. In [8] J. Koekoek and R. Koekoek proved that these polynomials $\{L_n^{\alpha, M}(x)\}_{n=0}^{\infty}$ in general satisfy an infinite order differential equation. For integer values of α this differential equation is of order $2\alpha+4$.

In [9] we studied a q-analogue of Koornwinder's generalized Laguerre polynomials. These polynomials $\{L_n^{\alpha, M}(x; q)\}_{n=0}^{\infty}$ are generalizations of Moak's q-Laguerre polynomials described in [17].

In [11] we studied further generalizations of these q-Laguerre polynomials. The polynomials described in [11] are q-analogues of the polynomials $\{L_n^{\alpha, M_0, M_1, \ldots, M_N}(x)\}_{n=0}^{\infty}$ in the special case $N=1$.

Now it is the aim of the present paper to find the q-analogues of the polynomials $\{L_n^{\alpha, M_0, M_1, \ldots, M_N}(x)\}_{n=0}^{\infty}$ in the general case. These q-orthogonal polynomials will be denoted by $\{L_n^{\alpha, M_0, M_1, \ldots, M_N}(x; q)\}_{n=0}^{\infty}$.

2. Some Basic Formulas

First we summarize some definitions and formulas we need from the q-theory. For details the reader is referred to [4].

We always take $0<q<1$ in the sequel.

The q-shifted factorial is defined by

$$\begin{cases} (a; q)_0 = 1 \\ (a; q)_n = (1-a)(1-aq)(1-aq^2) \cdots (1-aq^{n-1}), \qquad n = 1, 2, 3, \ldots. \end{cases}$$

For negative subscripts the q-shifted factorial is defined by

$$(a; q)_{-n} = \frac{1}{(1-aq^{-n})(1-aq^{-n+1}) \cdots (1-aq^{-1})}, \qquad a \neq q, q^2, q^3, \ldots, q^n,\ n = 1, 2, 3, \ldots. \tag{2.1}$$

Further we have for all integers n

$$(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

where

$$(a;q)_\infty:=\prod_{k=0}^{\infty}(1-aq^k).$$

We will use two simple formulas involving these q-shifted factorials:

$$(a;q)_{n+k}=(a;q)_n\,(aq^n;q)_k,\qquad k,n=0,1,2,... \tag{2.2}$$

and

$$(a^{-1}q^{1-n};q)_n=(-a^{-1})^n\,q^{-\binom{n}{2}}(a;q)_n,\qquad a\neq 0,\ n=0,1,2,.... \tag{2.3}$$

We have a q-analogue of the binomial coefficient given by

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}. \tag{2.4}$$

It is easy to see that

$$\lim_{q\uparrow 1}\begin{bmatrix} n\\ k\end{bmatrix}_q=\binom{n}{k}.$$

The basic hypergeometric series or q-hypergeometric series is defined by

$$\begin{aligned}&{}_r\phi_s\left(\left.\begin{matrix}a_1,a_2,...,a_r\\ b_1,b_2,...,b_s\end{matrix}\,\right|\,q;z\right)\\ &\quad=\sum_{n=0}^{\infty}\frac{(a_1,a_2,...,a_r;q)_n}{(b_1,b_2,...,b_s;q)_n}\,\frac{(-1)^{(1+s-r)n}\,q^{(1+s-r)\binom{n}{2}}z^n}{(q;q)_n},\end{aligned}$$

where

$$(a_1,a_2,...,a_r;q)_n:=(a_1;q)_n\,(a_2;q)_n\cdots(a_r;q)_n.$$

The q-hypergeometric series is a q-analogue of the hypergeometric series since

$$\begin{aligned}&\lim_{q\uparrow 1}{}_r\phi_s\left(\left.\begin{matrix}q^{\alpha_1},q^{\alpha_2},...,q^{\alpha_r}\\ q^{\beta_1},q^{\beta_2},...,q^{\beta_s}\end{matrix}\,\right|\,q;(q-1)^{1+s-r}z\right)\\ &\quad={}_rF_s\left(\left.\begin{matrix}\alpha_1,\alpha_2,...,\alpha_r\\ \beta_1,\beta_2,...,\beta_s\end{matrix}\,\right|\,z\right).\end{aligned}$$

The q-binomial theorem

$$ {}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix}\,\middle|\, q; z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z|<1 $$

is a q-analogue of Newton's binomial series. If $a=0$ this leads to

$$ e_q(z) := {}_1\phi_0\left(\begin{matrix} 0 \\ - \end{matrix}\,\middle|\, q; z\right) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \qquad |z|<1, \tag{2.5} $$

which can be seen as a q-analogue of the exponential function since

$$ \lim_{q\uparrow 1} e_q((1-q)z) = e^z. $$

We will use another summation formula

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix}\,\middle|\, q; \frac{cq^n}{b}\right) = \frac{(c/b;q)_n}{(c;q)_n} \tag{2.6} $$

which is often referred to as the q-Vandermonde summation formula.

The q-difference operator D_q is defined by

$$ D_q f(x) := \begin{cases} \dfrac{f(x)-f(qx)}{(1-q)x}, & x \neq 0 \\ f'(0), & x=0, \end{cases} \tag{2.7} $$

where the function f is differentiable in a neighbourhood of $x=0$. We easily see that

$$ \lim_{q\uparrow 1} D_q f(x) = f'(x). $$

For functions f analytic in a neighbourhood of $x=0$ this implies

$$ \begin{aligned} (D_q^n f)(0) &:= (D_q(D_q^{n-1}f))(0) \\ &= \frac{f^{(n)}(0)}{n!} \frac{(q;q)_n}{(1-q)^n}, \qquad n=1,2,3,.... \end{aligned} \tag{2.8} $$

An easy consequence of the definition (2.7) is

$$ D_q^n[f(\gamma x)] = \gamma^n (D_q^n f)(\gamma x), \qquad \gamma \text{ real and } n=0,1,2,.... \tag{2.9} $$

Further, we easily find from (2.7)

$$ D_q[f(x)\,g(x)] = f(qx)\,D_q g(x) + g(x)\,D_q f(x) \tag{2.10} $$

which is often referred to as the q-product rule. This q-product rule can be generalized to a q-analogue of Leibniz' rule

$$D_q^n[f(x)\,g(x)] = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (D_q^{n-k}f)(q^k x)(D_q^k g)(x), \tag{2.11}$$

where $\left[{n \atop k}\right]_q$ denotes the q-binomial coefficient defined by (2.4).

The q-integral of a function f on $(0, \infty)$ is defined by

$$\int_0^\infty f(t)\,d_q t := (1-q) \sum_{k=-\infty}^{\infty} f(q^k) q^k, \tag{2.12}$$

provided that the sum on the right-hand side converges. This definition of the q-integral on $(0, \infty)$ is due to F. H. Jackson. See [6]. For more details concerning q-integrals the reader is referred to Section 1.11 of the book [4]. It can be shown that

$$\lim_{q \uparrow 1} \int_0^\infty f(t)\,d_q t = \int_0^\infty f(t)\,dt$$

for functions f which satisfy suitable conditions. For details the reader is referred to [1] and to references given in [4].

In [5] Jackson defined a q-analogue of the gamma function:

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}. \tag{2.13}$$

Note that this q-gamma function $\Gamma_q(x)$ satisfies the functional equation

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x), \qquad \Gamma_q(1) = 1.$$

Jackson also showed that

$$\lim_{q \uparrow 1} \Gamma_q(x) = \Gamma(x).$$

For details the reader is referred to [1] and to Section 1.10 of [4].

In [2] R. Askey gave a proof of the following integral formula which is due to Ramanujan:

$$\int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty}\,dx = \frac{\Gamma(-\alpha)\,\Gamma(\alpha+1)}{\Gamma_q(-\alpha)} > 0, \qquad \alpha > -1. \tag{2.14}$$

If $\alpha=k$ is a nonnegative integer we have to take the analytic continuation

$$\begin{aligned}\lim_{\alpha\to k}\frac{\Gamma(-\alpha)\,\Gamma(\alpha+1)}{\Gamma_q(-\alpha)}&=\lim_{\alpha\to k}\frac{(-\alpha+k)\,\Gamma(-\alpha)}{(-\alpha+k)\,\Gamma_q(-\alpha)}\,\Gamma(\alpha+1)\\&=\frac{(-1)^k}{k!}\,\frac{(q^{-k};q)_k\ln q^{-1}}{(1-q)^{k+1}}\,\Gamma(k+1)\\&=\frac{(q;q)_k\,q^{-\binom{k+1}{2}}\ln q^{-1}}{(1-q)^{k+1}}.\end{aligned}$$

For the residue of the q-gamma function the reader is referred to formula (1.10.6) in [4]. We remark that we have in view of (2.5)

$$\frac{1}{(-(1-q)x;q)_\infty}=e_q(-(1-q)x)\to e^{-x}\qquad\text{as}\quad q\uparrow 1.$$

Finally we have a basic bilateral series which is defined by

$$\begin{aligned}&{}_r\psi_s\left(\begin{matrix}a_1,a_2,...,a_r\\b_1,b_2,...,b_s\end{matrix}\,\middle|\,q;z\right)\\&=\sum_{n=-\infty}^{\infty}\frac{(a_1,a_2,...,a_r;q)_n}{(b_1,b_2,...,b_s;q)_n}(-1)^{(s-r)n}\,q^{(s-r)\binom{n}{2}}z^n.\end{aligned}$$

The special case $r=s=1$ can be summed:

$$\begin{aligned}{}_1\psi_1\left(\begin{matrix}a\\b\end{matrix}\,\middle|\,q;z\right)&=\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n\\&=\frac{(q,a^{-1}b,az,a^{-1}z^{-1}q;q)_\infty}{(b,a^{-1}q,z,a^{-1}z^{-1}b;q)_\infty},\qquad |a^{-1}b|<|z|<1.\end{aligned}\tag{2.15}$$

This summation formula is due to Ramanujan. A proof of this summation formula can be found in [2, 4].

3. The Definition and Properties of the q-Laguerre Polynomials

In this section we state the definition and some properties of the q-Laguerre polynomials $\{L_n^{(\alpha)}(x;q)\}_{n=0}^{\infty}$. These q-Laguerre polynomials were studied in detail by D. S. Moak in [17]. For more details concerning these polynomials the reader is referred to [9, 17].

Let $\alpha>-1$.

The q-Laguerre polynomials $\{L_n^{(\alpha)}(x;q)\}_{n=0}^{\infty}$ are defined by

$$L_n^{(\alpha)}(x;q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n} \times \sum_{k=0}^{n} \frac{(q^{-n};q)_k\, q^{-\binom{k}{2}}(1-q)^k\,(q^{n+\alpha+1}x)^k}{(q^{\alpha+1};q)_k\,(q;q)_k}, \qquad n=0,1,2,\ldots \tag{3.1}$$

We easily see that

$$\lim_{q\uparrow 1} L_n^{(\alpha)}(x;q)=L_n^{(\alpha)}(x),$$

where $L_n^{(\alpha)}(x)$ denotes the classical Laguerre polynomial.

By using (2.3) we obtain

$$L_n^{(\alpha)}(x;q)=(-1)^n q^{n(n+\alpha)}\frac{(1-q)^n}{(q;q)_n}x^n + \text{lower order terms}, \qquad n=0,1,2,\ldots \tag{3.2}$$

The orthogonality relation for these q-Laguerre polynomials $\{L_n^{(\alpha)}(x;q)\}_{n=0}^{\infty}$ can be written as

$$\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\,\Gamma(\alpha+1)}\int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty} L_m^{(\alpha)}(x;q)\,L_n^{(\alpha)}(x;q)\,dx = \frac{(q^{\alpha+1};q)_n}{(q;q)_n\,q^n}\,\delta_{mn}. \tag{3.3}$$

This orthogonality relation can also be written as

$$\frac{1}{A}\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty} L_m^{(\alpha)}(cq^k;q)\,L_n^{(\alpha)}(cq^k;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n\,q^n}\,\delta_{mn}, \qquad c>0, \tag{3.4}$$

where the normalization factor A equals

$$A=\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty}.$$

This can be shown by proving that

$$\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\,\Gamma(\alpha+1)}\int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty} P(x)\,dx = \frac{1}{A}\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty} P(cq^k) \tag{3.5}$$

for every polynomial P. To do this take for instance $P(x) = (-(1-q)x; q)_m$ where m is a nonnegative integer. Then we easily see that both sides of (3.5) equal $q^{-(\alpha+1)m}$.

By using the fact that

$$(-c(1-q)q^k; q)_\infty = \frac{(-c(1-q); q)_\infty}{(-c(1-q); q)_k}$$

we obtain from Ramanujan's sum (2.15) with $a = -c(1-q)$, $b = 0$, and $z = q^{\alpha+1}$,

$$\begin{aligned} A &= \sum_{k=-\infty}^{\infty} \frac{q^{k\alpha+k}}{(-c(1-q)q^k; q)_\infty} \\ &= \frac{(q, -c(1-q)q^{\alpha+1}, -c^{-1}(1-q)^{-1} q^{-\alpha}; q)_\infty}{(q^{\alpha+1}, -c(1-q), -c^{-1}(1-q)^{-1} q; q)_\infty}. \end{aligned} \tag{3.6}$$

Note that (3.4) can also be stated in terms of the q-integral defined by (2.12):

$$\begin{aligned} &\frac{1}{A^*} \int_0^\infty \frac{t^\alpha}{(-c(1-q)t; q)_\infty} L_m^{(\alpha)}(ct; q)\, L_n^{(\alpha)}(ct; q)\, d_q t \\ &\qquad = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n\, q^n} \delta_{mn}, \qquad c > 0, \end{aligned} \tag{3.7}$$

where A^* equals

$$A^* := \int_0^\infty \frac{t^\alpha}{(-c(1-q)t; q)_\infty} d_q t. \tag{3.8}$$

We remark that the orthogonality relations (3.3), (3.4), and (3.7) are the same since we work in the space of polynomials. This allows us to define for polynomials f and g,

$$\begin{aligned} \langle f, g \rangle &= \frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\,\Gamma(\alpha+1)} \int_0^\infty \frac{x^\alpha}{(-(1-q)x; q)_\infty} f(x)\, g(x)\, dx \\ &= \frac{1}{A} \sum_{k=-\infty}^{\infty} \frac{q^{k\alpha+k}}{(-c(1-q)q^k; q)_\infty} f(cq^k)\, g(cq^k) \\ &= \frac{1}{A^*} \int_0^\infty \frac{t^\alpha}{(-c(1-q)t; q)_\infty} f(ct)\, g(ct)\, d_q t, \qquad c > 0, \end{aligned} \tag{3.9}$$

where A and A^* are defined by (3.6) and (3.8), respectively. However, since c is an arbitrary positive constant, the relations (3.4) and (3.7) give rise to infinite many different weight functions. So the Stieltjes moment problem for the q-Laguerre polynomials $\{L_n^{(\alpha)}(x; q)\}_{n=0}^{\infty}$ is indeterminate. For details the reader is referred to [17]. (In particular, see Moak's remarks on page 21 and page 25 in [17].)

As a q-analogue of $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$ we have

$$L_n^{(\alpha)}(0; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n}, \qquad n = 0, 1, 2, ... \tag{3.10}$$

The q-Laguerre polynomials satisfy a second order q-difference equation which can be stated in terms of the q-difference operator defined by (2.7) as

$$\begin{aligned} x\, D_q^2 L_n^{(\alpha)}(x; q) + \left[\frac{1-q^{\alpha+1}}{1-q} - q^{\alpha+2}x\right] (D_q L_n^{(\alpha)})(qx; q) \\ + \frac{1-q^n}{1-q}\, q^{\alpha+1} L_n^{(\alpha)}(qx; q) = 0. \end{aligned} \tag{3.11}$$

Further we have a three term recurrence relation

$$\begin{aligned} -x L_n^{(\alpha)}(x; q) &= \frac{1-q^{n+1}}{(1-q)\, q^{2n+\alpha+1}} L_{n+1}^{(\alpha)}(x; q) \\ &\quad - \left[\frac{1-q^{n+\alpha+1}}{(1-q)\, q^{2n+\alpha+1}} + \frac{1-q^n}{(1-q)\, q^{2n+\alpha}}\right] L_n^{(\alpha)}(x; q) \\ &\quad + \frac{1-q^{n+\alpha}}{(1-q)\, q^{2n+\alpha}} L_{n-1}^{(\alpha)}(x; q) \end{aligned}$$

and a Christoffel–Darboux formula

$$\begin{aligned} (x-y)\, \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{q^k (q; q)_k\, L_k^{(\alpha)}(x; q)\, L_k^{(\alpha)}(y; q)}{(q^{\alpha+1}; q)_k} \\ = \frac{1-q^{n+1}}{(1-q)\, q^{n+\alpha+1}} [L_n^{(\alpha)}(x; q)\, L_{n+1}^{(\alpha)}(y; q) \\ - L_{n+1}^{(\alpha)}(x; q)\, L_n^{(\alpha)}(y; q)]. \end{aligned} \tag{3.12}$$

If we divide by $x - y$ and let y tend to x we obtain the confluent form of the Christoffel–Darboux formula

$$\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n}\frac{q^k(q;q)_k\,\{L_k^{(\alpha)}(x;q)\}^2}{(q^{\alpha+1};q)_k}$$
$$=\frac{1-q^{n+1}}{(1-q)q^{n+\alpha+1}}\left[L_{n+1}^{(\alpha)}(x;q)\frac{d}{dx}L_n^{(\alpha)}(x;q)\right.$$
$$\left.-L_n^{(\alpha)}(x;q)\frac{d}{dx}L_{n+1}^{(\alpha)}(x;q)\right]. \tag{3.13}$$

The q-analogue of the well-known differentiation formula $D^kL_n^{(\alpha)}(x)=(-1)^k L_{n-k}^{(\alpha+k)}(x)$ yields

$$D_q^kL_n^{(\alpha)}(x;q)=(-1)^k q^{k(\alpha+k)}L_{n-k}^{(\alpha+k)}(q^kx;q),$$
$$k=0,1,2,...,n,\; n=0,1,2,.... \tag{3.14}$$

4. The Definition and the Orthogonality

We will try to determine the polynomials $\{L_n^{\alpha,M_0,M_1,...,M_N}(x;q)\}_{n=0}^{\infty}$ which are orthogonal with respect to the inner product

$$\begin{cases}\langle f,g\rangle_q=\langle f,g\rangle+\displaystyle\sum_{\nu=0}^{N}M_\nu(D_q^\nu f)(0)(D_q^\nu g)(0),\\ \alpha>-1,\; N\in\{0,1,2,...\},\text{ and } M_\nu\geqslant 0 \text{ for all } \nu\in\{0,1,2,...,N\},\end{cases} \tag{4.1}$$

where the inner product $\langle\ ,\ \rangle$ is defined by (3.9).

We will show that these orthogonal polynomials can be defined by

$$L_n^{\alpha,M_0,M_1,...,M_N}(x;q)=\sum_{k=0}^{N+1}q^{-k(\alpha+k)}A_k(D_q^kL_n^{(\alpha)})(q^{-k}x;q),$$
$$n=0,1,2,... \tag{4.2}$$

for some real coefficients $\{A_k\}_{k=0}^{N+1}$. Moreover, we will prove the orthogonality relation

$$\langle L_m^{\alpha,M_0,M_1,...,M_N}(x;q),L_n^{\alpha,M_0,M_1,...,M_N}(x;q)\rangle_q$$
$$=\frac{(q^{\alpha+1};q)_n}{(q;q)_n\,q^n}A_0\left(\sum_{k=0}^{N+1}q^{nk-\binom{k}{2}}A_k\right)\delta_{mn},\qquad m,n=0,1,2,.... \tag{4.3}$$

First we will determine the polynomials $\{L_n^{\alpha,M_0,M_1,...,M_N}(x;q)\}_{n=0}^{\infty}$ which are orthogonal with respect to the inner product (4.1). The Gram–Schmidt orthogonalization process assures us that such a set of polynomials exists with degree$[L_n^{\alpha, M_0, M_1, ..., M_N}(x; q)] = n$. So we may write by using (3.14)

$$\begin{aligned} L_n^{\alpha, M_0, M_1, ..., M_N}(x; q) &= \sum_{k=0}^{n} (-1)^k A_k L_{n-k}^{(\alpha+k)}(x; q) \\ &= \sum_{k=0}^{n} q^{-k(\alpha+k)} A_k (D_q^k L_n^{(\alpha)})(q^{-k}x; q), \qquad n = 0, 1, 2, ..., \end{aligned} \tag{4.4}$$

where $L_n^{(\alpha)}(x; q)$ denotes the q-Laguerre polynomial defined by (3.1) and the coefficients $\{A_k\}_{k=0}^n$ are real constants which may depend on n, α, M_0 $M_1, ..., M_N$, and q. Moreover, each polynomial $L_n^{\alpha, M_0, M_1, ..., M_N}(x; q)$ is unique except for a multiplicative constant. We will choose this constant such that

$$L_n^{\alpha, 0, 0, ..., 0}(x; q) = L_n^{(\alpha)}(x; q).$$

By using the representation (4.4) and (3.2) we easily see that the coefficient k_n of x^n in the polynomial $L_n^{\alpha, M_0, M_1, ..., M_N}(x; q)$ equals

$$k_n = (-1)^n q^{n(n+\alpha)} \frac{(1-q)^n}{(q; q)_n} A_0. \tag{4.5}$$

This implies that $A_0 \neq 0$.

Let $p(x) = x^m$. First of all we choose $L_0^{\alpha, M_0, M_1, ..., M_N}(x; q) = 1$ for the moment and we will try to determine the polynomials $\{L_n^{\alpha, M_0, M_1, ..., M_N}(x; q)\}_{n=1}^{\infty}$ in such a way that $\langle p(x), L_n^{\alpha, M_0, M_1, ..., M_N}(x; q)\rangle_q = 0$ for all $m \in \{0, 1, 2, ..., n-1\}$.

We use the definition (3.1) of the q-Laguerre polynomials and Ramanujan's integral formula (2.14) to obtain for $k = 0, 1, 2, ..., n$ and $m, n = 0, 1, 2, ...$

$$\begin{aligned} &\int_0^\infty \frac{x^{\alpha+m}}{(-(1-q)x; q)_\infty} L_{n-k}^{(\alpha+k)}(x; q)\, dx \\ &= \frac{(q^{\alpha+k+1}; q)_{n-k}}{(q; q)_{n-k}} \sum_{j=0}^{n-k} \frac{(q^{-n+k}; q)_j q^{\binom{j}{2}} (1-q)^j q^{(n+\alpha+1)j}}{(q^{\alpha+k+1}; q)_j (q; q)_j} \\ &\quad \times \int_0^\infty \frac{x^{\alpha+m+j}}{(-(1-q)x; q)_\infty}\, dx \\ &= \frac{(q^{\alpha+k+1}; q)_{n-k}}{(q; q)_{n-k}} \sum_{j=0}^{n-k} \frac{(q^{-n+k}; q)_j q^{\binom{j}{2}} (1-q)^j q^{(n+\alpha+1)j}}{(q^{\alpha+k+1}; q)_j (q; q)_j} \\ &\quad \times \frac{\Gamma(-\alpha-m-j)\,\Gamma(\alpha+m+j+1)}{\Gamma_q(-\alpha-m-j)}. \end{aligned}$$

Now we use the definition (2.13) of the q-gamma function and the identities (2.2) and (2.3) to find

$$\begin{aligned}
&\frac{\Gamma_q(-\alpha)\,\Gamma(-\alpha-m-j)\,\Gamma(\alpha+m+j+1)}{\Gamma(-\alpha)\,\Gamma(\alpha+1)\,\Gamma_q(-\alpha-m-j)}\\
&\quad=(-1)^{m+j}(1-q)^{-m-j}\frac{(q^{-\alpha-m-j};q)_\infty}{(q^{-\alpha};q)_\infty}\\
&\quad=(1-q)^{-m-j}q^{-(\alpha+1)m-\binom{m}{2}}q^{-(\alpha+m+1)j-\binom{j}{2}}(q^{\alpha+1};q)_m\,(q^{\alpha+m+1};q)_j.
\end{aligned}$$

Hence, by using the summation formula (2.6) we find

$$\begin{aligned}
&\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\,\Gamma(\alpha+1)}\int_0^\infty \frac{x^{\alpha+m}}{(-(1-q)x;q)_\infty}L_{n-k}^{(\alpha+k)}(x;q)\,dx\\
&\quad=\frac{(q^{\alpha+k+1};q)_{n-k}}{(q;q)_{n-k}}\frac{(q^{\alpha+1};q)_m}{(1-q)^m}q^{-(\alpha+1)m-\binom{m}{2}}\\
&\quad\quad\times{}_2\phi_1\left(\left.\begin{matrix}q^{-n+k},\ q^{\alpha+m+1}\\ q^{\alpha+k+1}\end{matrix}\ \right|\ q;q^{n-m}\right)\\
&\quad=\frac{(q^{k-m};q)_{n-k}}{(q;q)_{n-k}}\frac{(q^{\alpha+1};q)_m}{(1-q)^m}q^{-(\alpha+1)m-\binom{m}{2}},\\
&\quad\quad k=0,1,2,...,n,\quad m,n=0,1,2,.... \qquad (4.6)
\end{aligned}$$

Now we have by using (4.4) and (4.6)

$$\begin{aligned}
&\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\,\Gamma(\alpha+1)}\int_0^\infty \frac{x^{\alpha+m}}{(-(1-q)x;q)_\infty}L_n^{\alpha,M_0,M_1,...,M_N}(x;q)\,dx\\
&\quad=\frac{(q^{\alpha+1};q)_m}{(1-q)^m}q^{-(\alpha+1)m-\binom{m}{2}}\sum_{k=0}^n(-1)^k\\
&\quad\quad\times\frac{(q^{k-m};q)_{n-k}}{(q;q)_{n-k}}A_k,\qquad m,n=0,1,2,....
\end{aligned}$$

First we consider the case that $n\geqslant N+2$ and $N+1\leqslant m\leqslant n-1$. Then it is clear that

$$(D_q^\nu p)(0)=0\qquad\text{for all}\quad \nu\in\{0,1,2,...,N\}.$$

Since

$$(q^{k-m};q)_{n-k}=0\qquad\text{for}\quad k=0,1,2,...,m\ \text{and}\ m<n$$

we see that $\langle p(x), L_n^{\alpha, M_0, M_1, ..., M_N}(x; q)\rangle_q = 0$ is equivalent to

$$\frac{(q^{\alpha+1}; q)_m}{(1-q)^m} q^{-(\alpha+1)m-\binom{m}{2}} \sum_{k=m+1}^{n} (-1)^k \\ \times \frac{(q^{k-m}; q)_{n-k}}{(q; q)_{n-k}} A_k = 0, \qquad m = N+1, N+2, ..., n-1.$$

If we substitute $m = n-1, n-2, ..., N+1$ respectively we easily obtain

$$A_{N+2} = A_{N+3} = \cdots = A_n = 0 \qquad \text{for} \quad n \geqslant N+2.$$

Hence, the expression (4.4) reduces to (4.2) for $n \geqslant N+2$. For $n \leqslant N+1$, (4.2) is trivial. In that case the coefficients $\{A_k\}_{k=n+1}^{N+1}$ can be chosen arbitrarily. This proves that the polynomials $\{L_n^{\alpha, M_0, M_1, ..., M_N}(x; q)\}_{n=0}^{\infty}$ can be defined by (4.2) for all $n \in \{0, 1, 2, ...\}$.

In order to define the coefficients $\{A_k\}_{k=0}^{N+1}$ we now have to consider for $n = 1, 2, 3, ...$

$$\langle p(x), L_n^{\alpha, M_0, M_1, ..., M_N}(x; q)\rangle_q = 0 \qquad \text{for} \quad m = 0, 1, 2, ..., \min(n-1, N). \tag{4.7}$$

Since $p(x) = x^m$ we have by using (2.8)

$$(D_q^{\nu} p)(0) = \frac{(q; q)_m}{(1-q)^m} \delta_{m\nu}, \qquad \nu = 0, 1, 2, ..., N.$$

Hence, (4.7) implies, by using (4.1), (4.2), (4.6), (3.14), and (3.10), that

$$\frac{(q^{\alpha+1}; q)_m}{(1-q)^m} q^{-(\alpha+1)m-\binom{m}{2}} \sum_{k=m+1}^{\min(n, N+1)} (-1)^k \frac{(q^{k-m}; q)_{n-k}}{(q; q)_{n-k}} A_k \\ + (-1)^m \frac{(q; q)_m}{(1-q)^m} q^{m(m+\alpha)} M_m \sum_{k=0}^{\min(n, N+1)} (-1)^k \\ \times \frac{(q^{\alpha+k+m+1}; q)_{n-k-m}}{(q; q)_{n-k-m}} q^{mk} A_k = 0,$$

for $m = 0, 1, 2, ..., \min(n-1, N)$. We remark that the definition (2.1) implies that

$$\frac{(q^{\gamma}; q)_{-n}}{(q; q)_{-n}} = \frac{(1-q^{-n+1})(1-q^{-n+2}) \cdots (1-q^0)}{(1-q^{\gamma-n})(1-q^{\gamma-n+1}) \cdots (1-q^{\gamma-1})} = 0$$

for $\gamma - n > 0$ and $n = 1, 2, 3, \ldots$. Hence

$$\frac{(q^{k-m}; q)_{n-k}}{(q; q)_{n-k}} = \frac{(q^{\alpha+k+m+1}; q)_{n-k-m}}{(q; q)_{n-k-m}} = 0$$

for $k \geqslant n+1$ and $m = 0, 1, 2, \ldots, \min(n-1, N)$. Note that we have by using (2.4)

$$\frac{(q^{k-m}; q)_{n-k}}{(q; q)_{n-k}} = \begin{bmatrix} n-m-1 \\ n-k \end{bmatrix}_q = \begin{bmatrix} n-m-1 \\ k-m-1 \end{bmatrix}_q$$
$$= \frac{(q^{n-k+1}; q)_{k-m-1}}{(q; q)_{k-m-1}}, \qquad m < n.$$

This allows us to write

$$\frac{(q^{\alpha+1}; q)_m}{(1-q)^m} q^{-(\alpha+1)m - \binom{m}{2}} \sum_{k=m+1}^{N+1} (-1)^k \frac{(q^{n-k+1}; q)_{k-m-1}}{(q; q)_{k-m-1}} A_k$$
$$+ (-1)^m \frac{(q; q)_m}{(1-q)^m} q^{m(m+\alpha)} M_m \sum_{k=0}^{N+1} (-1)^k$$
$$\times \frac{(q^{\alpha+k+m+1}; q)_{n-k-m}}{(q; q)_{n-k-m}} q^{mk} A_k = 0,$$

for $m = 0, 1, 2, \ldots, \min(n-1, N)$. However, we will define the coefficients $\{A_k\}_{k=0}^{N+1}$ in such a way that

$$\frac{(q^{\alpha+1}; q)_m}{(q; q)_m} q^{-(\alpha+1)m - \binom{m}{2}} \sum_{k=m+1}^{N+1} (-1)^k \frac{(q^{n-k+1}; q)_{k-m-1}}{(q; q)_{k-m-1}} A_k$$
$$+ (-1)^m q^{m(m+\alpha)} M_m \sum_{k=0}^{N+1} (-1)^k \frac{(q^{\alpha+k+m+1}; q)_{n-k-m}}{(q; q)_{n-k-m}} q^{mk} A_k = 0, \tag{4.8}$$

for $m = 0, 1, 2, \ldots, N$ is valid for all $n \in \{0, 1, 2, \ldots\}$. For $n \geqslant N+1$ this is the same system of equations. For $n \leqslant N$ we have added the following conditions on the arbitrary coefficients $\{A_k\}_{k=n+1}^{N+1}$:

$$\frac{(q^{\alpha+1}; q)_m}{(q; q)_m} q^{-(\alpha+1)m - \binom{m}{2}} \sum_{k=m+1}^{N+1} (-1)^k \frac{(q^{n-k+1}; q)_{k-m-1}}{(q; q)_{k-m-1}} A_k$$
$$+ (-1)^m q^{m(m+\alpha)} M_m \sum_{k=0}^{N+1} (-1)^k \frac{(q^{\alpha+k+m+1}; q)_{n-k-m}}{(q; q)_{n-k-m}} q^{mk} A_k = 0,$$

where $m=n, n+1, n+2, ..., N$. Since we have by using (2.3) for $k\geqslant n+1$

$$(q^{n-k+1};q)_{k-n-1}=(-1)^{k-n-1}q^{-\binom{k-n}{2}}(q;q)_{k-n-1}$$
$$=(-1)^{k-n-1}q^{nk-\binom{k}{2}-\binom{n+1}{2}}(q;q)_{k-n-1}$$

this implies

$$\begin{cases}\dfrac{(q^{\alpha+1};q)_n}{(q;q)_n}q^{-(\alpha+1)n-\binom{n}{2}-\binom{n+1}{2}}\displaystyle\sum_{k=n+1}^{N+1}q^{nk-\binom{k}{2}}A_k=q^{n(n+\alpha)}M_nA_0\\ \dfrac{(q^{\alpha+1};q)_{n+i}}{(q;q)_{n+i}}q^{-(\alpha+1)(n+i)-\binom{n+i}{2}}\\ \quad\times\displaystyle\sum_{k=n+i+1}^{N+1}(-1)^k\dfrac{(q^{n-k+1};q)_{k-n-i-1}}{(q;q)_{k-n-i-1}}A_k=0,\qquad i=1,2,3,...,N-n.\end{cases}$$

This implies for $n\leqslant N$ that $A_{n+2}=A_{n+3}=\cdots=A_{N+1}=0$ and

$$\frac{(q^{\alpha+1};q)_n}{(q;q)_n}q^{-n(n+\alpha+1)+n(n+1)-\binom{n+1}{2}}A_{n+1}=q^{n(n+\alpha)}M_nA_0.$$

However, in the sequel we only need

$$q^{n(n+\alpha)}M_nA_0=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}q^{-n(n+\alpha+1)}\sum_{k=n+1}^{N+1}q^{nk-\binom{k}{2}}A_k\qquad\text{for}\quad n\leqslant N. \tag{4.9}$$

Now we have found the representation (4.2) where the coefficients $\{A_k\}_{k=0}^{N+1}$ satisfy (4.8). Note that we changed the choice of $L_0^{\alpha,M_0,M_1,...,M_N}(x;q)=1$ such that (4.2) also holds for $n=0$. We remark that the system (4.8) of equations for the coefficients $\{A_k\}_{k=0}^{N+1}$ can be solved for every N. For instance, in [11] we found an explicit representation in the case $N=1$. It would be a nice result to find an explicit formula for each coefficient A_k in general. However, in this paper we only need the property (4.9).

To complete the proof of the orthogonality relation (4.3) we note that it follows from (4.2), (3.2), and the orthogonality we just proved that

$$\langle L_n^{\alpha,M_0,M_1,...,M_N}(x;q),L_n^{\alpha,M_0,M_1,...,M_N}(x;q)\rangle_q$$
$$=(-1)^n q^{n(n+\alpha)}\frac{(1-q)^n}{(q;q)_n}A_0\langle x^n,L_n^{\alpha,M_0,M_1,...,M_N}(x;q)\rangle_q.$$

Now we obtain from (4.1), (4.2), and (4.6) for $m=n\geqslant N+1$

$$\begin{aligned}
&\langle x^n, L_n^{\alpha, M_0, M_1, \ldots, M_N}(x;q)\rangle_q \\
&\quad=\frac{(q^{\alpha+1};q)_n}{(1-q)^n}\, q^{-(\alpha+1)n-\binom{n}{2}} \sum_{k=0}^{N+1} (-1)^k \frac{(q^{k-n};q)_{n-k}}{(q;q)_{n-k}} A_k \\
&\quad=(-1)^n \frac{(q^{\alpha+1};q)_n}{(1-q)^n}\, q^{-n(n+\alpha+1)} \sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k.
\end{aligned}$$

This proves (4.3) in the case that $n\geqslant N+1$.

For $n\leqslant N$ we find by using (4.9)

$$\begin{aligned}
&\langle x^n, L_n^{\alpha, M_0, M_1, \ldots, M_N}(x;q)\rangle_q \\
&\quad=\frac{(q^{\alpha+1};q)_n}{(1-q)^n}\, q^{-(\alpha+1)n-\binom{n}{2}} \sum_{k=0}^{n} (-1)^k \frac{(q^{k-n};q)_{n-k}}{(q;q)_{n-k}} A_k \\
&\qquad+(-1)^n \frac{(q;q)_n}{(1-q)^n}\, q^{n(n+\alpha)} M_n A_0 \\
&\quad=(-1)^n \frac{(q^{\alpha+1};q)_n}{(1-q)^n}\, q^{-n(n+\alpha+1)} \sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k.
\end{aligned}$$

This proves (4.3).

5. Another Representation

The polynomials $\{L_n^{\alpha, M_0, M_1, \ldots, M_N}(x;q)\}_{n=0}^{\infty}$ given by (4.2) can also be written as

$$L_n^{\alpha, M_0, M_1, \ldots, M_N}(x;q)=\sum_{k=0}^{N+1} q^{-k(\alpha+2k)} B_k x^k (D_q^k L_n^{(\alpha+k)})(q^{-k}x;q), \tag{5.1}$$

where the coefficients $\{B_k\}_{k=0}^{N+1}$ are related to the coefficients $\{A_k\}_{k=0}^{N+1}$ found in the preceding section in the following way

$$\begin{aligned}
A_i=q^{\binom{i+1}{2}} &\sum_{k=i}^{N+1} q^{-k(\alpha+k+i)} \begin{bmatrix} k \\ i \end{bmatrix}_q \\
&\times \frac{(q^{n-k+1};q)_{k-i}\,(q^{\alpha+k};q)_i}{(1-q)^k} B_k, \qquad i=0,1,2,\ldots,N+1
\end{aligned}$$

and

$$B_k = \frac{(1-q)^k}{(q^{\alpha+k}; q)_k} q^{-\binom{k+1}{2}+k(\alpha+2k)} \sum_{j=k}^{N+1} (-1)^{j+k} \begin{bmatrix} j \\ k \end{bmatrix}_q$$
$$\times \frac{(q^{n-j+1}; q)_{j-k}}{(q^{\alpha+2k+1}; q)_{j-k}} A_j, \qquad k=0, 1, 2, ..., N+1,$$

where the q-binomial coefficient is defined by (2.4).

This can be shown by first proving and then using the following two relations involving q-Laguerre polynomials

$$x^k(D_q^k L_n^{(\alpha+k)})(q^{-k}x; q) = q^{k^2} \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{(q^{n-k+1}; q)_{k-i} (q^{\alpha+k}; q)_i}{(1-q)^k}$$
$$\times q^{-\binom{i}{2}-(\alpha+k)i} (D_q^i L_n^{(\alpha)})(q^{-i}x; q),$$

and

$$q^{-k(\alpha+k)}(D_q^k L_n^{(\alpha)})(q^{-k}x; q)$$
$$= \sum_{i=0}^{k} (-1)^{i+k} \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{(q^{n-k+1}; q)_{k-i} (1-q)^i}{(q^{\alpha+i}; q)_i (q^{\alpha+2i+1}; q)_{k-i}}$$
$$\times q^{-\binom{i+1}{2}} x^i (D_q^i L_n^{(\alpha+i)})(q^{-i}x; q),$$

respectively, for $k, n = 0, 1, 2, ...$

The proof can be found in [13, 14].

6. Representation as Basic Hypergeometric Series

If we write

$$L_n^{\alpha, M_0, M_1, ..., M_N}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{m=0}^{n} C_m q^{\binom{m}{2}} q^{(n+\alpha+1)m} \frac{(1-q)^m}{(q; q)_m} x^m$$

then it follows from (4.2) and (3.1), by using (2.2) and (2.3) that

$$C_m = \sum_{k=0}^{N+1} \frac{(q^{-n}; q)_{m+k}}{(q^{\alpha+1}; q)_{m+k}} q^{nk-\binom{k}{2}} A_k$$
$$= \frac{(q^{-n}; q)_m}{(q^{\alpha+1}; q)_{m+N+1}} \sum_{k=0}^{N+1} (q^{-n+m}; q)_k (q^{\alpha+k+m+1}; q)_{N+1-k} q^{nk-\binom{k}{2}} A_k.$$

Note that

$$F(z) := \sum_{k=0}^{N+1} (q^{-n}z; q)_k (q^{\alpha+k+1}z; q)_{N+1-k} q^{nk-\binom{k}{2}} A_k$$

is a polynomial in z of degree at most $N+1$. The coefficient of z^{N+1} in $F(z)$ equals

$$(-1)^{N+1} q^{(N+1)(\alpha+1)+\binom{N+1}{2}} \sum_{k=0}^{N+1} q^{-(\alpha+1)k-\binom{k}{2}} A_k.$$

Note that it follows from (4.3) that

$$F(0)=\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k \neq 0.$$

This implies that all zeros of $F(z)$ can be written as (complex) powers of q. If

$$\sum_{k=0}^{N+1} q^{-(\alpha+1)k-\binom{k}{2}} A_k \neq 0, \tag{6.1}$$

then the polynomial $F(z)$ has degree $N+1$. In that case we may write

$$\begin{aligned} F(q^m) &= \sum_{k=0}^{N+1} (q^{-n+m}; q)_k \, (q^{\alpha+k+m+1}; q)_{N+1-k} \, q^{nk-\binom{k}{2}} A_k \\ &= \left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k \right) (1-q^{\beta_0})(1-q^{\beta_1}) \cdots (1-q^{\beta_N}) \\ &\quad \times \frac{(q^{\beta_0+1}; q)_m \, (q^{\beta_1+1}; q)_m \cdots (q^{\beta_N+1}; q)_m}{(q^{\beta_0}; q)_m \, (q^{\beta_1}; q)_m \cdots (q^{\beta_N}; q)_m} \end{aligned}$$

for some complex β_j, $j=0, 1, 2, ..., N$. Hence, by using

$$(q^{\alpha+1}; q)_{m+N+1} = (q^{\alpha+1}; q)_{N+1} \, (q^{\alpha+N+2}; q)_m,$$

which follows directly from (2.2), we have

$$\begin{aligned} & L_n^{\alpha, M_0, M_1, ..., M_N}(x; q) \\ &\quad = \frac{(1-q^{\beta_0})(1-q^{\beta_1}) \cdots (1-q^{\beta_N})}{(q^{\alpha+1}; q)_{N+1}} \left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k \right) \\ &\quad \times \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \\ &\quad \times {}_{N+2}\phi_{N+2} \left(\begin{matrix} q^{-n}, q^{\beta_0+1}, q^{\beta_1+1}, ..., q^{\beta_N+1} \\ q^{\alpha+N+2}, q^{\beta_0}, q^{\beta_1}, ..., q^{\beta_N} \end{matrix} \middle| \, q; -(1-q) q^{n+\alpha+1} x \right). \end{aligned} \tag{6.2}$$

If (6.1) is not satisfied, then $F(z)$ is a polynomial of a degree less than $N+1$. In that case we find a representation as a ${}_k\phi_k$ basic hypergeometric series where $k<N+2$ in a similar way.

7. A Second Order q-Difference Equation

In this section we will show that the polynomials $\{L_n^{\alpha, M_0, M_1, \ldots, M_N}(x; q)\}_{n=0}^{\infty}$ satisfy a second order q-difference equation. The method found in [7] can be applied in this case too. We prove the following theorem.

Theorem 1. *The polynomials* $\{L_n^{\alpha, M_0, M_1, \ldots, M_N}(x; q)\}_{n=0}^{\infty}$ *satisfy a second order q-difference equation of the form*

$$xP_2(x)\, D_q^2 L_n^{\alpha, M_0, M_1, \ldots, M_N}(x; q) - P_1(x)(D_q L_n^{\alpha, M_0, M_1, \ldots, M_N})(qx; q) + \frac{1-q^n}{1-q} P_0(x)\, L_n^{\alpha, M_0, M_1, \ldots, M_N}(qx; q) = 0, \tag{7.1}$$

where $P_0(x)$, $P_1(x)$, *and* $P_2(x)$ *are polynomials with*

$$\begin{cases} P_0(x) = q^{\alpha+1} A_0 \left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k \right) x^{N+1} + \textit{lower order terms} \\ P_1(x) = q^{\alpha+2} A_0 \left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k \right) x^{N+2} + \textit{lower order terms} \\ P_2(x) = A_0 \left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k \right) x^{N+1} + \textit{lower order terms} \end{cases} \tag{7.2}$$

and

$$P_1(qx) = x\, D_q P_2(x) + \left[q^{\alpha+N+4} x - \frac{1-q^{\alpha+N+2}}{1-q} \right] P_2(x). \tag{7.3}$$

Proof. We consider the q-difference equation (3.11) for the q-Laguerre polynomials. By using the fact that

$$L_n^{(\alpha)}(q^{-1}x; q) = L_n^{(\alpha)}(x; q) + q^{-1}(1-q)\, x (D_q L_n^{(\alpha)})(q^{-1}x; q)$$

which follows directly from (2.7), we write this q-difference equation (3.11) in the form

$$q^{-2} x (D_q^2 L_n^{(\alpha)})(q^{-2}x; q) + \left[\frac{1-q^{\alpha+1}}{1-q} - q^{n+\alpha} x \right] (D_q L_n^{(\alpha)})(q^{-1}x; q) + \frac{1-q^n}{1-q} q^{\alpha+1} L_n^{(\alpha)}(x; q) = 0. \tag{7.4}$$

If we let D_q^k act on (7.4) and use the q-analogue of Leibniz' rule (2.11) we obtain

$$x(D_q^{k+2}L_n^{(\alpha)})(q^{-k-2}x;q)+q^{k+2}\left[\frac{1-q^{\alpha+k+1}}{1-q}-q^{n+\alpha}x\right]\times(D_q^{k+1}L_n^{(\alpha)})(q^{-k-1}x;q)+\frac{1-q^{n-k}}{1-q}q^{\alpha+3k+3}\times(D_q^kL_n^{(\alpha)})(q^{-k}x;q)=0,\qquad k=0,1,2,.... \tag{7.5}$$

Now we consider the definition (4.2). We multiply by x and use (7.5) for $k=N-1$ to find

$$xL_n^{\alpha,M_0,M_1,...,M_N}(x;q)=\sum_{k=0}^{N} b_k(x)(D_q^kL_n^{(\alpha)})(q^{-k}x;q), \tag{7.6}$$

where

$$\begin{cases} b_k(x)=q^{-k(\alpha+k)}A_kx, \qquad k=0,1,2,...,N-2 \\ b_{N-1}(x)=q^{-(N-1)(\alpha+N-1)}A_{N-1}x \\ \qquad -q^{\alpha+3N-(N+1)(\alpha+N+1)}\dfrac{1-q^{n-N+1}}{1-q}A_{N+1} \\ b_N(x)=q^{-N(\alpha+N)}A_Nx-q^{-(N+1)(\alpha+N)}\left[\dfrac{1-q^{\alpha+N}}{1-q}-q^{n+\alpha}x\right]A_{N+1}. \end{cases}$$

Now we multiply (7.6) by x and use (7.5) for $k=N-2$ to obtain

$$x^2L_n^{\alpha,M_0,M_1,...,M_N}(x;q)=\sum_{k=0}^{N-1}\tilde{b}_k(x)(D_q^kL_n^{(\alpha)})(q^{-k}x;q),$$

where

$$\begin{cases} \tilde{b}_k(x)=xb_k(x), \qquad k=0,1,2,...,N-3 \\ \tilde{b}_{N-2}(x)=xb_{N-2}(x)-\dfrac{1-q^{n-N+2}}{1-q}q^{\alpha+3N-3}b_N(x) \\ \tilde{b}_{N-1}(x)=xb_{N-1}(x)-q^N\left[\dfrac{1-q^{\alpha+N-1}}{1-q}-q^{n+\alpha}x\right]b_N(x). \end{cases}$$

Repeating this process we finally obtain by using (7.5) for $k=0$

$$x^NL_n^{\alpha,M_0,M_1,...,M_N}(x;q)=p_0(x)\,L_n^{(\alpha)}(x;q)+p_1(x)(D_qL_n^{(\alpha)})(q^{-1}x;q) \tag{7.7}$$

for some polynomials $p_0(x)$ and $p_1(x)$ which satisfy

$$\begin{cases} p_0(x) = A_0 x^N + \text{lower order terms} \\ p_1(x) = q^{-(n+\alpha+1)} \left(\sum_{k=1}^{N+1} q^{nk - \binom{k}{2}} A_k \right) x^N + \text{lower order terms.} \end{cases} \tag{7.8}$$

Now we use the q-product rule (2.10) to obtain from (7.7)

$$\frac{1-q^N}{1-q} x^{N-1} L_n^{\alpha, M_0, M_1, ..., M_N}(qx; q) + x^N D_q L_n^{\alpha, M_0, M_1, ..., M_N}(x; q)$$

$$= D_q p_0(x) L_n^{(\alpha)}(qx; q) + [p_0(x) + D_q p_1(x)](D_q L_n^{(\alpha)})(x; q)$$

$$+ q^{-1} p_1(x)(D_q^2 L_n^{(\alpha)})(q^{-1}x; q).$$

We multiply by x and replace x by $q^{-1}x$ to obtain

$$\begin{aligned} &\frac{1-q^N}{(1-q)q^N} x^N L_n^{\alpha, M_0, M_1, ..., M_N}(x; q) + q^{-N-1} x^{N+1} (D_q L_n^{\alpha, M_0, M_1, ..., M_N})(q^{-1}x; q) \\ &= q^{-1} x (D_q p_0)(q^{-1}x) L_n^{(\alpha)}(x; q) \\ &\quad + q^{-1} x [p_0(q^{-1}x) + (D_q p_1)(q^{-1}x)](D_q L_n^{(\alpha)})(q^{-1}x; q) \\ &\quad + q^{-2} x p_1(q^{-1}x)(D_q^2 L_n^{(\alpha)})(q^{-2}x; q). \end{aligned}$$

Now we use (7.4) and (7.7) to find

$$\begin{aligned} &x^{N+1}(D_q L_n^{\alpha, M_0, M_1, ..., M_N})(q^{-1}x; q) \\ &\quad = r_0(x) L_n^{(\alpha)}(x; q) + r_1(x)(D_q L_n^{(\alpha)})(q^{-1}x; q), \end{aligned} \tag{7.9}$$

where

$$\begin{cases} r_0(x) = q^{N+1} \left[q^{-1} x (D_q p_0)(q^{-1}x) - \dfrac{1-q^n}{1-q} q^{\alpha+1} p_1(q^{-1}x) \right] \\ \qquad - q \dfrac{1-q^N}{1-q} p_0(x) \\ r_1(x) = q^N x [p_0(q^{-1}x) + (D_q p_1)(q^{-1}x)] \\ \qquad - q^{N+1} \left[\dfrac{1-q^{\alpha+1}}{1-q} - q^{n+\alpha} x \right] p_1(q^{-1}x) - q \dfrac{1-q^N}{1-q} p_1(x). \end{cases} \tag{7.10}$$

By using (7.8) and (7.10) we easily see that

$$\begin{cases} r_0(x) = -\dfrac{1-q^n}{1-q} q^{-n+1} \left(\displaystyle\sum_{k=1}^{N+1} q^{nk-\binom{k}{2}} A_k \right) x^N + \text{lower order terms} \\ r_1(x) = \left(\displaystyle\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k \right) x^{N+1} + \text{lower order terms.} \end{cases} \tag{7.11}$$

In the same way we obtain from (7.9)

$$\begin{aligned} &\frac{1-q^{N+1}}{1-q} x^N D_q L_n^{\alpha, M_0, M_1, ..., M_N}(x; q) + q^{-1} x^{N+1} (D_q^2 L_n^{\alpha, M_0, M_1, ..., M_N})(q^{-1}x; q) \\ &\quad = D_q r_0(x) L_n^{(\alpha)}(qx; q) + [r_0(x) + D_q r_1(x)] D_q L_n^{(\alpha)}(x; q) \\ &\qquad + q^{-1} r_1(x) (D_q^2 L_n^{(\alpha)})(q^{-1}x; q). \end{aligned}$$

Multiplying by x and applying (7.4) again gives us by using (7.9)

$$\begin{aligned} &x^{N+2} (D_q^2 L_n^{\alpha, M_0, M_1, ..., M_N})(q^{-2}x; q) \\ &\quad = s_0(x) L_n^{(\alpha)}(x; q) + s_1(x) (D_q L_n^{(\alpha)})(q^{-1}x; q), \end{aligned} \tag{7.12}$$

where

$$\begin{cases} s_0(x) = q^{N+2} \left[x (D_q r_0)(q^{-1}x) - \dfrac{1-q^n}{1-q} q^{\alpha+2} r_1(q^{-1}x) \right] \\ \qquad - q^2 \dfrac{1-q^{N+1}}{1-q} r_0(x) \\ s_1(x) = q^{N+2} x [r_0(q^{-1}x) + (D_q r_1)(q^{-1}x)] \\ \qquad - q^{N+3} \left[\dfrac{1-q^{\alpha+1}}{1-q} - q^{n+\alpha} x \right] r_1(q^{-1}x) - q^2 \dfrac{1-q^{N+1}}{1-q} r_1(x). \end{cases} \tag{7.13}$$

By using (7.11) we easily see that

$$\begin{cases} s_0(x) = -\dfrac{1-q^n}{1-q} q^{\alpha+3} \left(\displaystyle\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k \right) x^{N+1} + \text{lower order terms} \\ s_1(x) = q^{n+\alpha+2} \left(\displaystyle\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k \right) x^{N+2} + \text{lower order terms.} \end{cases} \tag{7.14}$$

Elimination of $(D_q L_n^{(\alpha)})(q^{-1}x; q)$ from (7.7), (7.9), and (7.12) gives us in view of (3.10)

$$\begin{cases} p_0(x)\, r_1(x) - p_1(x)\, r_0(x) = x^N P_2^*(x) \\ p_0(x)\, s_1(x) - p_1(x)\, s_0(x) = x^N P_1^*(x) \\ r_0(x)\, s_1(x) - r_1(x)\, s_0(x) = \dfrac{1-q^n}{1-q}\, x^{N+1} P_0^*(x) \end{cases} \tag{7.15}$$

for some polynomials $P_2^*(x)$, $P_1^*(x)$, and $P_0^*(x)$. Here we used the fact that for $n=0$ it follows from (7.7) that $p_0(x) = A_0 x^N$. Therefore we have from (7.10) and (7.13), $r_0(x) = s_0(x) = 0$.

Now we conclude from (7.7), (7.9), and (7.12), by using (7.15)

$$\begin{aligned} 0 &= \begin{vmatrix} x^N L_n^{\alpha, M_0, M_1, \ldots, M_N}(x; q) & p_0(x) & p_1(x) \\ x^{N+1}(D_q L_n^{\alpha, M_0, M_1, \ldots, M_N})(q^{-1}x; q) & r_0(x) & r_1(x) \\ x^{N+2}(D_q^2 L_n^{\alpha, M_0, M_1, \ldots, M_N})(q^{-2}x; q) & s_0(x) & s_1(x) \end{vmatrix} \\ &= x^{2N+2} P_2^*(x)(D_q^2 L_n^{\alpha, M_0, M_1, \ldots, M_N})(q^{-2}x; q) \\ &\quad - x^{2N+1} P_1^*(x)(D_q L_n^{\alpha, M_0, M_1, \ldots, M_N})(q^{-1}x; q) \\ &\quad + \frac{1-q^n}{1-q}\, x^{2N+1} P_0^*(x)\, L_n^{\alpha, M_0, M_1, \ldots, M_N}(x; q). \end{aligned}$$

We divide by x^{2N+1} to obtain

$$\begin{aligned} &x P_2^*(x)(D_q^2 L_n^{\alpha, M_0, M_1, \ldots, M_N})(q^{-2}x; q) - P_1^*(x)(D_q L_n^{\alpha, M_0, M_1, \ldots, M_N})(q^{-1}x; q) \\ &\quad + \frac{1-q^n}{1-q} P_0^*(x)\, L_n^{\alpha, M_0, M_1, \ldots, M_N}(x; q) = 0. \end{aligned}$$

We replace x by $q^2 x$ and use the fact that

$$\begin{aligned} L_n^{\alpha, M_0, M_1, \ldots, M_N}(q^2 x; q) &= L_n^{\alpha, M_0, M_1, \ldots, M_N}(qx; q) \\ &\quad - q(1-q)\, x(D_q L_n^{\alpha, M_0, M_1, \ldots, M_N})(qx; q) \end{aligned}$$

which follows directly from (2.7), to find

$$\begin{aligned} &q^2 x P_2^*(q^2 x)\, D_q^2 L_n^{\alpha, M_0, M_1, \ldots, M_N}(x; q) \\ &\quad - [P_1^*(q^2 x) + q(1-q^n)\, x P_0^*(q^2 x)](D_q L_n^{\alpha, M_0, M_1, \ldots, M_N})(qx; q) \\ &\quad + \frac{1-q^n}{1-q} P_0^*(q^2 x)\, L_n^{\alpha, M_0, M_1, \ldots, M_N}(qx; q) = 0 \end{aligned}$$

which proves (7.1) if we define

$$\begin{cases} q^{2N+4} P_2(x) := q^2 P_2^*(q^2 x) \\ q^{2N+4} P_1(x) := P_2^*(q^2 x) + q(1-q^n)\, x P_0^*(q^2 x) \\ q^{2N+4} P_0(x) := P_0^*(q^2 x). \end{cases} \tag{7.16}$$

It easily follows from (7.15), (7.8), (7.11), and (7.14) that

$$\begin{cases} P_0^*(x)=q^{\alpha+3}A_0\left(\sum\limits_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^{N+1}+\text{lower order terms} \\ P_1^*(x)=q^{n+\alpha+2}A_0\left(\sum\limits_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^{N+2}+\text{lower order terms} \\ P_2^*(x)=A_0\left(\sum\limits_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^{N+1}+\text{lower order terms.} \end{cases} \tag{7.17}$$

Now (7.2) follows from (7.16) and (7.17).

It remains to show that (7.3) is true. To prove this we note, by using (2.7) and (7.16), that (7.3) is equivalent to

$$\begin{aligned}(1-q)[P_1^*(qx)&+(1-q^n)\,xP_0^*(qx)] \\ &=q^{\alpha+N+4}P_2^*(x)-q^2P_2^*(qx)+(1-q)\,q^{\alpha+N+4}xP_2^*(x).\end{aligned} \tag{7.18}$$

Now we will prove (7.18).

From (7.10) it follows by using the definition (2.7) that

$$\begin{cases}(1-q)\,r_0(qx)=q^{N+1}p_0(x)-qp_0(qx)-(1-q^n)\,q^{\alpha+N+2}p_1(x) \\ (1-q)\,r_1(qx)=(1-q)\,q^{N+1}xp_0(x)+q^{\alpha+N+2}p_1(x)-qp_1(qx) \\ \qquad\qquad +(1-q)\,q^{n+\alpha+N+2}xp_1(x).\end{cases} \tag{7.19}$$

Now we use (7.15) and (7.19) to see that

$$\begin{aligned}x^N[P_1^*(qx)&+(1-q^n)\,xP_0^*(qx)] \\ &=q^{-N}[p_0(qx)\,s_1(qx)-p_1(qx)\,s_0(qx)] \\ &\quad+(1-q)\,q^{-N-1}[r_0(qx)\,s_1(qx)-r_1(qx)\,s_0(qx)] \\ &=[p_0(x)-(1-q^n)\,q^{\alpha+1}p_1(x)]\,s_1(qx) \\ &\quad-[(1-q)\,xp_0(x)+q^{\alpha+1}p_1(x) \\ &\quad+(1-q)\,q^{n+\alpha+1}xp_1(x)]\,s_0(qx).\end{aligned} \tag{7.20}$$

By using (7.13) and (2.7) we find

$$\begin{cases}(1-q)\,s_0(qx)=q^{N+3}r_0(x)-q^2r_0(qx)-(1-q^n)\,q^{\alpha+N+4}r_1(x) \\ (1-q)\,s_1(qx)=(1-q)\,q^{N+3}xr_0(x)+q^{\alpha+N+4}r_1(x)-q^2r_1(qx) \\ \qquad\qquad +(1-q)\,q^{n+\alpha+N+4}xr_1(x).\end{cases} \tag{7.21}$$

Hence, by using (7.20) and (7.21) we obtain

$$\begin{aligned}
(1-q)\, x^N[P_1^*(qx)+(1-q^n)\, xP_0^*(qx)]
&= q^{\alpha+N+4}[p_0(x)\, r_1(x)-p_1(x)\, r_0(x)]\\
&\quad+(1-q)\, q^{\alpha+N+4}x[p_0(x)\, r_1(x)-p_1(x)\, r_0(x)]\\
&\quad+[(1-q)\, q^2xp_0(x)+q^{\alpha+3}p_1(x)+(1-q)\, q^{n+\alpha+3}xp_1(x)]\, r_0(qx)\\
&\quad-[q^2p_0(x)-(1-q^n)\, q^{\alpha+3}p_1(x)]\, r_1(qx).
\end{aligned}$$

Finally, we use (7.19) and (7.15) to find

$$\begin{aligned}
(1-q)\, x^N[P_1^*(qx)+(1-q^n)\, xP_0^*(qx)]
&= q^{\alpha+N+4}[p_0(x)\, r_1(x)-p_1(x)\, r_0(x)]\\
&\quad+(1-q)\, q^{\alpha+N+4}x[p_0(x)\, r_1(x)-p_1(x)\, r_0(x)]\\
&\quad+[(1-q)\, q^{-N+1}r_1(qx)+q^{-N+2}p_1(qx)]\, r_0(qx)\\
&\quad-[(1-q)\, q^{-N+1}r_0(qx)+q^{-N+2}p_0(qx)]\, r_1(qx)\\
&= q^{\alpha+N+4}[p_0(x)\, r_1(x)-p_1(x)\, r_0(x)]\\
&\quad+(1-q)\, q^{\alpha+N+4}x[p_0(x)\, r_1(x)-p_1(x)\, r_0(x)]\\
&\quad-q^{-N+2}[p_0(qx)\, r_1(qx)-p_1(qx)\, r_0(qx)]\\
&= x^N[q^{\alpha+N+4}P_2^*(x)+(1-q)\, q^{\alpha+N+4}xP_2^*(x)-q^2P_2^*(qx)].
\end{aligned}$$

This proves (7.18) and therefore (7.3).

This completes the proof of the theorem.

8. Recurrence Relation

In this section we will prove the following theorem.

Theorem 2. *The polynomials* $\{L_n^{\alpha, M_0, M_1, ..., M_N}(x; q)\}_{n=0}^{\infty}$ *satisfy a* $(2N+3)$*-term recurrence relation of the form*

$$x^{N+1}L_n^{\alpha, M_0, M_1, ..., M_N}(x; q) = \sum_{k=\max(0,n-N-1)}^{n+N+1} E_k^{(n)}L_k^{\alpha, M_0, M_1, ..., M_N}(x; q), \qquad n=0, 1, 2, ... \tag{8.1}$$

Proof. Since $x^{N+1}L_n^{\alpha,M_0,M_1,...,M_N}(x;q)$ is a polynomial of degree $n+N+1$ we have

$$x^{N+1}L_n^{\alpha,M_0,M_1,...,M_N}(x;q)=\sum_{k=0}^{n+N+1} E_k^{(n)}L_k^{\alpha,M_0,M_1,...,M_N}(x;q), \qquad n=0,1,2,... \tag{8.2}$$

for some real coefficients $E_k^{(n)}$, $k=0,1,2,...,n+N+1$.

Taking the inner product with $L_m^{\alpha,M_0,M_1,...,M_N}(x;q)$ on both sides of (8.2) we find by using (4.1) for $n=0,1,2,...$ and $m=0,1,2,...,n+N+1$,

$$\begin{aligned}\langle L_m^{\alpha,M_0,M_1,...,M_N}(x;q), L_m^{\alpha,M_0,M_1,...,M_N}(x;q)\rangle_q\cdot E_m^{(n)}\\ =\langle x^{N+1}L_n^{\alpha,M_0,M_1,...,M_N}(x;q), L_m^{\alpha,M_0,M_1,...,M_N}(x;q)\rangle_q\\ =\langle x^{N+1}L_m^{\alpha,M_0,M_1,...,M_N}(x;q), L_n^{\alpha,M_0,M_1,...,M_N}(x;q)\rangle_q.\end{aligned} \tag{8.3}$$

In view of the orthogonality property of the polynomials $\{L_n^{\alpha,M_0,M_1,...,M_N}(x;q)\}_{n=0}^{\infty}$ we conclude that $E_m^{(n)}=0$ for $m+N+1<n$. This proves (8.1).

The coefficients $\{A_k\}_{k=0}^{N+1}$ in the definition (4.2) depend on n. To distinguish two coefficients with the same index, but depending on a different value of n we will write $A_k(n)$ instead of A_k. Comparing the leading coefficients on both sides of (8.2) we obtain by using this notation and (4.5)

$$\begin{aligned}E_{n+N+1}^{(n)}&=\frac{k_n}{k_{n+N+1}}=(-1)^{N+1}q^{-(N+1)(2n+\alpha+N+1)}\\ &\quad\times\frac{(q^{n+1};q)_{N+1}}{(1-q)^{N+1}}\frac{A_0(n)}{A_0(n+N+1)}\neq 0, \qquad n=0,1,2,....\end{aligned}$$

If we define

$$\begin{aligned}\Lambda_n&:=\langle L_n^{\alpha,M_0,M_1,...,M_N}(x;q), L_n^{\alpha,M_0,M_1,...,M_N}(x;q)\rangle_q\\ &=\frac{(q^{\alpha+1};q)_n}{(q;q)_n\,q^n}A_0\left(\sum_{k=0}^{N+1}q^{nk-\binom{k}{2}}A_k\right)\end{aligned}$$

then we find by using (8.3), (4.5), and the orthogonality that

$$E_{n-N-1}^{(n)}=\frac{k_{n-N-1}\Lambda_n}{k_n\Lambda_{n-N-1}}\neq 0, \qquad n=N+1,N+2,....$$

9. A Christoffel–Darboux Type Formula

From the recurrence relation (8.1) we easily obtain

$$\begin{aligned}(x^{N+1}-y^{N+1})\,&L_k^{\alpha,M_0,M_1,\dots,M_N}(x;q)\,L_k^{\alpha,M_0,M_1,\dots,M_N}(y;q)\\ &=\sum_{m=\max(0,k-N-1)}^{k+N+1} E_m^{(k)}[L_m^{\alpha,M_0,M_1,\dots,M_N}(x;q)\,L_k^{\alpha,M_0,M_1,\dots,M_N}(y;q)\\ &\quad -L_m^{\alpha,M_0,M_1,\dots,M_N}(y;q)\,L_k^{\alpha,M_0,M_1,\dots,M_N}(x;q)],\qquad k=0,1,2,\dots.\end{aligned}\tag{9.1}$$

We divide by Λ_k and sum over $k=0,1,2,\dots,n$:

$$\begin{aligned}(x^{N+1}-y^{N+1})&\sum_{k=0}^{n}\frac{L_k^{\alpha,M_0,M_1,\dots,M_N}(x;q)\,L_k^{\alpha,M_0,M_1,\dots,M_N}(y;q)}{\Lambda_k}\\ &=\sum_{k=0}^{n}\sum_{m=\max(0,k-N-1)}^{k+N+1}\frac{E_m^{(k)}}{\Lambda_k}[L_m^{\alpha,M_0,M_1,\dots,M_N}(x;q)\,L_k^{\alpha,M_0,M_1,\dots,M_N}(y;q)\\ &\quad -L_m^{\alpha,M_0,M_1,\dots,M_N}(y;q)\,L_k^{\alpha,M_0,M_1,\dots,M_N}(x;q)]\end{aligned}$$

for $n=0,1,2,\dots$.

Now we use (8.3) to see that

$$\frac{E_m^{(k)}}{\Lambda_k}=\frac{E_k^{(m)}}{\Lambda_m},\qquad k-N-1\leqslant m\leqslant k+N+1,\ k,m=0,1,2,\dots.$$

Now we have the following situations. For $n\leqslant N$ we have

$$\sum_{k=0}^{n}\sum_{m=\max(0,k-N-1)}^{k+N+1}=\sum_{k=0}^{n}\sum_{m=0}^{n}+\sum_{k=0}^{n}\sum_{m=n+1}^{k+N+1}=\sum_{k=0}^{n}\sum_{m=n+1}^{k+N+1}$$

and for $n\geqslant N+1$ we have

$$\begin{aligned}\sum_{k=0}^{n}\sum_{m=\max(0,k-N-1)}^{k+N+1}&=\sum_{k=0}^{n}\sum_{m=\max(0,k-N-1)}^{\min(n,k+N+1)}+\sum_{k=n-N}^{n}\sum_{m=n+1}^{k+N+1}\\ &=\sum_{k=n-N}^{n}\sum_{m=n+1}^{k+N+1}.\end{aligned}$$

So it follows from (9.1) by using this observation that

$$\begin{aligned}(x^{N+1}-y^{N+1})&\sum_{k=0}^{n}\frac{L_k^{\alpha,M_0,M_1,\dots,M_N}(x;q)\,L_k^{\alpha,M_0,M_1,\dots,M_N}(y;q)}{\Lambda_k}\\ &=\sum_{k=\max(0,n-N)}^{n}\sum_{m=n+1}^{k+N+1}\frac{E_m^{(k)}}{\Lambda_k}[L_m^{\alpha,M_0,M_1,\dots,M_N}(x;q)\,L_k^{\alpha,M_0,M_1,\dots,M_N}(y;q)\\ &\quad -L_m^{\alpha,M_0,M_1,\dots,M_N}(y;q)\,L_k^{\alpha,M_0,M_1,\dots,M_N}(x;q)]\end{aligned}\tag{9.2}$$

for $n=0, 1, 2, \ldots$. This can be considered as a generalization of the Christoffel–Darboux formula (3.12) for the q-Laguerre polynomials.

If we divide the Christoffel–Darboux type formula (9.2) by $x-y$ and let y tend to x then we find the confluent form

$$(N+1)x^N \sum_{k=0}^{n} \frac{\{L_k^{\alpha, M_0, M_1, \ldots, M_N}(x; q)\}^2}{\Lambda_k}$$
$$= \sum_{k=\max(0,n-N)}^{n} \sum_{m=n+1}^{k+N+1} \frac{E_m^{(k)}}{\Lambda_k} \left[L_k^{\alpha, M_0, M_1, \ldots, M_N}(x; q) \frac{d}{dx} L_m^{\alpha, M_0, M_1, \ldots, M_N}(x; q) \right.$$
$$\left. - L_m^{\alpha, M_0, M_1, \ldots, M_N}(x; q) \frac{d}{dx} L_k^{\alpha, M_0, M_1, \ldots, M_N}(x; q) \right]$$

for $n=0, 1, 2, \ldots$. This formula can be considered as a generalization of (3.13).

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References

1. R. Askey, The q-gamma and q-beta functions, *Appl. Anal.* **8** (1978), 125–141.
2. R. Askey, Ramanujan's extension of the gamma and beta function, *Amer. Math. Monthly* **87** (1980), 346–359.
3. T. S. Chihara, An introduction to orthogonal polynomials, *in* "Mathematics and Its Applications," Vol. 13, Gordon & Breach, New York, 1978.
4. G. Gasper and M. Rahman, Basic hypergeometric series, *in* "Encyclopedia of Mathematics and its Applications," Vol. 35, Cambridge Univ. Press, London/New York, 1990.
5. F. H. Jackson, A generalization of the functions $\Gamma(n)$ and x^n, *Proc. Roy. Soc. London* **74** (1904), 64–72.
6. F. H. Jackson, On q-definite integrals, *Quart. J. Pure Appl. Math.* **41** (1910), 193–203.
7. J. Koekoek and R. Koekoek, "A Simple Proof of a Differential Equation for Generalizations of Laguerre Polynomials," Faculty of Technical Mathematics and Informatics, Report No. 89-15, Delft University of Technology, 1989.
8. J. Koekoek and R. Koekoek, On a differential equation for Koornwinder's generalized Laguerre polynomials, *Proc. Amer. Math. Soc.* **112** (1991), 1045–1054.
9. R. Koekoek, "Koornwinder's Generalized Laguerre Polynomials and Its q-Analogues," Faculty of Technical Mathematics and Informatics, Report No. 88-87, Delft University of Technology, 1988.
10. R. Koekoek, Koornwinder's Laguerre polynomials, *Delft Progr. Rep.* **12** (1988), 393–404.
11. R. Koekoek, A generalization of Moak's q-Laguerre polynomials, *Canad. J. Math.* **42** (1990), 280–303.

12. R. KOEKOEK, Generalizations of Laguerre polynomials, *J. Math. Anal. Appl.* **153** (1990), 576–590.
13. R. KOEKOEK, "Generalizations of the Classical Laguerre Polynomials and Some q-Analogues," Thesis, Delft University of Technology, 1990.
14. R. KOEKOEK, On q-analogues of generalizations of the Laguerre polynomials, *in* "Orthogonal Polynomials and their Applications," Vol. 9, IMACS Annals on Computing and Applied Mathematics (C. Brezinski, L. Gori, and A. Ronveaux, Eds.), J. C. Baltzer AG, Basel, 1991, 315–320.
15. R. KOEKOEK AND H. G. MEIJER, A generalization of Laguerre polynomials, *SIAM J. Math. Anal.*, in press.
16. T. H. KOORNWINDER, Orthogonal polynomials with weight function $(1-x)^{\alpha}(1+x)^{\beta}+M\delta(x+1)+N\delta(x-1)$, *Canad. Math. Bull.* **27**, No. 2 (1984), 205–214.
17. D. S. MOAK, The q-analogue of the Laguerre polynomials, *J. Math. Anal. Appl.* **81** (1981), 20–47.
18. G. SZEGŐ, "Orthogonal Polynomials," 4th ed., American Mathematical Society Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, RI, 1975.