JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 61, 136–141 (1977)

Yet Another Proof of the Addition Formula for Jacobi Polynomials

TOM KOORNWINDER

Mathematisch Centrum, 2e Boerhaavestraat 49, Amsterdam, The Netherlands

Submitted by R. P. Boas

Short proofs of the addition formulas for Gegenbauer polynomials and for Jacobi polynomials are given. The properties of certain special orthogonal polynomials in two, respectively three, variables are used.

1. INTRODUCTION

The addition formula for Jacobi polynomials was announced by the author in [2]. Afterward, three different proofs were published (cf. [3, 4, 5]). A special case was earlier obtained by Šapiro [6]. The addition formula is a central result in the theory of Jacobi polynomials which implies many other important formulas. Therefore, it seems worthwhile to publish yet another proof of this addition formula. Compared to the earlier proofs, the present proof is rather short and it does not involve many calculations. However, it would not have been easy to obtain this proof without knowing the addition formula.

The idea of the proof is as follows. Consider the three-dimensional region bounded by the cone $z^2 - 2xy = 0$ and by the plane $x + y = 1$. Let $\mathscr{H}_n$ denote the class of all nth-degree orthogonal polynomials on this region with respect to the weight function $(1 - x - y)^{\alpha-\beta-1}(2xy - z^2)^{\beta-\frac{1}{2}}$. Then an explicit orthogonal basis can be constructed for $\mathscr{H}_n$ in terms of products of certain Jacobi polynomials. The region and the weight function are invariant with respect to rotations around the axis of the cone. Therefore, the reproducing kernel of $\mathscr{H}_n$ is invariant under such rotations. The addition formula for Jacobi polynomials follows from this symmetry relation for the reproducing kernel. There exists a similar proof of the addition formula for Gegenbauer polynomials. It uses orthogonal polynomials in two variables on the unit disk.

It is of interest to compare the present proof of the addition formula for Jacobi polynomials with two earlier proofs by group theoretic methods (cf. [3, 4]). In these two references a much bigger symmetry group was used than the one-parameter group considered in the present paper. Furthermore, a restriction to integer or half-integer values of the parameters α and β is not required here.

ISSN 0022-247X

2. Preliminaries

For $\alpha, \beta > -1$ Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal polynomials of degree n on the interval $(-1, 1)$ with respect to the weight function $(1-x)^\alpha (1+x)^\beta$ and with the normalization $P_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n!$. The quadratic norm $h_n^{(\alpha,\beta)}$ of a Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is given by

$$\begin{aligned} h_n^{(\alpha,\beta)} &= \int_{-1}^{1} (P_n^{(\alpha,\beta)}(x))^2 (1-x)^\alpha (1+x)^\beta \, dx \\ &= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\, n!\, \Gamma(n+\alpha+\beta+1)}. \end{aligned} \tag{2.1}$$

For $\alpha = \beta$ Jacobi polynomials are called Gegenbauer polynomials. Note that

$$P_n^{(\alpha,\alpha)}(-x) = (-1)^n P_n^{(\alpha,\alpha)}(x).$$

Let R be a bounded region in the q-dimensional Euclidean space E_q and let $w(x) = w(x_1, x_2, \ldots, x_q)$ be a positive continuous integrable function on R. The class $\mathscr{H}_n$ of orthogonal polynomials of degree n on R with respect to the weight function $w(x)$ consists of all polynomials $p(x_1, x_2, \ldots, x_q)$ of degree n such that

$$\int_R p(x)\, q(x)\, w(x)\, dx = 0$$

if q is a polynomial of degree less than n. There are infinitely many ways to choose an orthogonal basis of $\mathscr{H}_n$. One possible method is to apply the Gram–Schmidt orthogonalization process to the monomials

$$x_1^{n_1-n_2} x_2^{n_2-n_3} \cdots x_{q-1}^{n_{q-1}-n_q} x_q^{n_q} \qquad (n_1 \geqslant n_2 \geqslant \cdots \geqslant n_q \geqslant 0),$$

which are arranged by lexicographic ordering of the q-tuples $(n_1, n_2, \ldots, n_q)$.

Let $p_1, p_2, \ldots, p_N$ be an arbitrary orthogonal basis of $\mathscr{H}_n$ and let

$$\| p_k \|^2 = \int_R (p_k(x))^2\, w(x)\, dx.$$

The function

$$K(x, y) = \sum_{k=1}^{N} \| p_k \|^{-2} p_k(x)\, p_k(y) \qquad (x, y \in R) \tag{2.2}$$

is called the reproducing kernel of $\mathscr{H}_n$. Note that $K(x, y)$ is independent of the choice of the orthogonal basis. In particular, if T is an isometric mapping of E_q onto itself such that $T(R) = R$ and $w(Tx) = w(x)$ $(x \in R)$ then

$$K(Tx, Ty) = K(x, y). \tag{2.3}$$

3. The Addition Formula for Gegenbauer Polynomials

Let $\alpha > -\frac{1}{2}$. The formula

$$P_n^{(\alpha,\alpha)}(\cos\theta\cos\tau + \sin\theta\sin\tau\cos\phi)$$
$$= \sum_{k=0}^{n} c_{n,k}^{\alpha}(\sin\theta)^k P_{n-k}^{(\alpha+k,\alpha+k)}(\cos\theta)\,(\sin\tau)^k P_{n-k}^{(\alpha+k,\alpha+k)}(\cos\tau)\, P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\cos\phi), \tag{3.1}$$

where

$$c_{n,k} = \frac{(\alpha+k)\,(n+2\alpha+1)_k\,(2\alpha+1)_k\,(n-k)!}{2^{2k}(\alpha+\frac{1}{2}k)\,(\alpha+\frac{1}{2})_k\,(\alpha+1)_n} \tag{3.2}$$

is called the addition formula for Gegenbauer polynomials (cf. [1, 3.15(19)]). For fixed θ, formula (3.1) can be considered as an expansion of the left-hand side in terms of the functions

$$(\sin\tau)^k P_{n-k}^{(\alpha+k,\alpha+k)}(\cos\tau)\, P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\cos\phi).$$

Lemma 3.1 below states that these functions are orthogonal polynomials in the two variables $x = \cos\tau$ and $y = \sin\tau\cos\phi$. A new short proof of (3.1) then follows very easily.

LEMMA 3.1. *Let $\mathscr{H}_n$ be the class of orthogonal polynomials of degree n on the disk* $R = \{(x, y) \mid x^2 + y^2 < 1\}$ *with respect to the weight function* $(1 - x^2 - y^2)^{\alpha-\frac{1}{2}}$, $\alpha > -\frac{1}{2}$. *Then the functions*

$$p_{n,k}(x, y) = P_{n-k}^{(\alpha+k,\alpha+k)}(x)\,(1-x^2)^{\frac{1}{2}k}\, P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(y(1-x^2)^{-\frac{1}{2}}) \tag{3.3}$$

$(k = 0, 1, 2,..., n)$ *form an orthogonal basis of $\mathscr{H}_n$ which is obtained by orthogonalization of the sequence* $1, x, y, x^2, xy, y^2, x^3, x^2y,....$

Proof. Clearly, $p_{n,k}(x, y)$ is a linear combination of the monomials $1, x, y, x^2, xy, y^2,..., x^n, x^{n-1}y,..., x^{n-k}y^k$, and the coefficient of $x^{n-k}y^k$ is nonzero. By substituting $u = x$, $v = y(1-x^2)^{-\frac{1}{2}}$ and by using the orthogonality properties of Jacobi polynomials it follows that

$$\iint_R p_{n,k}(x, y)\, p_{m,l}(x, y)\,(1 - x^2 - y^2)^{\alpha-\frac{1}{2}}\, dx\, dy = \delta_{n,m}\delta_{k,l} h_{n-k}^{(\alpha+k,\alpha+k)} h_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}. \quad \blacksquare$$

Next we prove (3.1). Any rotation T around the origin maps the disk R onto itself and leaves the weight function $(1 - x^2 - y^2)^{\alpha-\frac{1}{2}}$ invariant. Let

$$K((x, y), (x', y')) = \sum_{k=0}^{n} \| p_{n,k} \|^{-2} p_{n,k}(x, y)\, p_{n,k}(x', y'). \tag{3.4}$$

Hence, it follows from (2.3) that

$$K((x,y),(\cos\theta,\sin\theta)) = K((x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta),(1,0)). \tag{3.5}$$

Substitution of (3.3) and (3.4) in (3.5) gives

$$\begin{aligned} &\| p_{n,0}\|^{-2} P_n^{(\alpha,\alpha)}(1)\, P_n^{(\alpha,\alpha)}(x\cos\theta + y\sin\theta) \\ &\quad = \sum_{k=0}^{n} \| p_{n,k}\|^{-2} P_{n-k}^{(\alpha+k,\alpha+k)}(x)\,(1-x^2)^{\frac{1}{2}k} P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(y(1-x^2)^{-\frac{1}{2}}) \\ &\qquad \cdot P_{n-k}^{(\alpha+k,\alpha+k)}(\cos\theta)\,(\sin\theta)^k P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(1). \end{aligned} \tag{3.6}$$

Putting $x = \cos\tau$, $y = \sin\tau\cos\theta$ in (3.6) we obtain (3.1) with

$$c_{n,k}^{\alpha} = \frac{\| p_{n,0}\|^2 P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(1)}{\| p_{n,k}\|^2 P_n^{(\alpha,\alpha)}(1)}.$$

By using that $\| p_{n,k}\|^2 = h_{n-k}^{(\alpha+k,\alpha+k)} h_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}$, a straightforward calculation gives back (3.2).

4. The Addition Formula for Jacobi Polynomials

Let $\alpha > \beta > -\frac{1}{2}$. The formula

$$\begin{aligned} &P_n^{(\alpha,\beta)}(2\cos^2\theta\cos^2\tau + 2\sin^2\theta\sin^2\tau\, r^2 + \sin 2\theta\sin 2\tau\, r\cos\phi - 1) \\ &\quad = \sum_{k=0}^{n}\sum_{l=0}^{k} c_{n,k,l}^{(\alpha,\beta)} (\sin\theta)^{2k-l}(\cos\theta)^l P_{n-k}^{(\alpha+2k-l,\beta+l)}(\cos 2\theta) \\ &\qquad \cdot (\sin\tau)^{2k-l}(\cos\tau)^l P_{n-k}^{(\alpha+2k-l,\beta+l)}(\cos 2\tau) \\ &\qquad \cdot r^l P_{k-l}^{(\alpha-\beta-1,\beta+l)}(2r^2-1)\, P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\phi), \end{aligned} \tag{4.1}$$

where

$$c_{n,k,l}^{(\alpha,\beta)} = \frac{\begin{pmatrix}(\alpha+2k-l)(\beta+l)(n+\alpha+\beta+1)_k \\ \times (\beta+n-k+l+1)_{k-l}(2\beta+1)_l(n-k)!\end{pmatrix}}{(\alpha+k)(\beta+\frac{1}{2}l)(\beta+1)_k(\beta+\frac{1}{2})_l(\alpha+k+1)_{n-l}} \tag{4.2}$$

is called the addition formula for Jacobi polynomials (cf. [2, (3)]). It was pointed out in [5, Sect. 3] that for fixed θ and τ, formula (4.1) can be considered as an expansion of the left-hand side in terms of the functions

$$r^l P_{k-l}^{(\alpha-\beta-1,\beta+l)}(2r^2-1)\, P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\phi),$$

which are orthogonal polynomials in the two variables r^2 and $r\cos\phi$. However, for fixed θ, formula (4.1) can also be considered as an expansion of the left-hand side in terms of functions in τ, r, ϕ which are orthogonal polynomials in the three variables $x = \cos^2\tau$, $y = r^2\sin^2\tau$, $z = 2^{-\frac{1}{2}}r\sin 2\tau\cos\phi$. This will be proved in Lemma 4.1 below. Then the addition formula (4.1) follows in a similar way as the result in Section 3.

Let R be the three-dimensional region $\{(x, y, z) \mid 0 < x + y < 1,\ z^2 < 2xy\}$, which is bounded by the cone $z^2 = 2xy$ and by the plane $x + y = 1$ orthogonal to the axis of the cone. Let $\mathscr{H}_n$ be the class of orthogonal polynomials of degree n on the region R with respect to the weight function

$$w(x, y, z) = (1 - x - y)^{\alpha-\beta-1}(2xy - z^2)^{\beta-\frac{1}{2}}, \qquad \alpha > \beta > -\tfrac{1}{2}. \tag{4.3}$$

LEMMA 4.1. *The functions*

$$p_{n,k,l}(x, y, z) = P_{n-k}^{(\alpha+2k-l,\beta+l)}(2x - 1)(1 - x)^{k-l} \cdot P_{k-l}^{(\alpha-\beta-1,\beta+l)}((x + 2y - 1)/(1 - x))\,(xy)^{\frac{1}{2}l}\,P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}((2xy)^{-\frac{1}{2}}z) \tag{4.4}$$

$(n \geqslant k \geqslant l \geqslant 0)$ form an orthogonal basis of $\mathscr{H}_n$, which is obtained by orthogonalization of the sequence

$$1, x, y, z, x^2, xy, xz, y^2, yz, z^2, x^3, \ldots .$$

Proof. Clearly, function $p_{n,k,l}(x, y, z)$ is a polynomial of degree n in x, y, z, of degree k in y, z and of degree l in z. Hence, $p_{n,k,l}(x, y, z)$ is a linear combination of monomials $x^{m_1-m_2}y^{m_2-m_3}z^{m_3}$ with "highest" term const. $x^{n-k}y^{k-l}z^l$. Let $u = 2x - 1$, $v = (x + 2y - 1)/(1 - x)$, $w = z(2xy)^{-\frac{1}{2}}$. The mapping $(x, y, z) \to (u, v, w)$ is a diffeomorphism from R onto the cubic region $\{(u, v, w) \mid -1 < u < 1, -1 < v < 1, -1 < w < 1\}$. By making this substitution and by using the orthogonality properties of Jacobi polynomials it follows that

$$\iiint_R p_{n,k,l}(x, y, z)\,p_{n',k',l'}(x, y, z)\,w(x, y, z)\,dx\,dy\,dz = \delta_{n,n'}\delta_{k,k'}\delta_{l,l'}\,2^{-2\alpha-2k-l-1}h_{n-k}^{(\alpha+2k-l,\beta+l)}\,h_{k-l}^{(\alpha-\beta-1,\beta+l)}h_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}.$$ ∎

Next we prove the addition formula (4.1). Let

$$K((x, y, z), (x', y', z')) = \sum_{k=0}^{n}\sum_{l=0}^{k} \| p_{n,k,l} \|^{-2}\,p_{n,k,l}(x, y, z)\,p_{n,k,l}(x', y', z'). \tag{4.5}$$

It follows from (4.4) that $p_{n,k,l}(1, 0, 0) = 0$ if $(n, k, l) \neq (n, 0, 0)$. Hence

$$K((x, y, z), (1, 0, 0)) = \| p_{n,0,0} \|^{-2}\,P_n^{(\alpha,\beta)}(1)\,P_n^{(\alpha,\beta)}(2x - 1). \tag{4.6}$$

Any rotation around the axis $\{(x, y, z) \mid x = y, z = 0\}$ of the cone maps the region R onto itself and leaves the weight function $w(x, y, z)$ invariant. In particular, consider a rotation of this type over an angle -2θ. It maps point $(\cos^2\theta, \sin^2\theta, 2^{-\frac{1}{2}}\sin 2\theta)$ onto $(1, 0, 0)$ and point (x, y, z) onto a point (ξ, η, ζ) where $\xi = x\cos^2\theta + y\sin^2\theta + 2^{-\frac{1}{2}}z\sin 2\theta$. Hence, by (2.3), (4.4), (4.5), and (4.6) we have

$$\begin{aligned}
&\| p_{n,0,0} \|^{-2} P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(2(x\cos^2\theta + y\sin^2\theta + 2^{-\frac{1}{2}}z\sin 2\theta) - 1)\\
&\quad = K((x, y, z), (\cos^2\theta, \sin^2\theta, 2^{-\frac{1}{2}}\sin 2\theta))\\
&\quad = \sum_{k=0}^{n}\sum_{l=0}^{k} \| p_{n,k,l} \|^{-2} P_{k-l}^{(\alpha-\beta-1,\beta+l)}(1)\, P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(1)\\
&\qquad \cdot (\sin\theta)^{2k-l}(\cos\theta)^l\, P_{n-k}^{(\alpha+2k-l,\beta+l)}(\cos 2\theta)\\
&\qquad \cdot P_{n-k}^{(\alpha+2k-l,\beta+l)}(2x-1)\,(1-x)^{k-l}\, P_{k-l}^{(\alpha-\beta-1,\beta+l)}((x+2y-1)/(1-x))\\
&\qquad \cdot (xy)^{\frac{1}{2}l}\, P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}((2xy)^{-\frac{1}{2}}z).
\end{aligned} \tag{4.7}$$

Substitution of $x = \cos^2\tau$, $y = r\sin^2\tau$, $z = 2^{-\frac{1}{2}}r\sin 2\tau\cos\phi$ gives (4.1) with

$$c_{n,k,l}^{(\alpha,\beta)} = \frac{\| p_{n,0,0} \|^2 P_{k-l}^{(\alpha-\beta-1,\beta+l)}(1)\, P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(1)}{\| p_{n,k,l} \|^2 P_n^{(\alpha,\beta)}(1)}.$$

Using the expression for $\| p_{n,k,l} \|^2$ at the end of the proof of Lemma 4.1 we get back formula (4.2).

References

1. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, "Higher Transcendental Functions," Vol. I, McGraw–Hill, New York, 1953.
2. T. H. Koornwinder, The addition formula for Jacobi polynomials. I. Summary of results, *Nederl. Akad. Wetensch. Proc. A* **75**; *Indag. Math.* **34** (1972), 188–191.
3. T. H. Koornwinder, The addition formula for Jacobi polynomials, II, III, *Math. Centrum Amsterdam Reports TW* **133**, **135** (1972).
4. T. H. Koornwinder, The addition formula for Jacobi polynomials and spherical harmonics, *SIAM J. Appl. Math.* **25** (1973), 236–246.
5. T. H. Koornwinder, Jacobi polynomials, III. An analytic proof of the addition formula. *SIAM. J. Math. Anal.* **6** (1975), 533–543.
6. R. L. Šapiro, Special functions related to representations of the group $SU(n)$, of class I with respect to $SU(n-1)$ $(n \geq 3)$ (Russian). *Izv. Vysš. Učebn. Zaved. Matematika* **4** (1968), 97–107.