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Research Article

A Generalization of Poly-Cauchy Numbers and Their Properties

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In Komatsu's work (2013), the concept of poly-Cauchy numbers is introduced as an analogue of that of poly-Bernoulli numbers. Both numbers are extensions of classical Cauchy numbers and Bernoulli numbers, respectively. There are several generalizations of poly-Cauchy numbers, including poly-Cauchy numbers with a q parameter and shifted poly-Cauchy numbers. In this paper, we give a further generalization of poly-Cauchy numbers and investigate several arithmetical properties. We also give the corresponding generalized poly-Bernoulli numbers so that both numbers have some relations.

1. Introduction

Let $n \geq 0$, $k \geq 1$ be integers. *Poly-Cauchy numbers of the first kind* $c_n^{(k)}$ are defined by

$$c_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 x_2 \cdots x_k)(x_1 x_2 \cdots x_k - 1) \cdots (x_1 x_2 \cdots x_k - n + 1)\, dx_1 dx_2 \cdots dx_k \tag{1}$$

[1]. The concept of poly-Cauchy numbers is a generalization of that of the classical Cauchy numbers $c_n = c_n^{(1)}$ defined by

$$c_n = \int_0^1 x(x-1)\cdots(x-n+1)\, dx \tag{2}$$

(see, e.g., [2, 3]). The generating function of poly-Cauchy numbers ([1], Theorem 2) is given by

$$\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}, \tag{3}$$

where

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k} \tag{4}$$

is the kth *polylogarithm factorial* function. An explicit formula for $c_n^{(k)}$ ([1], Theorem 1) is given by

$$c_n^{(k)} = \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{(-1)^{n-m}}{(m+1)^k} \quad (n \geq 0, k \geq 1), \tag{5}$$

where $\left[{n \atop m} \right]$ are the (unsigned) Stirling numbers of the first kind, arising as coefficients of the rising factorial

$$x(x+1)\cdots(x+n-1) = \sum_{m=0}^{n} \left[{n \atop m} \right] x^m \tag{6}$$

(see, e.g., [4]). See ([5], A224094–A224101) for the sequences arising from poly-Cauchy numbers.

The concept of poly-Cauchy numbers is an analogue of that of poly-Bernoulli numbers $B_n^{(k)}$ [6] defined by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}, \tag{7}$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \tag{8}$$

is the kth polylogarithm function. When $k=1$, $B_n = B_n^{(1)}$ is the classical Bernoulli number with $B_1^{(1)} = 1/2$, defined by the generating function

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{9}$$

An explicit formula for $B_n^{(k)}$ ([6], Theorem 1) is given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-1)^m m!}{(m+1)^k} \quad (n \ge 0, k \ge 1), \tag{10}$$

where $\left\{ {n \atop m} \right\}$ are the Stirling numbers of the second kind, determined by

$$\left\{ {n \atop m} \right\} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (m-j)^n \tag{11}$$

(see, e.g., [4]).

There are some kinds of generalizations of poly-Cauchy numbers. One is the *poly-Cauchy number with a q parameter* $c_{n,q}^{(k)}$ [7] defined by

$$c_{n,q}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 \cdots x_k)(x_1 \cdots x_k - q) \cdots (x_1 \cdots x_k - (n-1)q)\, dx_1 \cdots dx_k. \tag{12}$$

Another is the *shifted poly-Cauchy number* $c_{n,a}^{(k)}$ [8] defined by

$$c_{n,a}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 \cdots x_k)^a (x_1 \cdots x_k - 1) \cdots (x_1 \cdots x_k - (n-1))\, dx_1 \cdots dx_k. \tag{13}$$

Notice that $c_{n,a}^{(k)}$ can be expressed as

$$c_{n,a}^{(k)} = \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{(-1)^{n-m}}{(m+a)^k}. \tag{14}$$

For example, if $n=5$ and $a=3$, then

$$\begin{aligned} c_5^{(k)} &= \frac{24}{2^k} - \frac{50}{3^k} + \frac{35}{4^k} - \frac{10}{5^k} + \frac{1}{6^k}, \\ c_{5,3}^{(k)} &= \frac{24}{4^k} - \frac{50}{5^k} + \frac{35}{6^k} - \frac{10}{7^k} + \frac{1}{8^k}. \end{aligned} \tag{15}$$

Therefore, such numbers are *shifted* from the original poly-Cauchy numbers. Remember that the Hurwitz zeta function $\zeta(s,q) = \sum_{n=0}^{\infty} 1/(q+n)^s$ is a generalization of the famous Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ since $\zeta(s) = \zeta(s,1)$.

In this paper, we give a further generalization of poly-Cauchy numbers, including both kinds of generalizations, and show several combinatorial and characteristic properties. We also give the corresponding poly-Bernoulli numbers so that both numbers have some relations.

2. Definitions and Basic Properties

Let $n \ge 0$, $k \ge 1$ be integers, and let a, q and $l_1, \ldots, l_k$ be nonzero real numbers. For simplicity, we write $L = (l_1, \ldots, l_k)$ and $\ell = l_1 \cdots l_k$. Define $c_{n,a,q,L}^{(k)}$ by

$$c_{n,a,q,L}^{(k)} = \underbrace{\int_0^{l_1} \cdots \int_0^{l_k}}_{k} (x_1 \cdots x_k)^a (x_1 \cdots x_k - q) \cdots (x_1 \cdots x_k - (n-1)q)\, dx_1 \cdots dx_k. \tag{16}$$

Then, $c_{n,a,q,L}^{(k)}$ can be expressed in terms of the Stirling numbers of the first kind $\left[{n \atop m} \right]$.

Theorem 1. *Let a be a positive real number. Then,*

$$c_{n,a,q,L}^{(k)} = \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{(-q)^{n-m} \ell^{m+a}}{(m+a)^k} \quad (n \ge 0, k \ge 1). \tag{17}$$

Remark 2. If $a = \ell = 1$, then $c_{n,1,q,(1,\ldots,1)}^{(k)} = c_{n,q}^{(k)}$ is the poly-Cauchy number with a q parameter ([7], Theorem 1). If $q = \ell = 1$, then $c_{n,a,1,(1,\ldots,1)}^{(k)} = c_{n,a}^{(k)}$ is the shifted poly-Cauchy number ([8], Theorem 2).

Proof. By

$$x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^{n-m} x^m, \tag{18}$$

we have

$$\begin{aligned} c_{n,a,q,L}^{(k)} &= \int_0^{l_1} \cdots \int_0^{l_k} \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^{n-m} \\ &\quad \times (x_1 \cdots x_k)^{m+a-1} q^{n-m} dx_1 \cdots dx_k \\ &= \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{(-q)^{n-m} \ell^{m+a}}{(m+a)^k}. \end{aligned} \tag{19}$$

□

For an integer k and a positive real number a, define the extended polylogarithm factorial function $\mathrm{Lif}_k(z;a)$ by

$$\mathrm{Lif}_k(z;a) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+a)^k} \tag{20}$$

[8]. When $a = 1$, $\mathrm{Lif}_k(z;1) = \mathrm{Lif}_k(z)$ is the *polylogarithm factorial* function [1]. The generating function of the number $c_{n,a,q,L}^{(k)}$ $(q \ne 0)$ is given by using the extended polylogarithm factorial function $\mathrm{Lif}_k(a;z)$.

Theorem 3. *One has*

$$\ell^a \,\mathrm{Lif}_k\left(\frac{\ell \ln(1+qx)}{q}; a \right) = \sum_{n=0}^{\infty} c_{n,a,q,L}^{(k)} \frac{x^n}{n!}. \tag{21}$$

Remark 4. If $a = \ell = 1$, then Theorem 3 is reduced to Theorem 2 in [7]. If $q = \ell = 1$, then Theorem 3 is reduced to Theorem 3 in [8].

Proof. Since

$$\frac{(\ln(1+x))^m}{m!} = (-1)^m \sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-x)^n}{n!}, \tag{22}$$

by Theorem 1 we have

$$\begin{aligned}
\sum_{n=0}^{\infty} c_{n,a,q,L}^{(k)} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-q)^{n-m} \ell^{m+a}}{(m+a)^k} \frac{x^n}{n!} \\
&= \ell^a \sum_{m=0}^{\infty} \frac{(-q)^{-m} \ell^m}{(m+a)^k} \sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-qx)^n}{n!} \\
&= \ell^a \sum_{m=0}^{\infty} \frac{(-q)^{-m} \ell^m}{(m+a)^k} (-1)^m \frac{(\ln(1+qx))^m}{m!} \\
&= \ell^a \sum_{m=0}^{\infty} \frac{1}{m!(m+a)^k} \left(\frac{\ell \ln(1+qx)}{q} \right)^m \\
&= \ell^a \mathrm{Lif}_k \left(\frac{\ell \ln(1+qx)}{q}; a \right).
\end{aligned} \tag{23}$$

□

The generating function of the number $c_{n,a,q,L}^{(k)}$ can be written in the form of iterated integrals.

Corollary 5. *Let a and q be real numbers with $a > 0$ and $q \neq 0$. For $k = 1$, one has*

$$\begin{aligned}
&\left(\frac{\ell q}{\ln(1+qx)} \right)^a \int_0^x \left(\frac{\ln(1+qx)}{q} \right)^{a-1} (1+qx)^{\ell/q-1} dx \\
&\quad = \sum_{n=0}^{\infty} c_{n,a,q,L}^{(1)} \frac{x^n}{n!}.
\end{aligned} \tag{24}$$

For $k > 1$, one has

$$\begin{aligned}
&\left(\frac{\ell q}{\ln(1+qx)} \right) \\
&\underbrace{\int_0^x \frac{q}{(1+qx)\ln(1+qx)} \int_0^x \cdots \frac{q}{(1+qx)\ln(1+qx)} \int_0^x}_{k} \\
&\left(\frac{\ln(1+qx)}{q} \right)^{a-1} (1+qx)^{\ell/q-1} \underbrace{dx \cdots dx}_{k} = \sum_{n=0}^{\infty} c_{n,a,q,L}^{(k)} \frac{x^n}{n!}.
\end{aligned} \tag{25}$$

Remark 6. If $a = \ell = 1$, then Corollary 5 is reduced to Corollary 1 in [7]. If $q = \ell = 1$, then Corollary 5 is reduced to Corollary 1 in [8].

Proof. For $k = 1$,

$$\begin{aligned}
\mathrm{Lif}_1(z; a) &= \sum_{m=0}^{\infty} \frac{z^m}{m!(m+a)} = \frac{1}{z^a} \sum_{m=0}^{\infty} \frac{z^{m+a}}{m!(m+a)} \\
&= \frac{1}{z^a} \int_0^z \sum_{m=0}^{\infty} \frac{z^{m+a-1}}{m!} = \frac{1}{z^a} \int_0^z z^{a-1} e^z dz \\
&= \frac{1}{z^a} \Bigg((-1)^a (a-1)! \\
&\quad + e^z \sum_{i=0}^{a-1} (-1)^i \frac{(a-1)!}{(a-i-1)!} z^{a-i-1} \Bigg).
\end{aligned} \tag{26}$$

Note that the last equation holds only if a is an integer. For $k > 1$, we have

$$\begin{aligned}
\mathrm{Lif}_k(z; a) &= \frac{1}{z^a} \sum_{m=0}^{\infty} \frac{z^{m+a}}{m!(m+a)^k} \\
&= \frac{1}{z^a} \int_0^z \sum_{m=0}^{\infty} \frac{z^{m+a-1}}{m!(m+a)^{k-1}} dz \\
&= \frac{1}{z^a} \int_0^z z^{a-1} \mathrm{Lif}_{k-1}(z; a) \, dz.
\end{aligned} \tag{27}$$

Hence,

$$\mathrm{Lif}_k(z; a) = \frac{1}{z^a} \underbrace{\int_0^z \frac{1}{z} \int_0^z \cdots \frac{1}{z} \int_0^z \frac{1}{z} \int_0^z}_{k} z^{a-1} e^z \underbrace{dz \cdots dz}_{k}. \tag{28}$$

Putting $z = \ell \ln(1+qx)/q$ and multiplying by ℓ^a, we get the result. □

3. Poly-Cauchy Numbers of the Second Kind

In [1], the concept of poly-Cauchy numbers of the second kind is also introduced. The poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ are defined by

$$\begin{aligned}
\hat{c}_n^{(k)} = &\underbrace{\int_0^1 \cdots \int_0^1}_{k} (-x_1 x_2 \cdots x_k)(-x_1 x_2 \cdots x_k - 1) \cdots \\
&(-x_1 x_2 \cdots x_k - n + 1) \, dx_1 dx_2 \cdots dx_k,
\end{aligned} \tag{29}$$

and the generating function is given by

$$\mathrm{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{x^n}{n!}. \tag{30}$$

Then, the poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ can also be expressed in terms of the Stirling numbers of the first kind ([1], Theorem 4). See ([5], A219247, A224102–A224107, A224109) for the sequences arising from poly-Cauchy numbers of the second kind.

Proposition 7. *One has*

$$\hat{c}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k}. \tag{31}$$

Let a be a positive real number. Similar to generalized poly-Cauchy numbers of the first kind $c_{n,a,q,L}^{(k)}$, define the poly-Cauchy numbers of the second kind $\widehat{c}_{n,a,q,L}^{(k)}$ ($n \geq 0, k \geq 1$) by

$$\widehat{c}_{n,a,q,L}^{(k)} = (-1)^{a-1} \int_0^{l_1} \cdots \int_0^{l_k} (-x_1 \cdots x_k)^a (-x_1 \cdots x_k - q) \cdots (-x_1 \cdots x_k - (n-1) q) dx_1 \cdots dx_k. \quad (32)$$

Then, similar to Theorem 1, $\widehat{c}_{n,a,q,L}^{(k)}$ can also be expressed in terms of the Stirling numbers of the first kind $\left[{n \atop m} \right]$.

Theorem 8. *One has*

$$\widehat{c}_{n,a,q,L}^{(k)} = (-1)^n \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{q^{n-m} \ell^{m+a}}{(m+a)^k} \quad (n \geq 0, k \geq 1). \quad (33)$$

Theorem 9. *The generating function of the number $\widehat{c}_{n,a,q,L}^{(k)}$ is given by*

$$\ell^a \operatorname{Lif}_k \left(-\frac{\ell \ln (1+qx)}{q}; a \right) = \sum_{m=0}^{\infty} \widehat{c}_{n,a,q,L}^{(k)} \frac{x^n}{n!}, \quad (34)$$

where

$$\operatorname{Lif}_k (z; a) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+a)^k}. \quad (35)$$

Remark 10. If $a = \ell = 1$, then Theorem 8 is reduced to Theorem 3 in [7] and Theorem 9 is reduced to Theorem 4 in [7]. If $q = \ell = 1$, then Theorem 8 is reduced to Theorem 5 in [8] and Theorem 9 is reduced to Theorem 6 in [8].

The generating function of the number $\widehat{c}_{n,a,q,L}^{(k)}$ can be written in the form of iterated integrals.

Corollary 11. *Let a be a positive real number. For $k = 1$, one has*

$$\left(\frac{\ell q}{\ln(1+qx)} \right)^a \int_0^x \left(\frac{\ln(1+qx)}{q} \right)^{a-1} (1+qx)^{-\ell/q-1} dx = \sum_{n=0}^{\infty} \widehat{c}_{n,a,q,L}^{(1)} \frac{x^n}{n!}. \quad (36)$$

For $k > 1$, one has

$$\left(\frac{\ell q}{\ln(1+qx)} \right) \underbrace{\int_0^x \frac{q}{(1+qx)\ln(1+qx)} \int_0^x \cdots \frac{q}{(1+qx)\ln(1+qx)} \int_0^x}_{k} \left(\frac{\ln(1+qx)}{q} \right)^{a-1} (1+qx)^{-\ell/q-1} \underbrace{dx \cdots dx}_{k} = \sum_{n=0}^{\infty} \widehat{c}_{n,a,q,L}^{(k)} \frac{x^n}{n!}. \quad (37)$$

Remark 12. When $a = q = k = \ell = 1$ in the first identity, we have the generating function of the classical Cauchy numbers of the second kind:

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \widehat{c}_n \frac{x^n}{n!}. \quad (38)$$

In addition, there are relations between both kinds of poly-Cauchy numbers if $q = 1$. For simplicity, we write $c_{n,a,L}^{(k)} = c_{n,a,1,L}^{(k)}$ and $\widehat{c}_{n,a,L}^{(k)} = \widehat{c}_{n,a,1,L}^{(k)}$.

Theorem 13. *Let k be an integer and a a positive real number. For $n \geq 1$, one has*

$$(-1)^n \frac{c_{n,a,L}^{(k)}}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{c}_{m,a,L}^{(k)}}{m!},$$
$$(-1)^n \frac{\widehat{c}_{n,a,L}^{(k)}}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{c_{m,a,L}^{(k)}}{m!}. \quad (39)$$

Remark 14. If $a = \ell = 1$, then Theorem 13 is reduced to Theorem 7 in [1].

Proof. We will prove the second identity. The first one is proved similarly and omitted. By using the identity (see, e.g., [4], Chapter 6)

$$\frac{(-1)^i}{n!} \left[{n \atop i} \right] = \sum_{m=1}^{n} \frac{(-1)^m}{m!} \binom{n-1}{m-1} \left[{m \atop i} \right] \quad (40)$$

and Theorems 1 and 8, we have

$$\begin{aligned} \text{RHS} &= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{1}{m!} \times \sum_{i=1}^{m} \left[{m \atop i} \right] \frac{(-1)^{m-i} \ell^{i+a}}{(i+a)^k} \\ &= \sum_{i=1}^{n} \frac{(-1)^i \ell^i}{(i+a)^k} \sum_{m=i}^{n} \frac{(-1)^m}{m!} \binom{n-1}{m-1} \left[{m \atop i} \right] \\ &= \sum_{i=1}^{n} \frac{(-1)^i \ell^{i+a}}{(i+a)^k} \frac{(-1)^i}{n!} \left[{n \atop i} \right] = \text{LHS}. \end{aligned} \quad (41)$$

□

4. Some Expressions of Poly-Cauchy Numbers with Negative Indices

It is known that poly-Bernoulli numbers satisfy the duality theorem $B_n^{(-k)} = B_k^{(-n)}$ for $n, k \geq 0$ ([6], Theorem 2) because of the symmetric formula

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}. \quad (42)$$

However, the corresponding duality theorem does not hold for poly-Cauchy numbers for any real number a, by the following results.

Proposition 15. *Suppose that $\ell = 1$. Then, for nonnegative integers n and k and a real number $a \neq 0$, one has*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{n,a,q,L}^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!} = e^{ay}(1+qx)^{e^y/q},$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \widehat{c}_{n,a,q,L}^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^{ay}}{(1+qx)^{e^y/q}}. \quad (43)$$

Remark 16. If $a = \ell = 1$, then Proposition 15 is reduced to Proposition 1 in [7]. If $q = \ell = 1$, then Proposition 15 is reduced to Proposition 3 in [8].

Proof. We will prove the first identity. The second identity is proved similarly. By Theorem 3, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{n,a,q,L}^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!} \\
&\quad = \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} c_{n,a,q,L}^{(-k)} \frac{x^n}{n!}\right)\frac{y^k}{k!} \\
&\quad = \sum_{k=0}^{\infty} \ell^a \sum_{m=0}^{\infty} \frac{(m+a)^k}{m!}\left(\frac{\ell \ln(1+qx)}{q}\right)^m \frac{y^k}{k!} \\
&\quad = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\ln(1+qx)}{q}\right)^m \sum_{k=0}^{\infty} \frac{((m+a)\,y)^k}{k!} \\
&\quad = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\ln(1+qx)}{q}\right)^m e^{(m+a)y} \\
&\quad = e^{ay}\sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{e^y}{q}\ln(1+qx)\right)^m = e^{ay}(1+qx)^{e^y/q}.
\end{aligned} \quad (44)$$

□

By using Proposition 15, we have explicit expressions of poly-Cauchy numbers with negative indices. For simplicity, we write $c_{n,a,q}^{(-k)} = c_{n,a,q,L}^{(-k)}$ and $\widehat{c}_{n,a,q}^{(-k)} = \widehat{c}_{n,a,q,L}^{(-k)}$ if $\ell = 1$.

Theorem 17. *For nonnegative integers n, k, and a real number $a \neq 0$, one has*

$$\begin{aligned}
c_{n,a,q}^{(-k)} &= \sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\lambda=0}^{n}\sum_{\nu=0}^{\lambda} j! \binom{k}{i} \left\{ {i \atop j} \right\} \binom{n}{\lambda} \left[{n-\lambda \atop j} \right] \\
&\quad \times \left[{\lambda \atop \nu} \right] a^{k-i}(-q)^{n-j-\nu}, \\
\widehat{c}_{n,a,q}^{(-k)} &= \sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\lambda=0}^{n}\sum_{\nu=0}^{\lambda} (-1)^n j! \binom{k}{i} \left\{ {i \atop j} \right\} \binom{n}{\lambda} \\
&\quad \times \left[{n-\lambda \atop j} \right] \left[{\lambda \atop \nu} \right] a^{k-i} q^{n-j-\nu}.
\end{aligned} \quad (45)$$

Remark 18. If $a = q = 1$, by

$$\sum_{i=0}^{k} \binom{k}{i} \left\{ {i \atop j} \right\} = \left\{ {k+1 \atop j+1} \right\} \quad (46)$$

[4], the above identities become

$$\begin{aligned}
c_{n,1,1}^{(-k)} &= c_n^{(-k)} \\
&= \sum_{j=0}^{k} (-1)^{n+j} j! \\
&\quad \times \left(\left[{n \atop j} \right] - n \left[{n-1 \atop j} \right] \right) \left\{ {k+1 \atop j+1} \right\}, \\
\widehat{c}_{n,1,1}^{(-k)} &= \widehat{c}_n^{(-k)} = \sum_{j=0}^{k} (-1)^n j! \left[{n+1 \atop j+1} \right] \left\{ {k+1 \atop j+1} \right\}.
\end{aligned} \quad (47)$$

Proof. By Proposition 15 together with

$$\frac{(e^y - 1)^j}{j!} = \sum_{k=j}^{\infty} \left\{ {k \atop j} \right\} \frac{y^k}{k!},$$
$$\frac{(-\ln(1+x))^j}{j!} = \sum_{n=j}^{\infty} \left[{n \atop j} \right] \frac{(-x)^n}{n!} \quad (48)$$

[4], we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{n,a,q}^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!} \\
&\quad = \left((1+qx)^{1/q}\right)^{e^y - 1}(1+qx)^{1/q} e^{ay} \\
&\quad = \exp\left((e^y - 1)\left(\ln(1+qx)\right)^{1/q}\right) \\
&\qquad \times (1+qx)^{1/q} e^{ay} \\
&\quad = \sum_{j=0}^{\infty} \frac{j!}{q^j} \frac{(e^y - 1)^j}{j!} \frac{(\ln(1+qx))^j}{j!} \\
&\qquad \times (1+qx)^{1/q} e^{ay} \\
&\quad = \sum_{j=0}^{\infty} (-1)^j \frac{j!}{q^j} e^{ay} \sum_{k=j}^{\infty} \left\{ {k \atop j} \right\} \frac{y^k}{k!} (1+qx)^{1/q} \\
&\qquad \times \sum_{n=j}^{\infty} \left[{n \atop j} \right] \frac{(-qx)^n}{n!}.
\end{aligned} \quad (49)$$

Since

$$e^{ay}\sum_{k=j}^{\infty}\begin{Bmatrix} k \\ j \end{Bmatrix}\frac{y^k}{k!}=\sum_{l=0}^{\infty}\frac{(ay)^l}{l!}\sum_{k=j}^{\infty}\begin{Bmatrix} k \\ j \end{Bmatrix}\frac{y^k}{k!}$$
$$=\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\frac{a^{k-i}}{(k-i)!}\begin{Bmatrix} i \\ j \end{Bmatrix}\frac{1}{i!}\right)y^k$$
$$=\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i}\begin{Bmatrix} i \\ j \end{Bmatrix}a^{k-i}\right)\frac{y^k}{k!},$$
$$(1+qx)^{1/q}\sum_{n=j}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-qx)^n}{n!}$$
$$=\sum_{l=0}^{\infty}\binom{\frac{1}{q}}{l}(qx)^l\sum_{n=j}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-qx)^n}{n!}$$
$$=\sum_{l=0}^{\infty}(-1)^l\sum_{\nu=0}^{l}\begin{bmatrix} l \\ \nu \end{bmatrix}\left(-\frac{1}{q}\right)^{\nu}\frac{(qx)^l}{l!}$$
$$\times\sum_{n=j}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-qx)^n}{n!}$$
$$=\sum_{l=0}^{\infty}\sum_{\nu=0}^{l}\begin{bmatrix} l \\ \nu \end{bmatrix}(-q)^{l-\nu}\frac{x^l}{l!}\sum_{n=0}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-qx)^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{\lambda=0}^{n}\sum_{\nu=0}^{\lambda}\begin{bmatrix} \lambda \\ \nu \end{bmatrix}\frac{(-q)^{\lambda-\nu}}{\lambda!}\begin{bmatrix} n-\lambda \\ j \end{bmatrix}\frac{(-q)^{n-\lambda}}{(n-\lambda)!}x^n$$
$$=\sum_{n=0}^{\infty}\sum_{\lambda=0}^{n}\binom{n}{\lambda}\begin{bmatrix} n-\lambda \\ j \end{bmatrix}\sum_{\nu=0}^{\lambda}\begin{bmatrix} \lambda \\ \nu \end{bmatrix}(-q)^{n-\nu}\frac{x^n}{n!}, \tag{50}$$

we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}c_{n,a,q}^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\frac{(-1)^j j!}{q^j}\binom{k}{i}\begin{Bmatrix} i \\ j \end{Bmatrix}a^{k-i}$$
$$\times\sum_{\lambda=0}^{n}\binom{n}{\lambda}\begin{bmatrix} n-\lambda \\ j \end{bmatrix}\sum_{\nu=0}^{\lambda}\begin{bmatrix} \lambda \\ \nu \end{bmatrix}(-q)^{n-\nu}\frac{x^n}{n!}\frac{y^k}{k!} \tag{51}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\lambda=0}^{n}\sum_{\nu=0}^{\lambda}j!\binom{k}{i}\begin{Bmatrix} i \\ j \end{Bmatrix}$$
$$\times\binom{n}{\lambda}\begin{bmatrix} n-\lambda \\ j \end{bmatrix}\begin{bmatrix} \lambda \\ \nu \end{bmatrix}a^{k-i}$$
$$\times(-q)^{n-j-\nu}\frac{x^n}{n!}\frac{y^k}{k!}.$$

Similarly, by

$$(1+qx)^{-1/q}\sum_{n=j}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-qx)^n}{n!}$$
$$=\sum_{l=0}^{\infty}(-1)^l\sum_{\nu=0}^{l}\begin{bmatrix} l \\ \nu \end{bmatrix}\left(\frac{1}{q}\right)^{\nu}\frac{(qx)^l}{l!}$$
$$\times\sum_{n=j}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-qx)^n}{n!}$$
$$=\sum_{l=0}^{\infty}\sum_{\nu=0}^{l}\begin{bmatrix} l \\ \nu \end{bmatrix}q^{l-\nu}(-1)^l\frac{x^l}{l!}\sum_{n=0}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-qx)^n}{n!} \tag{52}$$
$$=\sum_{n=0}^{\infty}\sum_{\lambda=0}^{n}\sum_{\nu=0}^{\lambda}\begin{bmatrix} \lambda \\ \nu \end{bmatrix}\frac{q^{\lambda-\nu}(-1)^{\lambda}}{\lambda!}$$
$$\times\begin{bmatrix} n-\lambda \\ j \end{bmatrix}\frac{(-q)^{n-\lambda}}{(n-\lambda)!}x^n$$
$$=\sum_{n=0}^{\infty}(-1)^n\sum_{\lambda=0}^{n}\binom{n}{\lambda}\begin{bmatrix} n-\lambda \\ j \end{bmatrix}\sum_{\nu=0}^{\lambda}\begin{bmatrix} \lambda \\ \nu \end{bmatrix}q^{n-\nu}\frac{x^n}{n!},$$

we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\widehat{c}_{n,a,q}^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}$$
$$=(1+qx)^{-e^y/q}e^{ay}$$
$$=\exp\left(\left(e^y-1\right)\left(\ln\left(1+qx\right)\right)^{-1/q}\right)$$
$$\times(1+qx)^{-1/q}e^{ay}$$
$$=\sum_{j=0}^{\infty}\frac{j!}{q^j}\frac{(e^y-1)^j}{j!}\frac{(-\ln(1+qx))^j}{j!}$$
$$\times(1+qx)^{-1/q}e^{ay}$$
$$=\sum_{j=0}^{\infty}\frac{j!}{q^j}e^{ay}\sum_{k=j}^{\infty}\begin{Bmatrix} k \\ j \end{Bmatrix}\frac{y^k}{k!}(1+qx)^{-1/q}$$
$$\times\sum_{n=j}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-qx)^n}{n!}$$
$$=\sum_{j=0}^{\infty}\frac{j!}{q^j}\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i}\begin{Bmatrix} i \\ j \end{Bmatrix}a^{k-i}\right)\frac{y^k}{k!}$$

$$\times \sum_{n=0}^{\infty}(-1)^n \sum_{\lambda=0}^{n}\binom{n}{\lambda}\begin{bmatrix} n-\lambda \\ j \end{bmatrix}\sum_{\nu=0}^{\lambda}\begin{bmatrix} \lambda \\ \nu \end{bmatrix} q^{n-\nu}\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\lambda=0}^{n}\sum_{\nu=0}^{\lambda}(-1)^n j! \times \binom{k}{i}\begin{Bmatrix} i \\ j \end{Bmatrix}\binom{n}{\lambda}\begin{bmatrix} n-\lambda \\ j \end{bmatrix} \times \begin{bmatrix} \lambda \\ \nu \end{bmatrix} a^{k-i} q^{n-j-\nu}\frac{x^n}{n!}\frac{y^k}{k!}. \quad (53)$$

□

5. Poly-Bernoulli Numbers Corresponding to Poly-Cauchy Numbers

In this section, we will consider the corresponding generalized poly-Bernoulli numbers to the generalized poly-Cauchy numbers discussed in the previous sections. Let k be an integer and a a positive real number. An explicit form of poly-Bernoulli number $B_n^{(k)}$ is given by

$$B_n^{(k)} = \sum_{m=0}^{n}\begin{Bmatrix} n \\ m \end{Bmatrix}\frac{(-1)^{n-m} m!}{(m+1)^k} \quad (54)$$

([6], Theorem 1). In ([1], Theorem 8), one expression of $B_n^{(k)}$ in terms of poly-Cauchy numbers $c_n^{(k)}$ is given.

Proposition 19. *One has*

$$B_n^{(k)} = \sum_{l=1}^{n}\sum_{m=1}^{n} m! \begin{Bmatrix} n \\ m \end{Bmatrix}\begin{Bmatrix} m-1 \\ l-1 \end{Bmatrix} c_l^{(k)} \quad (n \geq 1). \quad (55)$$

On the contrary, in ([9], Theorem 2.2), one expression of $c_n^{(k)}$ in terms of $B_n^{(k)}$ is given.

Proposition 20. *One has*

$$c_n^{(k)} = \sum_{l=1}^{n}\sum_{m=1}^{n}\frac{(-1)^{n-m}}{m!}\begin{bmatrix} n \\ m \end{bmatrix}\begin{bmatrix} m \\ l \end{bmatrix} B_l^{(k)} \quad (n \geq 1). \quad (56)$$

As a counterpart of a generalized poly-Cauchy number, we will define a generalized poly-Bernoulli number $B_{n,a,L}^{(k)}$ by

$$\frac{\ell^{a-1}\mathrm{Li}_k\left(\ell\left(1-e^{-t}\right); a-1\right)}{1-e^{-t}} = \sum_{n=0}^{\infty} B_{n,a,L}^{(k)}\frac{t^n}{n!}, \quad (57)$$

where $\mathrm{Li}_k(z; a)$ is the generalized polylogarithm function defined by

$$\mathrm{Li}_k(z; a) = \sum_{m=0}^{\infty}\frac{z^m}{(m+a)^k}, \quad (58)$$

so that $\mathrm{Li}_k(z; 0) = \mathrm{Li}_k(z)$.

Then, $B_{n,a,L}^{(k)}$ can be expressed explicitly in terms of the Stirling numbers of the second kind. Note that $B_{n,1,(1,\ldots,1)}^{(k)} = B_n^{(k)}$.

Proposition 21. *One has*

$$B_{n,a,L}^{(k)} = \sum_{m=0}^{n}\begin{Bmatrix} n \\ m \end{Bmatrix}\frac{(-1)^{n-m} m! \ell^{m+a}}{(m+a)^k} \quad (n \geq 0). \quad (59)$$

Proof. By

$$\frac{\left(e^t-1\right)^m}{m!} = \sum_{n=m}^{\infty}\begin{Bmatrix} n \\ m \end{Bmatrix}\frac{t^n}{n!}, \quad (60)$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,a,L}^{(k)}\frac{t^n}{n!} &= \frac{\ell^a}{\ell\left(1-e^{-t}\right)}\sum_{m=1}^{\infty}\frac{\left(\ell(1-e^{-t})\right)^m}{(m+a-1)^k} \\ &= \ell^a\sum_{m=0}^{\infty}\frac{\left(\ell(1-e^{-t})\right)^m}{(m+a)^k} \\ &= \ell^a\sum_{m=0}^{\infty}\frac{(-\ell)^m m!}{(m+a)^k}\sum_{n=m}^{\infty}\begin{Bmatrix} n \\ m \end{Bmatrix}\frac{(-t)^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\begin{Bmatrix} n \\ m \end{Bmatrix}\frac{(-1)^{n-m} m! \ell^{m+a}}{(m+a)^k}\right)\frac{t^n}{n!}.\end{aligned} \quad (61)$$

Comparing the coefficients on both sides, we get the result. □

For simplicity, we write $c_{n,a,L}^{(k)} = c_{n,a,1,L}^{(k)}$ and $\widehat{c}_{n,a,L}^{(k)} = \widehat{c}_{n,a,1,L}^{(k)}$. If $a = \ell = 1$, then our results below are reduced to those previous ones.

Theorem 22. *For $n \geq 0$, one has*

$$\begin{aligned} B_{n,a,L}^{(k)} &= \sum_{j=1}^{n}\sum_{m=1}^{n} m! \begin{Bmatrix} n \\ m \end{Bmatrix}\begin{Bmatrix} m-1 \\ j-1 \end{Bmatrix} c_{j,a,L}^{(k)}, \\ c_{n,a,L}^{(k)} &= \sum_{j=1}^{n}\sum_{m=1}^{n}\frac{(-1)^{n-m}}{m!}\begin{bmatrix} n \\ m \end{bmatrix}\begin{bmatrix} m \\ j \end{bmatrix} B_{j,a,L}^{(k)}. \end{aligned} \quad (62)$$

Proof. For the first identity,

$$\begin{aligned}\mathrm{RHS} &= \sum_{j=1}^{n}\sum_{m=j}^{n} m! \begin{Bmatrix} n \\ m \end{Bmatrix}\begin{Bmatrix} m-1 \\ j-1 \end{Bmatrix} \times \sum_{i=0}^{j}\begin{bmatrix} j \\ i \end{bmatrix}\frac{(-1)^{j-i}\ell^{i+a}}{(i+a)^k} \\ &= \sum_{i=1}^{n}\frac{(-1)^i \ell^{i+a}}{(i+a)^k} \times \sum_{j=i}^{n}\sum_{m=j}^{n} m! \begin{Bmatrix} n \\ m \end{Bmatrix}\begin{Bmatrix} m-1 \\ j-1 \end{Bmatrix}(-1)^j\begin{bmatrix} j \\ i \end{bmatrix} \\ &= \sum_{i=1}^{n}\frac{(-1)^i \ell^{i+a}}{(i+a)^k}\sum_{m=i}^{n} m! \begin{Bmatrix} n \\ m \end{Bmatrix}\end{aligned}$$

$$\times \sum_{j=i}^{m} (-1)^j \left\{ {m-1 \atop j-1} \right\} \left[{j \atop i} \right]$$
$$= \sum_{i=1}^{n} \frac{(-1)^i \ell^{i+a}}{(i+a)^k} \sum_{m=i}^{n} m! \left\{ {n \atop m} \right\} (-1)^m \binom{m-1}{i-1}$$
$$= \sum_{i=1}^{n} \frac{(-1)^i \ell^{i+a}}{(i+a)^k} (-1)^n i! \left\{ {n \atop i} \right\} = \text{LHS}. \tag{63}$$

For the second identity,

$$\text{RHS} = \sum_{j=1}^{n} \sum_{m=1}^{n} \frac{(-1)^{n-m}}{m!} \left[{n \atop m} \right] \left[{m \atop j} \right] \times \sum_{i=0}^{j} \left\{ {j \atop i} \right\} \frac{(-1)^{j-i} i! \ell^{i+a}}{(i+a)^k}$$
$$= \sum_{m=1}^{n} \frac{(-1)^{n-m}}{m!} \left[{n \atop m} \right] \times \sum_{j=0}^{n} \left[{m \atop j} \right] \sum_{i=0}^{j} \left\{ {j \atop i} \right\} \frac{(-1)^{j-i} i! \ell^{i+a}}{(i+a)^k}$$
$$= \sum_{m=1}^{n} \frac{(-1)^{n-m}}{m!} \left[{n \atop m} \right] \times \sum_{i=0}^{n} \frac{(-1)^i i! \ell^{i+a}}{(i+a)^k} \sum_{j=i}^{n} (-1)^j \left[{m \atop j} \right] \left\{ {j \atop i} \right\}$$
$$= \sum_{m=0}^{n} \frac{(-1)^{n-m}}{m!} \left[{n \atop m} \right] \times \frac{(-1)^m m! \ell^{m+a}}{(m+a)^k} (-1)^m$$
$$= \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{(-1)^{n-m} \ell^{m+a}}{(m+a)^k} = \text{LHS}. \tag{64}$$

Note that $\left[{m \atop 0} \right] = 0 \ (m \geq 1)$ and $\left[{m \atop j} \right] = 0 \ (j > m)$, and

$$\sum_{j=i}^{m} (-1)^{m-j} \left[{m \atop j} \right] \left\{ {j \atop i} \right\} = \begin{cases} 1 & (i = m); \\ 0 & (i \neq m). \end{cases} \tag{65}$$

□

Similarly, concerning

$$\widehat{c}_{n,a,L}^{(k)} = (-1)^n \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{\ell^{m+a}}{(m+a)^k} \quad (n \geq 0) \tag{66}$$

as a generalization of poly-Cauchy numbers of the second kind $\widehat{c}_n^{(k)}$, we have the following.

Theorem 23. *One has*

$$B_{n,a,L}^{(k)} = (-1)^n \sum_{j=1}^{n} \sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m \atop j} \right\} \widehat{c}_{j,a,L}^{(k)},$$
$$\widehat{c}_{n,a,L}^{(k)} = (-1)^n \sum_{j=1}^{n} \sum_{m=1}^{n} \frac{1}{m!} \left[{n \atop m} \right] \left[{m \atop j} \right] B_{j,a,L}^{(k)}. \tag{67}$$

Remark 24. If $a = \ell = 1$, these results are reduced to the identities in Theorems 3.2 and 3.1 in [9], respectively.

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References

[1] T. Komatsu, "Poly-Cauchy numbers," *Kyushu Journal of Mathematics*, vol. 67, no. 1, pp. 143–153, 2013.

[2] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Co., Dordrecht, The Netherlands, 1974.

[3] D. Merlini, R. Sprugnoli, and M. C. Verri, "The Cauchy numbers," *Discrete Mathematics*, vol. 306, no. 16, pp. 1906–1920, 2006.

[4] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, Mass, USA, 2nd edition, 1994.

[5] "The On-Line Encyclopedia of Integer Sequences," http://oeis.org

[6] M. Kaneko, "Poly-Bernoulli numbers," *Journal de Théorie des Nombres de Bordeaux*, vol. 9, no. 1, pp. 221–228, 1997.

[7] T. Komatsu, "Poly-Cauchy numbers with a q parameter," *Ramanujan Journal*, vol. 31, no. 3, pp. 353–371, 2013.

[8] T. Komatsu and L. Szalay, "Shifted poly-Cauchy numbers," (preprint).

[9] T. Komatsu and F. Luca, "Some relationships between poly-Cauchy numbers and poly-Bernoulli numbers," *Annales Mathematicae et Informaticae*, vol. 41, pp. 99–105, 2013.