Askey-Wilson polynomial

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1.00 - **Tom H. Koornwinder**

Askey-Wilson polynomial refers to a four-parameter family of q-hypergeometric orthogonal polynomials which contains all families of classical orthogonal polynomials (in the wide sense) as special or limit cases.

Contents [hide]

Orthogonal polynomials

See Szegő [Sz].

General orthogonal polynomials

Let $w(x)$ be a nonnegative function on an open real interval (a,b) such that the integral $\int_a^b |x|^n w(x)\,dx$ is well-defined and finite for all nonnegative integers n. A system of real-valued polynomials $p_n(x)$ $(n=0,1,2,\ldots)$ is called **orthogonal** on the interval (a,b) with respect to the *weight function* $w(x)$ if $p_n(x)$ has degree n and if $\int_a^b p_n(x)p_m(x)w(x)\,dx = 0$ for $n \neq m$.

More generally we can replace in this definition $w(x)dx$ by a positive measure $d\mu(x)$ on $\mathbb{R}$. Then the orthogonality relation becomes $\int_{\mathbb{R}} p_n(x)p_m(x)\,d\mu(x) = 0$ $(n \neq m)$. If the measure is *discrete* then this takes the form $\sum_{j=0}^{\infty} p_n(x_j)p_m(x_j)w_j = 0$ $(n \neq m)$, where the *weights* w_j are positive. The *finite* case $\sum_{j=0}^{N} p_n(x_j)p_m(x_j)w_j = 0$ $(n \neq m;\ n,m = 0,1,\ldots,N)$ also occurs.

Three-term recurrence relation

$$xp_n(x;d) = A_{n+d}\,p_{n+1}(x;d) + B_{n+d}\,p_n(x;d) + C_{n+d}\,p_{n-1}(x;d).$$

Any system of orthogonal polynomials $p_n(x)$ satisfies a three-term recurrence relation of the form

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x), \tag{1}$$

where $p_{-1}(x) := 0$ and $A_{n-1}C_n > 0$. One may also consider *associated orthogonal polynomials* $p_n(x;d)$, which satisfy the recurrence relation

Here d is a positive integer or, more generally, a positive real number as long as this makes sense by analyticity in n of the coefficients A_n, B_n, C_n.

Classical orthogonal polynomials (in the strict sense)

A system of orthogonal polynomials $p_n(x)$ is called **classical** (in the strict sense) if there is a second order linear differential operator L, not depending on n, such that p_n is an eigenfunction of L for each n:

$$Lp_n = \lambda_n p_n. \tag{2}$$

By *Bochner's theorem* [Bo] there are three families of orthogonal polynomials which are classical in the strict sense:

- *Jacobi polynomials* $P_n^{(\alpha,\beta)}(x)$, where $\alpha,\beta > -1$, $w(x) := (1-x)^{\alpha}(1+x)^{\beta}$, $(a,b) := (-1,1)$;
- *Laguerre polynomials* $L_n^{\alpha}(x)$, where $\alpha > -1$, $w(x) := e^{-x}x^{\alpha}$, $(a,b) := (0,\infty)$;
- $-x^2$

Hermite polynomials $H_n(x)$, where $w(x) := e$, $(a,b) := (-\infty, \infty)$.

Classical orthogonal polynomials (in the wide sense)

More generally, a system of orthogonal polynomials $p_n(x)$ is called **classical** (in the wide sense) if there is a second order linear difference or q-difference operator L, not depending on n, such that (2) holds.

- *difference operator*, for instance $(Lf)(x) := a(x)f(x-1) + b(x)f(x) + c(x)f(x+1)$. An example are the *Charlier polynomials* $C_n(x;a)$ $(a>0)$ which are orthogonal with respect to the weights $a^x/x!$ on the points x $(x = 0,1,2,\dots)$.
- *q-difference operator*, for instance $(Lf)(x) := a(x)f(q^{-1}x) + b(x)f(x) + c(x)f(qx)$. An example are the *discrete q-Hermite I polynomials* $h_n(x;q)$, which are orthogonal with respect to the weights $q^j \prod_{k=1}^{\infty}(1-q^{2k+2j+2})$ on the points $\pm q^j$ $(j = 0,1,2,\dots)$.

Hypergeometric and basic hypergeometric Series

Hypergeometric series

For complex a and nonnegative integer n let $(a)_n := a(a+1)\dots(a+n-1),\ (a)_0 := 1$ be the **Pochhammer symbol**.

A **hypergeometric series** with r upper parameters $a_1,\dots,a_r$ and s lower parameters $b_1,\dots,b_s$ is formally defined as

$$ {}_rF_s\left({a_1,\dots,a_r \atop b_1,\dots,b_s};z\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_r)_k}{(b_1)_k \dots (b_s)_k}\frac{z^k}{k!}. \tag{3}$$

If a_1 is equal to a nonpositive integer $-n$ then the series on the right-hand side of (3) terminates after the term with $k=n$.

Basic hypergeometric series

See Gasper & Rahman [GR]. Let q be a complex number not equal to 0 or 1.

For complex a and nonnegative integer n let $(a;q)_n := (1-a)(1-aq)\dots(1-aq^{n-1}),\ (a;q)_0 := 1$ be the q-**Pochhammer symbol**.

Also let $(a_1,a_2,\dots,a_r;q)_n := (a_1;q)_n(a_2;q)_n\dots(a_r;q)_n$.

For $|q|<1$ let $(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k)$, a convergent infinite product.

A **basic** or q-**hypergeometric series** with r upper parameters $a_1,\dots,a_r$ and s lower parameters $b_1,\dots,b_s$ is formally defined as

$$ {}_r\phi_s\left({a_1,\dots,a_r \atop b_1,\dots,b_s};q,z\right) := \sum_{k=0}^{\infty}\left((-1)^k q^{k(k-1)/2}\right)^{s-r+1}\frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k}\frac{z^k}{(q;q)_k}. \tag{4}$$

If $a_1 = q^{-n}$ for a nonnegative integer n then the series on the right-hand side of (4) terminates after the term with $k=n$.

Askey-Wilson polynomials

Askey-Wilson polynomials were introduced by Askey & Wilson [AW] in 1985.

Definition

$$p_n(\cos\theta) = p_n(\cos\theta;a,b,c,d \mid q) := \frac{(ab,ac,ad;q)_n}{a^n}\, {}_4\phi_3\left({q^{-n},q^{n-1}abcd,ae^{i\theta},ae^{-i\theta} \atop ab,ac,ad};q,q\right). \tag{5}$$

This is a polynomial of degree n in $\cos\theta$.

Symmetry, special value and duality

$$\frac{p_n(\frac{1}{2}(a^{-1}q^{-m}+aq^m);a,b,c,d\mid q)}{p_n(\frac{1}{2}(a^{-1}+a);a,b,c,d\mid q)}=\frac{p_m(\frac{1}{2}(\breve{a}^{-1}q^{-n}+\breve{a}q^n);\breve{a},\breve{b},\breve{c},\breve{d}\mid q)}{p_m(\frac{1}{2}(\breve{a}^{-1}+\breve{a});\breve{a},\breve{b},\breve{c},\breve{d}\mid q)}$$

The polynomials $p_n(x;a,b,c,d\mid q)$ are symmetric in the parameters a,b,c,d. They have special value $p_n(\frac{1}{2}(a+a^{-1});a,b,c,d\mid q)=\frac{(ab,ac,ad;q)_n}{a^n}$, and similarly for arguments $\frac{1}{2}(b+b^{-1})$, $\frac{1}{2}(c+c^{-1})$ and $\frac{1}{2}(d+d^{-1})$. For nonnegative integers m,n there is the duality

for $a=q^{-1/2}(\breve{a}\breve{b}\breve{c}\breve{d})^{1/2}$ and $ab=\breve{a}\breve{b}$, $ac=\breve{a}\breve{c}$, $ad=\breve{a}\breve{d}$.

Orthogonality relation

$$2\pi\sin\theta\, w(\cos\theta):=\left|\frac{(e^{2i\theta};q)_\infty}{(ae^{i\theta},be^{i\theta},ce^{i\theta},de^{i\theta};q)_\infty}\right|^2,$$

Let $0<q<1$. Assume that a,b,c,d are four reals, or two reals and one pair of complex conjugates, or two pairs of complex conjugates. Also assume that $|a|,|b|,|c|,|d|<1$. Then

$$\int_{-1}^{1}p_n(x)p_m(x)w(x)\,dx=h_n\,\delta_{n,m}, \qquad (6)$$

where

$$h_0:=\frac{(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty},\quad \frac{h_n}{h_0}:=\frac{1-abcdq^{n-1}}{1-abcdq^{2n-1}}\,\frac{(q,ab,ac,ad,bc,bd,cd;q)_n}{(abcd;q)_n}.$$

and For more general parameter values the orthogonality relation (6) can be given as the contour integral

$$\frac{1}{2\pi i}\oint_C p_n((z+z^{-1})/2)\,p_m((z+z^{-1})/2)\,\frac{(z^2,z^{-2};q)_\infty}{(az,az^{-1},bz,bz^{-1},cz,cz^{-1},dz,dz^{-1};q)_\infty}\,\frac{dz}{z}=2h_n\,\delta_{n,m}, \qquad (7)$$

where C is the unit circle traversed in positive direction with suitable deformations to separate the sequences of poles converging to zero from the sequences of poles diverging to ∞. The left-hand side of (7) can be rewritten as the left-hand side of (6) with finitely many terms added of the form $p_n(x_j)p_m(x_j)w_j$, where x_j is in $\mathbb{R}$ outside $[-1,1]$. The case $n=m=0$ of (6) or (7) is called the **Askey-Wilson integral**.

q-Difference equation

Let $P_n(z):=p_n((z+z^{-1})/2)$. Then

$$LP_n=(q^{-n}-1)(1-q^{n-1}abcd)P_n, \qquad (8)$$

where

$$(Lf)(z):=A(z)f(qz)-(A(z)+A(z^{-1}))f(z)+A(z^{-1})f(q^{-1}z) \qquad (9)$$

with $A(z):=(1-az)(1-bz)(1-cz)(1-dz)/((1-z^2)(1-qz^2))$. By (8) the Askey-Wilson polynomials $P_n(z)$ are eigenfunctions of a second order q-difference operator. Thus they are classical orthogonal polynomials in the wide sense.

Discretization, specializations and limit cases

See Chapter 14 in the book by Koekoek et al. [KLS] or see the earlier online Koekoek & Swarttouw report [KS].

q-Racah polynomials

$$R_n(q^{-y}+\gamma\delta q^{y+1};\alpha,\beta,\gamma,\delta \quad q) := {}_4\,{}_3\left(\begin{matrix} q^{-n}, q^{n+1}\alpha\beta, q^{-y}, \gamma\delta q^{y+1} \\ q\alpha, q\beta\delta, q\gamma \end{matrix}; q, q\right).$$

The q -Racah polynomials $R_n(x;\alpha,\beta,\gamma,\delta \quad q)$ form a family of finite ($n = 0, 1, \ldots, N$) sytems of orthogonal polynomials depending on four parameters $\alpha,\beta,\gamma,\delta$, where $q\alpha = q^{-N}$ or $q\beta\delta = q^{-N}$ or $q\gamma = q^{-N}$. They have essentially the same analytic expression as the Askey-Wilson polynomials:

Hence there is the duality $R_n(q^{-m}+\gamma\delta q^{m+1};\alpha,\beta,\gamma,\delta \quad q) = R_m(q^{-n}+\alpha\beta q^{n+1};\gamma,\delta,\alpha,\beta \quad q)$.

$$\sum_{y=0}^{N} R_n(q^{-y}+\gamma\delta q^{y+1}) R_m(q^{-y}+\gamma\delta q^{y+1})\, w_y = h_n\,\delta_{n,m} \quad (n, m = 0, 1, \ldots, N).$$

They satisfy an orthogonality relation of the form

Selected special cases

We obtain special subfamilies of the Askey-Wilson polynomials by specialization of parameters.

$$Q_n(x;a,b \quad q) := p_n(x;a,b,0,0 \quad q)\ .$$

Al-Salam-Chihara polynomials

$$P_n^{(\alpha,\beta)}(x;q) := \text{const.}\, p_n(x;q^{\frac12}, q^{\alpha+\frac12}, -q^{\beta+\frac12}, -q^{\frac12} \quad q) = \text{const.}\, p_n(x;q^{\alpha+\frac12}, q^{\alpha+\frac32}, -q^{\beta+\frac12}, -q^{\beta+\frac32} \quad q^2)\ .$$

Continuous q -Jacobi polynomials

$$C_n(\cos\theta;\beta \quad q) := \frac{(\beta;q)_n}{(q;q)_n}\, p_n(\cos\theta;\beta^{\frac12}, \beta^{\frac12} q^{\frac12}, -\beta^{\frac12}, -\beta^{\frac12} q^{\frac12} \quad q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}}\, e^{i(n-2k)\theta}\ .$$

Continuous q -ultraspherical polynomials

$$H_n(\cos\theta \quad q) := (q;q)_n\, C_n(\cos\theta;0 \quad q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\, e^{i(n-2k)\theta}\ .$$

Continuous q -Hermite polynomials

$$p_n(\cos\theta;1,-1,q^{\frac12},-q^{\frac12} \quad q) = \text{const.}\, \cos n\theta, \quad p_n(\cos\theta;q,-q,q^{\frac12},-q^{\frac12} \quad q) = \text{const.}\, \frac{\sin(n+1)\theta}{\sin\theta}\ ,$$

Chebyshev polynomials

$$p_n(\cos\theta;q,-1,q^{\frac12},-q^{\frac12} \quad q) = \text{const.}\, \frac{\sin(n+\frac12)\theta}{\sin\frac12\theta}\ , \quad p_n(\cos\theta;1,-q,q^{\frac12},-q^{\frac12} \quad q) = \text{const.}\, \frac{\cos(n+\frac12)\theta}{\cos\frac12\theta}\ .$$

Selected limit cases preserving q

$$P_n(x;a,b,c;q) := {}_3\phi_2\left({q^{-n}, q^{n+1}ab, x \atop qa, qc}; q, q\right) = \text{const.} \lim_{\lambda\downarrow 0} \lambda^n p_n(\tfrac{1}{2}\lambda^{-1}x; \lambda, \lambda^{-1}qa, \lambda^{-1}qc, \lambda bc^{-1} \mid q).$$

Big q-Jacobi polynomials

$$p_n(x;a,b;q) := {}_2\phi_1\left({q^{-n}, q^{n+1}ab \atop qa}; q, qx\right) = \text{const.} \lim_{\lambda\downarrow 0} \lambda^n p_n(\tfrac{1}{2}\lambda^{-1}x; -q^{\frac{1}{2}}a, qb\lambda, -q^{\frac{1}{2}}, \lambda^{-1} \mid q).$$

Little q-Jacobi polynomials

Selected limit cases for q to 1

$$W_n(y^2;a,b,c,d) = \lim_{q\uparrow 1}(1-q)^{-3n} p_n(\tfrac{1}{2}(q^{iy}+q^{-iy}); q^a, q^b, q^c, q^d \mid q).$$

Wilson polynomials

$$P_n^{(\alpha,\beta)}(x) = \lim_{q\uparrow 1} P_n^{(\alpha,\beta)}(x;q).$$

Jacobi polynomials

$$C_n^{\lambda}(x) = \lim_{q\uparrow 1} C_n(x;q^{\lambda} \mid q).$$

Ultraspherical polynomials

$$H_n(x) = \lim_{q\uparrow 1}(1-q)^{-n/2} H_n((1-q)^{1/2}x \mid q^2).$$

Hermite polynomials

A limit case for q to 0

$$\lim_{q\downarrow 0} C_n(\cos\theta;\beta \mid q) = (1-\beta)\frac{\sin(n+1)\theta}{\sin\theta} - \beta(1-\beta)\frac{\sin(n-1)\theta}{\sin\theta} \quad (n=1,2\ldots) \quad \text{and} \quad = 1 \quad (n=0).$$

Special Bernstein-Szegö polynomials

Askey scheme, characterization theorems and unusual limit cases

(q-)Askey scheme

The q-Askey scheme is a directed graph with Askey-Wilson and q-Racah polynomials on top, all other q-hypergeometric orthogonal polynomials as further nodes, and the specializations and limits as arrows. The Askey scheme is a similar and older graph for the hypergeometric orthogonal polynomials (the $q=1$ case). It was first given in [AW], and it was slightly improved soon after. A slightly further extended version is from [KLS].

Characterization theorems

Leonard [Le] showed the following. Let a finite system $\{p_n\}_{n=0,1,\ldots,N}$ of orthogonal polynomials have a dual system of orthogonal polynomials $\{q_m\}_{m=0,1,\ldots,N}$ (i.e., $p_n(x_m) = q_m(y_n)$, where the p_n are orthogonal on $\{x_0, x_1, \ldots, x_N\}$ and the q_m are orthogonal on $\{y_0, y_1, \ldots, y_N\}$). Then the p_n are q-Racah polynomials or one of its limit cases. Leonard's theorem was generalized by Bannai & Ito [BaIt]. They also included infinite systems ($N = \infty$) and they replaced the orthogonality assumption concerning the p_n and q_m by the assumption that these two systems satisfy a three-term recurrence relations of the form (1), however without the positivity assumption $A_{n-1} C_n > 0$. They explicitly gave all systems and dual systems of polynomials satisfying the conditions of the theorem.

$$A(s)p_n(z(s+1)) + B(s)p_n(z(s)) + C(s)p_n(z(s-1)) = \lambda_n p_n(z(s))$$

Generalized Bochner theorems characterizing Askey-Wilson polynomials and all their limit cases were successively proved in increasing generality by Grünbaum & Haine, by M.E.H. Ismail, and finally by Vinet & Zhedanov [ViZh], who showed that monic polynomials p_n of degree n ($n = 0, 1, 2, \ldots$) satisfying a second-order difference equation

together with some non-degeneracy conditions, must be Askey-Wilson polynomials or their limit cases with the grid given by the $z(s)$ being at most a q-quadratic grid $z(s) = aq^s + bq^{-s} + c$.

Unusual limit cases

As a surprise of the classification in [BaIt] came out a limit for $q \downarrow -1$ of the q- Racah polymials. Tsujimoto, Vinet & Zhedanov [TsViZh] gave the orthogonality relations and studied these polynomials from a wider perspective. Further limit cases of these Bannai-Ito polynomials were shown to be the big and little -1 Jacobi polynomials, which were studied in other recent papers by Vinet & Zhedanov, and which were in the little -1 Jacobi case already obtained by T.S. Chihara (1968, 1971). All these orthogonal polynomials obtained as $q \downarrow -1$ limit turned out to be eigenfunctions of certain Dunkl type operators.

Suitable limits of orthogonal polynomials in the q- Askey scheme for $q = s\omega \to \omega = e^{2\pi i/k}$ $(0 < s < 1)$ are known as sieved orthogonal polynomials. A prototype, the sieved ultraspherical polynomials, was studied by Al-Salam, Allaway and Askey [Al-SAllA]. This was followed by a long series of papers by M.E.H. Ismail and coauthors.

Analogues in several variables

Macdonald polynomials of type A

$$P_{m,n}(x,y;q,t) = \frac{(q;q)_{m-n}}{(t;q)_{m-n}} (xy)^{\frac{1}{2}(m+n)} \, C_{m-n}\Big(\frac{x+y}{2(xy)^{1/2}};t \quad q\Big) \quad (m \ge n \ge 0).$$

The A_ℓ type **Macdonald polynomials** $P_\lambda(z;q,t)$, see [M1], are certain symmetric homogeneous polynomials in $\ell + 1$ variables of degree $|\lambda|$ which form an orthogonal system. They can be expressed in terms of q-ultraspherical polynomials for $\ell = 1$:

$$P_{m,0}(e^{i\theta}, e^{-i\theta}; q, t) = \frac{(q;q)_m}{(t;q)_m} C_m(\cos\theta; t \quad q).$$

In particular, In the limit for q to 0 the A_ℓ type Macdonald polynomials are known as **Hall-Littlewood polynomials**.

Macdonald-Koornwinder polynomials

Macdonald [M2] introduced **Macdonald polynomials** for all irreducible root systems. They are certain Weyl group invariant trigonometric polynomials forming an orthogonal system and depending on as many parameters (apart from q) as there are root lengths. Thus the Macdonald polynomials for root system BC_ℓ depend on three parameters. For root system BC_1 this turns down to the two-parameter family of continuous q-Jacobi polynomials. Koornwinder [Ko1] extended the BC_ℓ type Macdonald polynomials to a family depending on five parameters a, b, c, d, t : the **Macdonald-Koornwinder polynomials**. For $\ell = 1$ they no longer depend on t and reduce to Askey-Wilson polynomials.

Algebraic aspects

Nonsymmetric Askey-Wilson polynomials

In 1992 Cherednik [Ch] introduced **double affine Hecke algebras** (DAHA's) as a natural habitat for **nonsymmetric Macdonald polynomials** from which the Macdonald polynomials themselves can be obtained by Weyl group symmetrization. In 1999 Sahi [Sa] extended this approach to Macdonald-Koornwinder polynomials, see also Macdonald's book [M3]. Here the DAHA is associated with the affine root system of type $(C_\ell^\vee, C_\ell)$. Its one-variable case led to nonsymmetric Askey-Wilson polynomials in the context of the rather simple DAHA of type $(C_1^\vee, C_1)$. In the so-called basic representation of this DAHA on the space of Laurent polynomials in one variable a certain element Y acts on a Laurent polynomial $f(z)$ as a q-difference-reflection operator, sending $f(z)$ to a linear combination of terms $f(z), f(qz), f(z^{-1}), f(qz^{-1})$ with rational functions in z as coefficients. It has eigenfunctions $E_n(z)$ for each integer n, where $E_n(z)$ is a linear combination of $z^{-n},\ldots,z^n$ for $n>0$, $E_{-n}(z)$ is a linear combination of $z^{-n},\ldots,z^{n-1}$ for $n>0$, and $E_0(z)=1$. The operator Y has an inverse which is also a q-difference-reflection operator and the operator $Y+Y^{-1}$ has two-dimensional eigenspaces spanned by $E_{\pm n}(z)$. A certain symmetrization operator projects these eigenspaces on one-dimensional spaces of symmetric Laurent polynomials spanned by the Askey-Wilson polynomials $p_n((z+z^{-1})/2)$. See Noumi & Stokman [NoSt].

Askey-Wilson algebra

$$[K_0,K_1]_q=K_2,\quad [K_1,K_2]_q=BK_1+C_0K_0+D_0,\quad [K_2,K_0]_q=BK_0+C_1K_1+D_1,$$

Zhedanov [Zh] introduced an associative algebra $AW(3)$ with identity over the complex numbers with generators K_0,K_1,K_2 and with relations

where $[X,Y]_q:=q^{1/2}XY-q^{-1/2}YX$ and B,C_0,C_1,D_0,D_1 are constants. There is a central element Q which is explicitly given as a polynomial of degree 3 in the generators. With B,C_0,C_1,D_0,D_1 suitably expressed in terms of a,b,c,d,q this algebra has a representation on the space of symmetric Laurent polynomials such that K_0 is the operator given by (9) and K_1 is the operator of multiplying $f(z)$ by $z+z^{-1}$. In this representation Q is equal to a constant Q_0 . Denote the quotient of $AW(3)$ with respect to the relation $Q=Q_0$ by $AW(3,Q_0)$.

A central extension of $AW(3,Q_0)$ can be embedded in the DAHA associated with the Askey-Wilson polynomials (see [Ko2], later very elegantly phrased in [Te]). The algebra $AW(3,Q_0)$ itself is isomorphic with the spherical subalgebra of the mentioned DAHA (see [Ko3]).

Various interpretations

Interpretations on quantum groups

Corresponding to many group theoretic interpretations of special functions there are interpretations of q-special functions on quantum groups. See Vilenkin & Klimyk [VKl] for both kinds of interpretations. The interpretation of little q-Jacobi polynomials as matrix elements of irreducible representations of the quantum group $SU_q(2)$ is a rather straightforward analogue of the classical situation, which was independently found by several authors during the late eighties of the 20th century. Here the matrix elements are taken with respect to the quantum subgroup which is the quantum analogue of the diagonal subgroup of $SU(2)$. It is not possible to obtain other quantum subgroups from this one by conjugation. However, there exist quantum analogues of the Lie subalgebras of the Lie algebra $su(2)$, and matrix elements of irreducible representations of $SU_q(2)$ can be defined with respect to these. By work of Koornwinder, Noumi & Mimachi, and Koelink these matrix elements could be expressed in terms of Askey-Wilson polynomials, with all four parameters of these polynomials being used. See Koelink [Koe] for a survey. Rosengren [Ro1] obtained Askey-Wilson polynomials and q-Racah polynomials as matrix elements of representations by using other special bases of representation space. Koelink & Rosengren [KoeRo] gave interpretations of Askey-Wilson polynomials and q-Racah polynomials in connection with the $SU(2)$ dynamical quantum group.

q-Racah polynomials have an interpretation as quantum $6j$-symbols for $SU_q(2)$. This was first established by A.N. Kirillov & Reshetikhin in 1989. There are many important applications, notably to invariants of links and 3-manifolds. See a survey and further work on $6j$-symbols by Rosengren [Ro2].

Combinatorial interpretation

Uchiyama, Sasamoto and Wadati related the stationary state of the one-dimensional *asymmetric simple exclusion process* (ASEP) with open boundary conditions to Askey-Wilson polynomials. Here all Askey-Wilson parameters, including q have an interpretation in the ASEP. Next Corteel & Williams [CoWi] introduced combinatorial objects called *staircase tableaux* in connection with the stationary measure for the ASEP. With the aid of this they could give a combinatorial formula for the moments of the Askey-Wilson polynomials.

Probabilistic interpretation

Bryc & Wesołowski [BrWe] constructed an auxiliary Markov process which has Askey-Wilson polynomials as orthogonal martingale polynomials. By using this they were able to construct a large class of Markov processes with linear regressions and quadratic conditional variances, that includes most of previously known cases either as special cases or as boundary cases.

Beyond the Askey-Wilson polynomials

- **Biorthogonal rational functions**. Rahman [R1] extended the four-parameter family of Askey-Wilson polynomials to a five-parameter biorthogonal system of rational functions, which are expressed in terms of a very well poised ${}_{10}\ {}_{9}$ series. In the same year Wilson [W] gave the corresponding finite biorthogonal system which extends the q-Racah polynomials.
- **Askey-Wilson functions**. Koelink & Stokman [KoeSt] gave a continuous analogue of the orthogonal system of Askey-Wilson polynomials in the form of the *Askey-Wilson function transform*, which has as its kernel the *Askey-Wilson functions*: a four-parameter family of functions expressed in terms of very well poised ${}_{8}\ {}_{7}$ series.
- **Associated Askey-Wilson polynomials**. These were studied by Ismail & Rahman [IR]. They gave explicit expressions for the polynomials and for the absolutely continuous part of the orthogonality measure.
- **Elliptic $6j$-symbols**. These were introduced by Date et al. and they led Frenkel & Turaev to the definition of *elliptic hypergeometric series*. Next Spiridonov & Zhedanov [SpZh] showed that they form a biorthogonal system having the q-Racah polynomials as a limit case. See also [Ro2].

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See also

- Ruijsenaars-Schneider model

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