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POLY-CAUCHY NUMBERS

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Abstract. We define the poly-Cauchy numbers by generalizing the Cauchy numbers of the first kind (the Bernoulli numbers of the second kind when divided by a factorial). We study their characteristic properties which are analogous to those of the poly-Bernoulli numbers introduced by Kaneko, and generalize those of the classical Cauchy numbers of the first kind as well as of the second kind.

1. Introduction

The Cauchy numbers of the first kind c_n are introduced by the integral of the falling factorial:

$$c_n = \int_0^1 x(x-1)\dots(x-n+1)\,dx = n!\int_0^1 \binom{x}{n}\,dx$$

(see [**1**, Ch. VII]). The number $c_n/n!$ is sometimes referred to as the Bernoulli number of the second kind. They are not so well known, although they seem to have similar properties to those of the Bernoulli numbers of the first kind. The classical Bernoulli numbers (or the Bernoulli numbers of the first kind) B_n (with $B_1 = 1/2$) are defined by the generating function (see [**1**, Ch. I] and [**3**])

$$\frac{x}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

There are many identities involving the Bernoulli numbers. The Bernoulli numbers are closely related to the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$ defined by

$$x(x+1)\dots(x+n-1) = \sum_{m=0}^{n} \left[{n \atop m}\right] x^m,$$

and the Stirling numbers of the second kind $\left\{{n \atop m}\right\}$ determined by

$$\left\{{n \atop m}\right\} = \frac{1}{m!}\sum_{j=0}^{m}(-1)^j \binom{m}{j}(m-j)^n$$

(see [**2**]). Some of such relations are

$$\frac{1}{n!}\sum_{m=0}^{n}(-1)^m \left[{n+1 \atop m+1}\right] B_m = \frac{1}{n+1},$$

$$B_n = (-1)^n \sum_{m=0}^{n} \left\{{n \atop m}\right\} \frac{(-1)^m m!}{m+1}.$$

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The Bernoulli numbers also are related to the binomial coefficients. Some of the known identities are

$$B_n = -\sum_{m=1}^{n+1} \frac{(-1)^m}{m} \binom{n+1}{m} \sum_{j=1}^{m} j^n,$$

$$\sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m B_m}{n-m+2} = \frac{B_{n+1}}{n+1},$$

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} B_{n+r} = \sum_{s=0}^{n} (-1)^s \binom{n}{s} B_{m+s},$$

$$\sum_{m=0}^{n} (-1)^{n+m} \binom{n+1}{m} (n+m+1) B_{n+m} = 0 \quad (n \ge 1),$$

$$\frac{1}{n} \sum_{m=1}^{n} \binom{n}{m} B_m B_{n-m} - B_{n-1} = -B_n \quad (n \ge 1).$$

They are also connected with the Riemann-zeta functions $\zeta(n)$ (n even) as

$$\zeta(n) = \frac{(-1)^{n/2-1} B_n (2\pi)^n}{2n!}.$$

In 1997 Kaneko [**3**] introduced the poly-Bernoulli numbers $B_n^{(k)}$ by

$$\frac{1}{z}\mathrm{Li}_k(z)\Big|_{z=1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!},$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

is the kth polylogarithm function. When $k = 1$, $B_n^{(1)}$ is the classical Bernoulli number with $B_1^{(1)} = 1/2$. The generating function of the poly-Bernoulli numbers can also be written in terms of iterated integrals:

$$e^x \cdot \underbrace{\frac{1}{e^x-1} \int_0^x \frac{1}{e^x-1} \int_0^x \cdots \frac{1}{e^x-1} \int_0^x}_{k-1} \frac{x}{e^x-1} \underbrace{dx\, dx \ldots dx}_{k-1} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}.$$

An explicit formula for $B_n^{(k)}$ is given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-1)^m m!}{(m+1)^k} \quad (n \ge 0, k \ge 1). \tag{1}$$

In this paper, we introduce the poly-Cauchy numbers as an analog of the poly-Bernoulli numbers, generalizing the classical Cauchy numbers.

2. Poly-Cauchy numbers of the first kind

Let n and k be integers with $n \ge 0$ and $k \ge 1$ throughout the paper. Define the *poly-Cauchy numbers* $c_n^{(k)}$ as follows.

$$c_n^{(k)} = n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{x_1 x_2 \dots x_k}{n} dx_1\, dx_2 \dots dx_k.$$

The Cauchy numbers $c_n = c_n^{(1)}$ can be expressed in terms of the (unsigned) Stirling numbers of the first kind $\left[{n \atop m} \right]$:

$$c_n^{(1)} = (-1)^n \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{(-1)^m}{m+1}$$

(see [**1**, Ch. VII] and [**4**, p. 1908]). The poly-Cauchy numbers $c_n^{(k)}$ can also be expressed in terms of the Stirling numbers of the first kind $\left[{n \atop m} \right]$. This can be considered as an analog of the identity (1).

THEOREM 1. *We have*

$$c_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{(-1)^m}{(m+1)^k} \quad (n \ge 0,\ k \ge 1).$$

Proof. Denote the falling factorial $(x)_n = x(x-1)\dots(x-n+1)$ $(n \ge 1)$ with $(x)_0 = 1$. Then by the identity

$$\binom{x}{n} = \frac{(x)_n}{n!} = \frac{1}{n!} \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^{n-m} x^m$$

(see e.g. [**2**, Ch. 6]), we have

$$\begin{aligned} c_n^{(k)} &= n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{x_1 x_2 \dots x_k}{n} dx_1\, dx_2 \dots dx_k \\ &= \underbrace{\int_0^1 \cdots \int_0^1}_{k} \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^{n-m} (x_1 x_2 \dots x_k)^m dx_1\, dx_2 \dots dx_k \\ &= (-1)^n \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{(-1)^m}{(m+1)^k}. \end{aligned}$$

□

By using Theorem 1, we obtain

$$\begin{aligned} c_0^{(k)} &= 1, \\ c_1^{(k)} &= \frac{1}{2^k}, \\ c_2^{(k)} &= -\frac{1}{2^k} + \frac{1}{3^k}, \\ c_3^{(k)} &= \frac{2}{2^k} - \frac{3}{3^k} + \frac{1}{4^k}, \end{aligned}$$

$$c_4^{(k)} = -\frac{6}{2^k} + \frac{11}{3^k} - \frac{6}{4^k} + \frac{1}{5^k},$$
$$c_5^{(k)} = \frac{24}{2^k} - \frac{50}{3^k} + \frac{35}{4^k} - \frac{10}{5^k} + \frac{1}{6^k},$$
$$c_6^{(k)} = -\frac{120}{2^k} + \frac{274}{3^k} - \frac{225}{4^k} + \frac{85}{5^k} - \frac{15}{6^k} + \frac{1}{7^k}.$$

We list the poly-Cauchy numbers $c_n^{(k)}$ for $n \le 6$ and $k \le 6$ in Table 1.

Similarly to the polylogarithm function $\mathrm{Li}_k(z)$, we define $\mathrm{Lif}_k(z)$ ($k = 1, 2, \dots$) by

$$\mathrm{Lif}_k(z) := \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}$$

and we call it the kth *polylogarithm factorial function*. Then, we get a generating function of the poly-Cauchy numbers. We may define the poly-Cauchy numbers by this generating function.

THEOREM 2. *The generating function of the poly-Cauchy numbers $c_n^{(k)}$ is given by the following:*

$$\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}.$$

Proof. Since

$$\frac{(\ln(1+x))^m}{m!} = (-1)^m \sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-x)^n}{n!},$$

we have by Theorem 1

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^{m+n}}{(m+1)^k} \frac{x^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^k} \sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-x)^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{(\ln(1+x))^m}{m!(m+1)^k} = \mathrm{Lif}_k(\ln(1+x)). \qquad \square
\end{aligned}$$

Remark. The poly-Cauchy numbers can be defined also for negative k, if one uses Theorem 2 as the definition, because the series has a meaning as a formal power series even if k is non-positive.

The generating function of the Cauchy numbers of the first kind c_n is also given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$$

(see [**1**, Ch. VII]). The generating function of the poly-Cauchy numbers in Theorem 2 can be written in the form of iterated integrals.

TABLE 1. $c_n^{(k)}$ $(1 \le n, k \le 6)$.

	n						
	0	1	2	3	4	5	6
$c_n^{(1)}$	1	1/2	−1/6	1/4	−19/30	9/4	−863/84
$c_n^{(2)}$	1	1/4	−5/36	11/48	−1103/1800	1627/720	−374473/35280
$c_n^{(3)}$	1	1/8	−19/216	89/576	−46261/108000	23323/14400	−114895757/14817600
$c_n^{(4)}$	1	1/16	−65/1296	635/6912	−1691507/6480000	2602903/2592000	−30316306813/6223392000
$c_n^{(5)}$	1	1/32	−211/7776	4241/82944	−57453709/388800000	29825987/51840000	−7362684132917/2613824640000
$c_n^{(6)}$	1	1/64	−665/46656	27251/995328	−1867678883/23328000000	2933162407/9331200000	−1700444184290653/1097806348800000

COROLLARY 1. *For $k \ge 2$ we have*

$$\underbrace{\frac{1}{\ln(1+x)}\int_0^x \frac{1}{(1+x)\ln(1+x)}\int_0^x \cdots \frac{1}{(1+x)\ln(1+x)}\int_0^x}_{k-1} \frac{x}{(1+x)\ln(1+x)} \underbrace{dx\, dx \ldots dx}_{k-1}$$

$$= \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}.$$

Proof. For $k \ge 2$ we have

$$\begin{aligned}
\mathrm{Lif}_k(z) &= \sum_{m=0}^{\infty} \frac{z^m}{(m+1)^{k-1}(m+1)!} \\
&= \frac{1}{z}\sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+1)^{k-1}(m+1)!} \\
&= \frac{1}{z}\int_0^z \sum_{m=0}^{\infty} \frac{z^m}{(m+1)^{k-2}(m+1)!}\, dz \\
&= \frac{1}{z}\int_0^z \frac{1}{z}\int_0^z \sum_{m=0}^{\infty} \frac{z^m}{(m+1)^{k-3}(m+1)!}\, dz\, dz \\
&= \cdots \\
&= \underbrace{\frac{1}{z}\int_0^z \frac{1}{z}\int_0^z \cdots \frac{1}{z}\int_0^z}_{k-1} \sum_{m=0}^{\infty} \frac{z^m}{(m+1)!} \underbrace{dz\, dz \ldots dz}_{k-1}.
\end{aligned}$$

For $k = 1$ we have

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{z^m}{(m+1)!} &= \frac{1}{z}\sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+1)!} \\
&= \frac{e^z - 1}{z}.
\end{aligned}$$

Therefore,

$$\mathrm{Lif}_k(z) = \underbrace{\frac{1}{z}\int_0^z \frac{1}{z}\int_0^z \cdots \frac{1}{z}\int_0^z}_{k-1} \frac{e^z-1}{z} \underbrace{dz\, dz \ldots dz}_{k-1}.$$

Putting $z = \ln(1+x)$, we obtain the result. □

THEOREM 3. *We have*

$$\sum_{m=0}^{n} \left\{ {n \atop m} \right\} c_m^{(k)} = \frac{1}{(n+1)^k}.$$

Remark. If $k = 1$, then this identity is the same as that in [**4**, Theorem 2.3].

Proof. Using the identity

$$\sum_{m=0}^{\max\{l,n\}}(-1)^{m-n}\begin{bmatrix} m \\ l \end{bmatrix}\begin{Bmatrix} n \\ m \end{Bmatrix}=\delta_{ln}$$

(see, e.g., [**2**, Ch. 6]), where δ_{ln} is the Kronecker delta defined by

$$\delta_{ln}=\begin{cases}1 & (l=n),\\ 0 & (l\neq n),\end{cases}$$

we have by Theorem 1

$$\begin{aligned}\sum_{m=0}^{n}\begin{Bmatrix} n \\ m \end{Bmatrix}c_m^{(k)} &= \sum_{m=0}^{n}\begin{Bmatrix} n \\ m \end{Bmatrix}(-1)^m\sum_{l=0}^{m}\begin{bmatrix} m \\ l \end{bmatrix}\frac{(-1)^l}{(l+1)^k}\\ &= \sum_{l=0}^{n}\frac{(-1)^l}{(l+1)^k}\sum_{m=l}^{n}(-1)^m\begin{bmatrix} m \\ l \end{bmatrix}\begin{Bmatrix} n \\ m \end{Bmatrix}\\ &= \sum_{l=0}^{n}\frac{(-1)^l}{(l+1)^k}\cdot(-1)^n\delta_{ln}\\ &= \frac{(-1)^n}{(n+1)^k}\cdot(-1)^n\cdot 1=\frac{1}{(n+1)^k}.\end{aligned}$$

□

3. Poly-Cauchy numbers of the second kind

The Cauchy numbers of the second kind $\hat{c}_n$ is defined by

$$\begin{aligned}\hat{c}_n &= n!\int_0^1\binom{-x}{n}dx\\ &= \int_0^1(-x)(-x-1)\dots(-x-n+1)\,dx=(-1)^n\int_0^1\langle x\rangle_n\,dx,\end{aligned}$$

where $\langle x\rangle_n=x(x+1)\dots(x+n-1)$ $(n\geq 1)$ is the rising factorial with $\langle x\rangle_0=1$ (see [**1**, Ch. VII]). The numbers c_n are called the Cauchy numbers of the first kind, in order to distinguish them from those of the second kind. Similarly to the poly-Cauchy numbers of the first kind, we define the poly-Cauchy numbers of the second kind as follows:

$$\hat{c}_n^{(k)}=n!\underbrace{\int_0^1\cdots\int_0^1}_{k}\binom{-x_1x_2\dots x_k}{n}dx_1\,dx_2\dots dx_k.$$

The Cauchy numbers of the second kind $\hat{c}_n=\hat{c}_n^{(1)}$ are expressed in terms of the Stirling numbers of the first kind:

$$\hat{c}_n^{(1)}=(-1)^n\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\frac{1}{m+1}$$

(see [**1**, Ch. VII] and [**4**]). The poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ can be also expressed in terms of the Stirling numbers of the first kind.

THEOREM 4. *We have*

$$\hat{c}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k}.$$

Proof. Note that

$$\binom{-x}{n} = (-1)^n \frac{\langle x \rangle_n}{n!} = \frac{(-1)^n}{n!} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m$$

(see, e.g., [**2**, Ch. 6]). Hence, we have

$$\begin{aligned} \hat{c}_n^{(k)} &= n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{-x_1 x_2 \ldots x_k}{n} \, dx_1 \, dx_2 \ldots dx_k \\ &= \underbrace{\int_0^1 \cdots \int_0^1}_{k} (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (x_1 x_2 \ldots x_k)^m \, dx_1 \, dx_2 \ldots dx_k \\ &= (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k}. \end{aligned}$$ □

By using Theorem 4, we obtain

$$\begin{aligned} \hat{c}_0^{(k)} &= 1, \\ \hat{c}_1^{(k)} &= -\frac{1}{2^k}, \\ \hat{c}_2^{(k)} &= \frac{1}{2^k} + \frac{1}{3^k}, \\ \hat{c}_3^{(k)} &= -\frac{2}{2^k} - \frac{3}{3^k} - \frac{1}{4^k}, \\ \hat{c}_4^{(k)} &= \frac{6}{2^k} + \frac{11}{3^k} + \frac{6}{4^k} + \frac{1}{5^k}, \\ \hat{c}_5^{(k)} &= -\frac{24}{2^k} - \frac{50}{3^k} - \frac{35}{4^k} - \frac{10}{5^k} - \frac{1}{6^k}. \end{aligned}$$

In a similar manner to the results in Theorem 2, Corollary 1 and Theorem 3 regarding the poly-Cauchy numbers of the first kind $c_n^{(k)}$, we can obtain the following corresponding results about the poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$. The proofs are similar and are omitted.

THEOREM 5. *The generating function of the poly-Cauchy numbers $\hat{c}_n^{(k)}$ is as follows:*

$$\mathrm{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{x^n}{n!},$$

where

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}.$$

The generating function of the Cauchy numbers of the second kind $\hat{c}_n = \hat{c}_n^{(1)}$ is given by

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \hat{c}_n^{(1)} \frac{x^n}{n!}$$

(see [**1**, Ch. VII] and [**4**, p. 1910]). The generating function of the poly-Cauchy numbers of the second kind can be also written in the form of iterated integrals by putting $z = -\ln(1+x)$ in

$$\mathrm{Lif}_k(z) = \underbrace{\frac{1}{z}\int_0^z \frac{1}{z}\int_0^z \cdots \frac{1}{z}\int_0^z}_{k-1} \frac{e^z - 1}{z} \underbrace{dz\, dz \ldots dz}_{k-1}.$$

COROLLARY 2. *For $k \ge 2$ we have*

$$\underbrace{\frac{1}{\ln(1+x)}\int_0^x \frac{1}{(1+x)\ln(1+x)}\int_0^x \cdots \frac{1}{(1+x)\ln(1+x)}\int_0^x}_{k-1} \frac{x}{(1+x)^2\ln(1+x)} \underbrace{dx\, dx \ldots dx}_{k-1}$$
$$= \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{x^n}{n!}.$$

We also have the corresponding identity to that in Theorem 3. If $k = 1$, the result is reduced to the second identity in [**4**, Theorem 2.6]. The proof is similar and is omitted.

THEOREM 6. *We have*

$$\sum_{m=0}^{n} \left\{ {n \atop m} \right\} \hat{c}_m^{(k)} = \frac{(-1)^n}{(n+1)^k}.$$

4. Relations between two kinds of poly-Cauchy numbers

There are some relations between the poly-Cauchy numbers of the first kind and those of the second kind.

THEOREM 7. *For $n \ge 1$ we have*

$$(-1)^n \frac{c_n^{(k)}}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\hat{c}_m^{(k)}}{m!},$$
$$(-1)^n \frac{\hat{c}_n^{(k)}}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{c_m^{(k)}}{m!}.$$

Proof. We shall prove the first identity. The second identity is proved similarly. By using the identity (see, e.g., [**2**, Ch. 6])

$$\frac{(-1)^l}{n!} \left[{n \atop l} \right] = \sum_{m=l}^{n} \frac{(-1)^m}{m!} \binom{n-1}{m-1} \left[{m \atop l} \right]$$

and Theorems 1 and 4, we have

$$\begin{aligned}\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{\hat{c}_m^{(k)}}{m!}&=\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{(-1)^m}{m!}\sum_{l=1}^{m}\left[{m \atop l}\right]\frac{1}{(l+1)^k}\\&=\sum_{l=1}^{n}\frac{1}{(l+1)^k}\sum_{m=l}^{n}\frac{(-1)^m}{m!}\binom{n-1}{m-1}\left[{m \atop l}\right]\\&=\sum_{l=1}^{n}\frac{1}{(l+1)^k}\frac{(-1)^l}{n!}\left[{n \atop l}\right]=(-1)^n\frac{c_n^{(k)}}{n!}.\qquad\square\end{aligned}$$

Remark. As an alternative proof of the theorem, one can use the generating series. Both generating series can be transformed to each other by the same substitution $-x/(1+x)$ for x.

5. A relation with poly-Bernoulli numbers

There is a relation with poly-Bernoulli numbers.

THEOREM 8. *For $n \ge 1$ we have*

$$B_n^{(k)}=\sum_{l=1}^{n}\sum_{m=1}^{n}m!\left\{{n \atop m}\right\}\left\{{m-1 \atop l-1}\right\}c_l^{(k)}.$$

Proof. Note that

$$\sum_{l=i}^{m}(-1)^l\left\{{m-1 \atop l-1}\right\}\left[{l \atop i}\right]=(-1)^m\binom{m-1}{i-1}$$

(see, e.g. [**2**, (6.26)]) and

$$\sum_{m=i}^{n}m!\left\{{n \atop m}\right\}(-1)^m\binom{m-1}{i-1}=(-1)^n i!\left\{{n \atop i}\right\}$$

(see, e.g. [**2**, (6.19), (6.21)]). Then by Theorem 1 we have

$$\begin{aligned}&\sum_{l=1}^{n}\sum_{m=1}^{n}m!\left\{{n \atop m}\right\}\left\{{m-1 \atop l-1}\right\}c_l^{(k)}\\&=\sum_{l=1}^{n}\sum_{m=l}^{n}m!\left\{{n \atop m}\right\}\left\{{m-1 \atop l-1}\right\}(-1)^l\sum_{i=0}^{l}\left[{l \atop i}\right]\frac{(-1)^i}{(i+1)^k}\\&=\sum_{i=1}^{n}\frac{(-1)^i}{(i+1)^k}\sum_{l=i}^{n}\sum_{m=l}^{n}m!\left\{{n \atop m}\right\}\left\{{m-1 \atop l-1}\right\}(-1)^l\left[{l \atop i}\right]\\&=\sum_{i=1}^{n}\frac{(-1)^i}{(i+1)^k}\sum_{m=i}^{n}m!\left\{{n \atop m}\right\}\sum_{l=i}^{m}(-1)^l\left\{{m-1 \atop l-1}\right\}\left[{l \atop i}\right]\\&=\sum_{i=1}^{n}\frac{(-1)^i}{(i+1)^k}\sum_{m=i}^{n}m!\left\{{n \atop m}\right\}(-1)^m\binom{m-1}{i-1}\\&=\sum_{i=1}^{n}\frac{(-1)^i}{(i+1)^k}(-1)^n i!\left\{{n \atop i}\right\}=B_n^{(k)}.\qquad\square\end{aligned}$$

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