Poly-Cauchy numbers and poly-Bernoulli numbers

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1 Introduction

The *Cauchy numbers* of the first kind, denoted by c_n ([5]), are defined by the integral of the falling factorial:

$$c_n = \int_0^1 x(x-1)\dots(x-n+1)dx\,.$$

The generating function of the Cauchy numbers of the first kind c_n is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^\infty c_n\frac{x^n}{n!}$$

([23]).

Cauchy numbers are not so famous, though they seem to have similar properties to those of the *Bernoulli numbers*. The classical Bernoulli numbers B_n are defined by the generating function

$$\frac{x}{e^x-1} = \sum_{n=0}^\infty B_n\frac{x^n}{n!}\quad\left(B_1=-\frac{1}{2}\right)\,.$$

Before the terminology of Cauchy numbers appeared in Comtet's book ([5]), the concept of the Cauchy numbers was first introduced by Nörlund ([24, pp.146–147]) in 1924. Here, the higher order Bernoulli numbers $B_n^{(r)}$ are defined by

$$\left(\frac{x}{e^x-1}\right)^r = \sum_{n=0}^\infty B_n^{(r)}\frac{x^n}{n!}\quad(|x|<2\pi)$$

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or

$$\left(\frac{\ln(1+x)}{x}\right)^r = r\sum_{n=0}^\infty \frac{B_n^{(r+n)}}{r+n}\frac{x^n}{n!} \quad (|x|<1)\,.$$

See also [8, p.257,p.259]. Then

$$B_n^{(n)} = \int_0^1 (x-1)(x-2)\cdots(x-n)dx$$

or

$$B_{n+1}^{(n)} = -n\int_0^1 x(x-1)\cdots(x-n)dx\,.$$

Hence, $c_n = -B_n^{(n-1)}/(n-1)$. Ch. Jordan studied the Bernoulli numbers of the second kind b_n ([13, p.131]), defined by

$$b_n = \psi_{n+1}(1) - \psi_{n+1}(0) = \int_0^1 \binom{x}{n}dx\,.$$

Hence, $b_n = c_n/n!$. In 1961 Carlitz ([4]) introduced the numbers β_n, defined by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^\infty \beta_n \frac{x^n}{n!}$$

Namely, $\beta_n = c_n$.

Cauchy numbers and Bernoulli numbers are much related to the Stirling numbers of the first kind and of the second kind. The (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$ arise as coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{m=0}^n \left[{n \atop m}\right] x^m\,.$$

The Stirling numbers of the second kind $\left\{{n \atop m}\right\}$ are determined by

$$\left\{{n \atop m}\right\} = \frac{1}{m!}\sum_{j=0}^m (-1)^j\binom{m}{j}(m-j)^n\,.$$

There are many identities about the Bernoulli numbers. They are much related to the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$ and the

Stirling numbers of the second kind $\left\{{n \atop m}\right\}$. Some of them are

$$\frac{1}{n!}\sum_{m=0}^n(-1)^m\left[{n+1 \atop m+1}\right]B_m=\frac{1}{n+1}\,,$$

$$B_n=(-1)^n\sum_{m=0}^n\left\{{n \atop m}\right\}\frac{(-1)^m m!}{m+1}\,.$$

The corresponding identities of the classical Cauchy numbers are

$$\sum_{m=0}^n\left\{{n \atop m}\right\}c_m=\frac{1}{n+1}\,,$$

$$c_n=(-1)^n\sum_{m=0}^n\left[{n \atop m}\right]\frac{(-1)^m}{m+1}\,.$$

2 Polylogarithms

The k-th *polylogarithm* function is defined by

$$\mathrm{Li}_k(x)=\sum_{m=1}^\infty\frac{x^m}{m^k}\,.$$

The k-th *polylogarithm factorial* function is defined by

$$\mathrm{Lif}_k(x)=\sum_{m=0}^\infty\frac{x^m}{m!(m+1)^k}\,.$$

For $k\ge 2$

$$x\frac{d}{dx}\mathrm{Li}_k(x)=\mathrm{Li}_{k-1}(x)\,,$$

so

$$\mathrm{Li}_k(x)=\int_0^x\frac{\mathrm{Li}_{k-1}(t)}{t}dt\,;$$

on the other hand,

$$\frac{d}{dx}(x\mathrm{Lif}_k(x))=\mathrm{Lif}_{k-1}(x)\,,$$

so

$$\mathrm{Lif}_k(x)=\frac{1}{x}\int_0^x\mathrm{Lif}_{k-1}(t)dt\,.$$

In special, for $k = 0, 1$ we have

$$\mathrm{Li}_0(x) = \frac{x}{1-x}, \quad \mathrm{Li}_1(x) = -\ln(1-x)$$

and

$$\mathrm{Lif}_0(x) = e^x, \quad \mathrm{Lif}_1(x) = (e^x - 1)/x\,.$$

For $k = -r$ we have

$$\mathrm{Li}_{-r}(x) = \frac{1}{(1-x)^{r+1}} \sum_{j=0}^{r} \left\langle {r \atop j} \right\rangle x^{r-j} \quad (r = 0, 1, 2, \dots)$$

([3]), where

$$\left\langle {r \atop j} \right\rangle = \sum_{l=0}^{j+1} (-1)^l \binom{r+1}{l} (j - l + 1)^r$$

are the Eulerian numbers.

On the other hand, for $k = -r$ we have

$$\mathrm{Lif}_{-r}(x) = e^x \sum_{j=0}^{r} \left\{ {r+1 \atop j+1} \right\} x^j \quad (r = 0, 1, 2, \dots)\,.$$

We have the record for the first some values r.

$$\begin{aligned}
\mathrm{Lif}_0(x) &= e^x, \\
\mathrm{Lif}_{-1}(x) &= (1 + x)e^x, \\
\mathrm{Lif}_{-2}(x) &= (1 + 3x + x^2)e^x, \\
\mathrm{Lif}_{-3}(x) &= (1 + 7x + 6x^2 + x^3)e^x, \\
\mathrm{Lif}_{-4}(x) &= (1 + 15x + 25x^2 + 10x^3 + x^4)e^x, \\
\mathrm{Lif}_{-5}(x) &= (1 + 31x + 90x^2 + 65x^3 + 15x^4 + x^5)e^x.
\end{aligned}$$

In 1997 M. Kaneko ([18]) introduced the *poly-Bernoulli numbers* $B_n^{(k)}$ by

$$\frac{\mathrm{Li}_k(1 - e^{-x})}{1 - e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}\,.$$

When $k = 1$, $B_n^{(1)}$ is the classical Bernoulli number with $B_1^{(1)} = 1/2$.

Recently, we [19] introduced the *poly-Cauchy numbers* $c_n^{(k)}$ by

$$\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}\,.$$

When $k=1$, $c_n^{(1)} = c_n$ is the classical Cauchy number.

Poly-Cauchy numbers of the first kind $c_n^{(k)}$ may be defined by

$$c_n^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (x_1 x_2 \dots x_k)(x_1 x_2 \dots x_k - 1) \\ \cdots (x_1 x_2 \dots x_k - n + 1) dx_1 dx_2 \dots dx_k\,.$$

In addition, *poly-Cauchy numbers of the second kind* $\hat{c}_n^{(k)}$ are defined by

$$\hat{c}_n^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (-x_1 x_2 \dots x_k)(-x_1 x_2 \dots x_k - 1) \\ \cdots (-x_1 x_2 \dots x_k - n + 1) dx_1 dx_2 \dots dx_k\,.$$

The generating function of the poly-Bernoulli numbers are written in terms of iterated integrals:

$$\frac{e^x}{e^x - 1} \underbrace{\int_0^x \frac{1}{e^x - 1} \cdots \int_0^x \frac{1}{e^x - 1}}_{k-1} \times x \underbrace{dx \dots dx}_{k-1} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}\,.$$

An explicit formula for $B_n^{(k)}$ is given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left\{ n \atop m \right\} \frac{(-1)^m m!}{(m+1)^k} \quad (n \ge 0, k \ge 1)\,. \tag{1}$$

The generating function of the poly-Cauchy numbers can be also written in the form of iterated integrals:

$$\frac{1}{\ln(1+x)} \underbrace{\int_0^x \frac{1}{(1+x)\ln(1+x)} \cdots \int_0^x \frac{1}{(1+x)\ln(1+x)}}_{k-1} \times x \underbrace{dx \dots dx}_{k-1} \\ = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}\,.$$

An explicit formula for $c_n^{(k)}$ is given by

$$c_n^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m}{(m+1)^k} \,. \tag{2}$$

There are some relations between poly-Cauchy numbers and poly-Bernoulli numbers.

Theorem 1. *For $n \ge 1$ we have*

$$\begin{aligned} B_n^{(k)} &= \sum_{l=1}^{n} \sum_{m=1}^{n} m! \left\{ n \atop m \right\} \left\{ m-1 \atop l-1 \right\} c_l^{(k)} \,, \\ c_n^{(k)} &= \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{(-1)^{n-m}}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} B_l^{(k)} \,. \end{aligned}$$

3 Duality theorem

It is known that the duality theorem holds for poly-Bernoulli numbers ([18]). Namely,

$$B_n^{(-k)} = B_k^{(-n)} \quad (n,\ k \ge 0)\,.$$

It is due to the symmetric formula:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}} \,.$$

It follows that

$$\begin{aligned} B_n^{(-k)} &= \sum_{m=0}^{n} (-1)^{m+n} m! \left\{ n \atop m \right\} (m+1)^k \,, \\ B_n^{(-k)} &= \sum_{j=0}^{k} (j!)^2 \left\{ n+1 \atop j+1 \right\} \left\{ k+1 \atop j+1 \right\} \,. \end{aligned}$$

However, the duality theorem does not hold for poly-Cauchy numbers. In fact, we have

Proposition 1.

$$\sum_{n=0}^\infty\sum_{k=0}^\infty c_n^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}=e^y(1+x)^{e^y}\,,$$
$$\sum_{n=0}^\infty\sum_{k=0}^\infty \hat{c}_n^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}=\frac{e^y}{(1+x)^{e^y}}\,.$$

By using Proposition 1 we have explicit expressions of the poly-Cauchy numbers with negative indices.

Theorem 2 ([16])**.**

$$c_n^{(-k)}=\sum_{j=0}^k(-1)^{n+j}j!\left(\begin{bmatrix}n\\j\end{bmatrix}-n\begin{bmatrix}n-1\\j\end{bmatrix}\right)\begin{Bmatrix}k+1\\j+1\end{Bmatrix}\,,$$
$$\hat{c}_n^{(-k)}=\sum_{j=0}^k(-1)^{n}j!\begin{bmatrix}n+1\\j+1\end{bmatrix}\begin{Bmatrix}k+1\\j+1\end{Bmatrix}\,.$$

Moreover, using Theorem 2 with (2) we have the following congruence results.

Theorem 3. *For any positive integer k, $c_n^{(-k)}\equiv c_n^{(-k-4)}\pmod{10}$ and $\hat{c}_n^{(-k)}\equiv\hat{c}_n^{(-k-4)}\pmod{10}$. In special, when $n=1$, for $k\ge 1$, $c_1^{(-k-4)}\equiv c_1^{(-k)}\pmod{30}$ and $\hat{c}_1^{(-k-4)}\equiv\hat{c}_1^{(-k)}\pmod{30}$.*

Theorem 4. *For $k\ge 1$ we have*

$$c_n^{(-k)}\equiv\begin{cases}0\pmod 2 & \textit{if } n=1 \textit{ or } n\ge 4;\\ 1\pmod 2 & \textit{if } n=2,3\,,\end{cases}$$
$$\hat{c}_n^{(-k)}\equiv\begin{cases}0\pmod 2 & \textit{if } n=1 \textit{ or } n\ge 4;\\ 1\pmod 2 & \textit{if } n=2,3\,.\end{cases}$$

4 Sums of products

Sums of products of Bernoulli numbers

$$\sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}} \frac{n!}{i_1!\cdots i_m!} B_{i_1}\cdots B_{i_m} \quad (m\ge 1,\ n\ge 0)$$

have been considered by many authors (see, e.g. [1, 2, 6]). When $m=2$, one has the famous Euler's identity:

$$\sum_{i=0}^{n}\binom{n}{i} B_i B_{n-i} = -nB_{n-1}-(n-1)B_n \quad (n\ge 1)\,. \tag{3}$$

Kamano ([14]) considered the sums of products of Bernoulli numbers, including poly-Bernoulli numbers

$$S_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}} \frac{n!}{i_1!\cdots i_m!} B_{i_1}\cdots B_{i_{m-1}} B_{i_m}^{(k)} \quad (m\ge 1,\ n\ge 0)\,.$$

Then, $S_m^{(k)}(n)$ satisfies the following relation:

Proposition 2.

$$\sum_{l=0}^{m}(-1)^{m-l}\begin{bmatrix} m+1\\ l+1\end{bmatrix} S_{m+1}^{(k-l)}(n) = \begin{cases} \dfrac{n!}{(n-m)!}\displaystyle\sum_{r=0}^{m}\begin{bmatrix} m\\ r\end{bmatrix} B_{n-m+r}^{(k)} & (n\ge m),\\ 0 & (0\le n\le m-1)\,. \end{cases}$$

Kamano also showed the explicit formulae $S_m^{(k)}(n)$ for $m=2,3$. For example, when $m=2$ we have

Proposition 3. *For $k\ge 1$ and $n\ge 0$,*

$$S_2^{(0)}(n) = B_n^{(1)},$$

$$S_2^{(k)}(n) = B_n^{(1)} - n\sum_{j=1}^{k} B_n^{(j)},$$

$$S_2^{(-k)}(n) = B_n^{(1)} + n\sum_{j=0}^{k-1} B_n^{(-j)}.$$

It seemed to be difficult to give an explicit formula for $S_m^{(k)}(n)$ for $m \ge 4$, but recently a general formula for all $m \ge 1$ is given.

Theorem 5 ([20]). *For $m \ge 1$, $n \ge 0$ and $k \ge 1$, we have*

$$S_{m+1}^{(0)}(n) = S_m^{(1)}(n),$$

$$S_{m+1}^{(k)}(n) = \sum_{r=0}^{m-1}(-1)^r r!\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{(i+1)^k}S_{m+1-r}^{(0)}(n-r)$$
$$+ (-1)^m\binom{n}{m}\sum_{\substack{j_1+\cdots+j_m\le k-1\\ j_1,\dots,j_m\ge 0}}\frac{1}{2^{j_2}\cdots m^{j_m}}\sum_{\nu=1}^{m}\left[{m \atop \nu}\right]B_{n-m+\nu}^{(1+j_1)},$$

$$S_{m+1}^{(-k)}(n)$$
$$= \sum_{r=0}^{m-1}(-1)^r r!\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}(-1)^i(i+1)^k S_{m+1-r}^{(0)}(n-r)$$
$$+ \binom{n}{m}\sum_{\substack{j_1+\cdots+j_m\le k\\ j_1,\dots,j_m\ge 1}}2^{j_2}\cdots m^{j_m}\sum_{\nu=1}^{m}\left[{m \atop \nu}\right]B_{n-m+\nu}^{(1-j_1)}.$$

Sums of products of Cauchy numbers

$$\sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}}\frac{n!}{i_1!\cdots i_m!}c_{i_1}\cdots c_{i_m}\quad (m\ge 1,\ n\ge 0)$$

were studied by Zhao ([25]). Consider the sums of products of Cauchy numbers, including poly-Cauchy numbers

$$T_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}}\frac{n!}{i_1!\cdots i_m!}c_{i_1}\cdots c_{i_{m-1}}c_{i_m}^{(k)}\quad (m\ge 1,\ n\ge 0)\,.$$

Then, $T_m^{(k)}(n)$ satisfies the following relation:

Proposition 4 ([22]).

$$\sum_{l=0}^{m}(-1)^{m-l}\left[{m+1 \atop l+1}\right]T_{m+1}^{(k-l)}(n)$$
$$= \begin{cases}\displaystyle\sum_{l=0}^{m}\sum_{i=0}^{n-m}\frac{n!}{i!}\binom{l}{n-m-i}\left\{{m \atop l}\right\}c_{l+i}^{(k)} & (n\ge m);\\ 0 & (0\le n\le m-1)\,.\end{cases}$$

When $m=2$, we have the following explicit formulae.

Proposition 5 ([22]). *For $n\ge 0$ and $k\ge 1$ we have*

$$\begin{aligned} T_2^{(0)}(n)&=c_n+nc_{n-1}\,,\\ T_2^{(k)}(n)&=T_2^{(0)}(n)-n\sum_{j=1}^k\bigl(c_n^{(j)}+(n-1)c_{n-1}^{(j)}\bigr)\,,\\ T_2^{(-k)}(n)&=T_2^{(0)}(n)+n\sum_{j=0}^{k-1}\bigl(c_n^{(-j)}+(n-1)c_{n-1}^{(-j)}\bigr)\,. \end{aligned}$$

Putting $k=1$ in the second identity, we have

Corollary 1 ([25]).

$$\sum_{i=0}^n\binom{n}{i}c_i c_{n-i}=-n(n-2)c_{n-1}-(n-1)c_n\quad(n\ge 0)\,.$$

This is an analogue of Euler's identity (3).
In general, we can obtain the following explicit expression of $T_m^{(k)}(n)$ for any general $m\ge 2$.

Theorem 6. *For $n\ge 0$ and $k>0$ we have*

$$\begin{aligned} &T_m^{(0)}(n)=T_{m-1}^{(1)}(n)+nT_{m-1}^{(1)}(n-1)\,,\\ &T_m^{(k)}(n)=\sum_{r=0}^{m-2}(-1)^r\binom{n}{r}\sum_{i=0}^r\binom{r}{i}\frac{(-1)^i}{(i+1)^k}T_{m-r}^{(0)}(n-r)\\ &+\frac{(-1)^{m-1}n!}{(n-m+1)!}\sum_{\substack{j_1+j_2+\dots+j_{m-1}=k+m-2\\ j_1,j_2,\dots,j_{m-1}\ge 1}}2^{-j_2}3^{-j_3}\dots(m-1)^{-j_{m-1}}\sum_{j=1}^{j_1}\sum_{\kappa=0}^{m-1}P_{m,\kappa}(n)c_{n-\kappa}^{(j)}\,, \end{aligned}$$

$$\begin{aligned} &T_m^{(-k)}(n)=\sum_{r=0}^{m-2}(-1)^r\binom{n}{r}\sum_{i=0}^r\binom{r}{i}(-1)^i(i+1)^k T_{m-r}^{(0)}(n-r)\\ &+\frac{n!}{(n-m+1)!}\sum_{\substack{j_1+j_2+\dots+j_{m-1}=k-m+1\\ j_1,j_2,\dots,j_{m-1}\ge 0}}2^{j_2}3^{j_3}\dots(m-1)^{j_{m-1}}\sum_{j=0}^{j_1}\sum_{\kappa=0}^{m-1}P_{m,\kappa}(n)c_{n-\kappa}^{(-j)}\,. \end{aligned}$$

where

$$P_{m,\kappa}(n) = \sum_{t=0}^{\kappa} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \binom{m-t-1}{m-\kappa-1} \frac{(n-m+1)!}{(n-m-\kappa+t+1)!}$$

$$(\kappa = 0, 1, \dots, m-2)$$

and

$$\begin{aligned} P_{m,m-1}(n) &= \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \frac{(n-m+1)!}{(n-2m+t+2)!} \\ &= (n-m+1)^{m-1} . \end{aligned}$$

5 Hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers

Hypergeometric Bernoulli numbers $B_{N,n}$ $(N \geq 1,\ n \geq 0)$ ([7, 9, 10, 11, 12]) are defined by

$$\frac{1}{{}_1F_1(1; N+1; x)} = \frac{x^N/N!}{e^x - \sum_{n=0}^{N-1} x^n/n!} = \sum_{n=0}^{\infty} B_{N,n} \frac{x^n}{n!} ,$$

where ${}_1F_1(a; b; z)$ is the confluent hypergeometric function defined by

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

with the Pochhammer symbol $(x)_n = x(x+1)\dots(x+n-1)$ $(n \geq 1)$ and $(x)_0 = 1$. When $N = 1$, $B_{1,n} = B_n$ are classical Bernoulli numbers.

Hypergeometric Cauchy numbers $c_{N,n}$ $(N \geq 1,\ n \geq 0)$ ([21]) are defined by

$$\frac{1}{{}_2F_1(1, N; N+1; -x)} = \frac{(-1)^{N-1} x^N/N}{\ln(1+x) - \sum_{n=1}^{N-1} (-1)^{n-1} x^n/n} = \sum_{n=0}^{\infty} c_{N,n} \frac{x^n}{n!} ,$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} .$$

When $N=1$, $c_{1,n}=c_n$ are classical Cauchy numbers.

We record of the first few values of $c_{N,n}$:

$$
\begin{aligned}
c_{N,0}&=1,\\
c_{N,1}&=\frac{N}{N+1},\\
c_{N,2}&=-\frac{2N}{(N+1)^2(N+2)},\\
c_{N,3}&=\frac{6N(N^2+N+2)}{(N+1)^3(N+2)(N+3)}\\
c_{N,4}&=-\frac{4!N(N^5+5N^4+14N^3+24N^2+20N+12)}{(N+1)^4(N+2)^2(N+3)(N+4)},\\
c_{N,5}&=\frac{5!N(N^7+8N^6+35N^5+96N^4+160N^3+184N^2+116N+48)}{(N+1)^5(N+2)^2(N+3)(N+4)(N+5)}\,.
\end{aligned}
$$

The sums of products of hypergeometric Bernoulli numbers were studied by Kamano ([15]) and those of hypergeometric Cauchy numbers are also studied in [21].

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