

STANDARD L -FUNCTIONS ATTACHED TO ALTERNATING TENSOR VALUED SIEGEL MODULAR FORMS

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(Received May 23, 2000)

1. Introduction

Let (ρ, V_ρ) be an irreducible rational representation of $GL(n, \mathbb{C})$ on a finite-dimensional complex vector space V_ρ such that the signature of ρ is $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. Let f be a V_ρ -valued Siegel cuspform of type ρ with respect to $Sp(n, \mathbb{Z})$ (size $2n$). Suppose f is an eigenform, i.e., a non-zero common eigenfunction of the Hecke algebra. Then we define the standard L -function attached to f by

$$(1.1) \quad L(s, f, \underline{St}) := \prod_p \left\{ (1 - p^{-s}) \prod_{j=1}^n (1 - \alpha_j(p)p^{-s})(1 - \alpha_j(p)^{-1}p^{-s}) \right\}^{-1},$$

where p runs over all prime numbers and $\alpha_j(p)$ ($1 \leq j \leq n$) are the Satake p -parameters of f . The right-hand side of (1.1) converges absolutely and locally uniformly for $\operatorname{Re}(s) > n + 1$. We put

$$\Lambda(s, f, \underline{St}) := \Gamma_{\mathbb{R}}(s + \varepsilon) \prod_{j=1}^n \Gamma_{\mathbb{C}}(s + \lambda_j - j) L(s, f, \underline{St})$$

with

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s} \Gamma(s),$$

and

$$\varepsilon := \begin{cases} 0 & \text{for } n \text{ even,} \\ 1 & \text{for } n \text{ odd.} \end{cases}$$

Then by Takayanagi [15], we expect the following:

Conjecture. $\Lambda(s, f, \underline{St})$ has a meromorphic continuation to the whole s -plane and satisfies a functional equation.

For $\rho = \det^k$ (cf. Andrianov and Kalinin [1], Böcherer [2] and Mizumoto [12]), $\rho = \det^k \otimes \text{sym}^l$ (cf. Takayanagi [15]), $\rho = \det^k \otimes \text{alt}^{n-1}$ (cf. Takayanagi [16]), this conjecture holds. In this paper, for $\rho = \det^k \otimes \text{alt}^l$ ($1 \leq l \leq n - 1$), we show the conjecture holds.

We note that the signature of $\det^k \otimes \text{alt}^l$ is $(\underbrace{k + 1, \dots, k + 1}_l, \underbrace{k, \dots, k}_{n-l})$. Then the main result of this paper is the following (cf. Piatetski-Shapiro and Rallis [14], Weis-sauer [17]).

Theorem 1. *Let $n \in \mathbb{Z}_{>0}$, $k, l \in 2\mathbb{Z}$, and $2k \geq n > 2$. Let f be a cuspidal eigenform of type ρ . Then $\Lambda(s, f, \text{St})$ has a meromorphic continuation to the whole s -plane and satisfies the functional equation*

$$\Lambda(s, f, \text{St}) = \Lambda(1 - s, f, \text{St}).$$

Moreover, $\Lambda(s, f, \text{St})$ is holomorphic except for possible simple poles at $s = 0$ and $s = 1$. If n is odd, then $\Lambda(s, f, \text{St})$ is entire.

2. Preliminaries

Let $n \in \mathbb{Z}_{>0}$. Let (ρ, V_ρ) be a finite-dimensional irreducible representation of $GL(n, \mathbb{C})$. We fix a Hermitian inner product $\langle \cdot, \cdot \rangle$ on V_ρ such that

$$\langle \rho(g)v, w \rangle = \langle v, \rho(\bar{g})w \rangle \quad \text{for } g \in GL(n, \mathbb{C}), v, w \in V_\rho.$$

Let $\Gamma^n := Sp(n, \mathbb{Z})$ be the Siegel modular group of degree n , and \mathfrak{H}_n the Siegel upper half space of degree n . For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n$ and $Z \in \mathfrak{H}_n$, we put

$$M\langle Z \rangle := (AZ + B)(CZ + D)^{-1}, \quad j(M, Z) := \det(CZ + D),$$

and for $f : \mathfrak{H}_n \rightarrow V_\rho$,

$$(f|_\rho M)(Z) := \rho((CZ + D)^{-1})f(M\langle Z \rangle).$$

We write $|_k$ for $\rho = \det^k$ and we omit subscripts ρ, k when there is no fear of confusion.

A C^∞ -function $f : \mathfrak{H}_n \rightarrow V_\rho$ is called a V_ρ -valued C^∞ -modular form of type ρ if it satisfies $f|_\rho M = f$ for all $M \in \Gamma^n$. The space of all such functions is denoted by $M^n(V_\rho)^\infty$. The space of V_ρ -valued Siegel modular forms of type ρ is defined by

$$M^n(V_\rho) := \{f \in M^n(V_\rho)^\infty \mid f \text{ is holomorphic on } \mathfrak{H}_n \text{ (and its cusps)}\},$$

and the space of cuspforms by

$$S^n(V_\rho) := \left\{ f \in M^n(V_\rho) \mid \lim_{\lambda \rightarrow \infty} f \left(\begin{pmatrix} Z & 0 \\ 0 & i\lambda \end{pmatrix} \right) = 0 \text{ for all } Z \in \mathfrak{H}_{n-1} \right\}.$$

If $\rho = \det^k$, we write $M_k^{n\infty}$, M_k^n , and S_k^n for $M^n(V_\rho)^\infty$, $M^n(V_\rho)$, and $S^n(V_\rho)$, respectively.

For $f, g \in M^n(V_\rho)^\infty$, the Petersson inner product of f and g is defined by

$$(f, g) := \int_{\Gamma^n \backslash \mathfrak{H}_n} \left\langle \rho(\sqrt{\text{Im}(Z)}) f(Z), \rho(\sqrt{\text{Im}(Z)}) g(Z) \right\rangle \det(\text{Im}(Z))^{-n-1} dZ$$

if the right-hand side is convergent.

If (ρ, V_ρ) is an irreducible rational representation, ρ is equivalent to an irreducible rational representation $(\tilde{\rho}, V_{\tilde{\rho}})$ satisfying the following condition: There exists a unique one-dimensional vector subspace $\mathbb{C}\tilde{v}$ of $V_{\tilde{\rho}}$ such that for any upper triangular matrix of $GL(n, \mathbb{C})$,

$$\tilde{\rho} \left(\begin{pmatrix} g_{11} & & * \\ & \ddots & \\ 0 & & g_{nn} \end{pmatrix} \right) \tilde{v} = \left(\prod_{j=1}^n g_{jj}^{\lambda_j} \right) \tilde{v},$$

where $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then we call $(\lambda_1, \lambda_2, \dots, \lambda_n)$ the signature of ρ .

Now, we put

$$G^+ Sp(n, \mathbb{Q}) := \left\{ M \in GL(2n, \mathbb{Q}) \mid {}^t M \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} M = \mu(M) \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \mu(M) > 0 \right\}.$$

For $g \in G^+ Sp(n, \mathbb{Q})$, let $\Gamma^n g \Gamma^n = \bigcup_{j=1}^r \Gamma^n g_j$ be a decomposition of the double coset $\Gamma^n g \Gamma^n$ into left cosets. For $f \in M^n(V_\rho)$ (resp. $S^n(V_\rho)$, $M^n(V_\rho)^\infty$), we define the Hecke operator $(\Gamma^n g \Gamma^n)$ by

$$f|(\Gamma^n g \Gamma^n) := \sum_{j=1}^r f|g_j.$$

Let $f \in S^n(V_\rho)$ be an eigenform. We define the standard L -function attached to f by (1.1). We also define the following series:

$$(2.1) \quad D(s, f) := \sum_{T \in \mathbb{T}^{(n)}} \lambda(f, T) \det(T)^{-s},$$

where

$$\mathbb{T}^{(n)} := \left\{ \begin{pmatrix} t_1 & & & \mathbf{0} \\ & t_2 & & \\ & & \ddots & \\ \mathbf{0} & & & t_n \end{pmatrix} \mid t_j \in \mathbb{Z}_{>0} (1 \leq j \leq n), \quad t_1 \mid \dots \mid t_n \right\}.$$

and $\lambda(f, T)$ is the eigenvalue on f of the Hecke operator $\left(\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n \right)$, $T \in \mathbb{T}^{(n)}$. By Böcherer [3], we have:

$$(2.2) \quad \zeta(s) \prod_{j=1}^n \zeta(2s - 2j) D(s, f) = L(s - n, f, \text{St}).$$

For $k \in 2\mathbb{Z}_{>0}$, $s \in \mathbb{C}$ and $Z \in \mathfrak{H}_n$, we define the Eisenstein series by

$$E_k^n(Z, s) := \det(\text{Im}(Z))^s \sum_{M \in P_{n,0} \backslash \Gamma^n} j(M, Z)^{-k} |j(M, Z)|^{-2s},$$

where

$$P_{n,r} := \left\{ \begin{pmatrix} * & * \\ \mathbf{0}^{(n+r, n-r)} & * \end{pmatrix} \in \Gamma^n \right\}.$$

Then $E_k^n(Z, s)$ converges absolutely and locally uniformly for $k + 2 \text{Re}(s) > n + 1$, and $E_k^n(Z, s) \in M_k^{n\infty}$. Moreover, we have the following:

Theorem 2 (Langlands [11], Kalinin [8] and Mizumoto [12, 13]). *Let $n \in \mathbb{Z}_{>0}$, $k \in 2\mathbb{Z}_{>0}$. For $Z \in \mathfrak{H}_n$, we put*

$$\mathbb{E}_k^n(Z, s) := \frac{\Gamma_n(s + k/2)}{\Gamma_n(s)} \xi(2s) \prod_{j=1}^{\lfloor n/2 \rfloor} \xi(4s - 2j) E_k^n \left(Z, s - \frac{k}{2} \right),$$

where

$$\Gamma_n(s) := \prod_{j=1}^n \Gamma \left(s - \frac{j-1}{2} \right), \quad \xi(s) := \Gamma_{\mathbb{R}}(s) \zeta(s).$$

Then $\mathbb{E}_k^n(Z, s)$ is invariant under $s \rightarrow (n + 1)/2 - s$ and it is an entire function in s .

It is also known that every partial derivative (in the entries of Z) of the Eisenstein series $E_k^n(Z, s)$ is slowly increasing (locally uniformly in s).

Theorem 3 (Mizumoto [13]). *Let $n \in \mathbb{Z}_{>0}$, $k \in 2\mathbb{Z}_{>0}$.*

(i) *For each $s_0 \in \mathbb{C}$, there exist constants $\delta > 0$ and $d \in \mathbb{Z}_{\geq 0}$, depending only on n , k and s_0 , such that*

$$(s - s_0)^d E_k^n(Z, s)$$

is holomorphic in s for $|s - s_0| < \delta$, and is C^∞ in $(\operatorname{Re}(Z), \operatorname{Im}(Z))$.

(ii) *Furthermore, for given $\varepsilon > 0$ and $N \in \mathbb{Z}_{\geq 0}$, there exist constants $\alpha > 0$ and $\beta > 0$ depending only on n , k , d , s_0 , ε , δ and N such that*

$$\left| (s - s_0)^d \frac{\partial^N}{\partial z_{\mu_1 \nu_1} \cdots \partial z_{\mu_N \nu_N}} E_k^n(Z, s) \right| \leq \alpha \det(\operatorname{Im}(Z))^\beta$$

for $\operatorname{Im}(Z) \geq \varepsilon 1_n$, $|s - s_0| < \delta$, and $1 \leq \mu_j, \nu_j \leq n$.

The assertion above for the case $N = 0$ has been proved by Langlands [11] and Kalinin [8].

3. Differential operator and the pullback formula

Let V be a finite-dimensional vector space. For a finite subset I of $\mathbb{Z}_{>0}$, we define V^I by

$$V^I := \underbrace{V \otimes \cdots \otimes V}_{\#I}.$$

Moreover for disjoint finite subsets I, J of $\mathbb{Z}_{>0}$, we identify $V^{I \cup J}$ with $V^I \otimes V^J$ by the following:

$$V^I \otimes V^J \ni (v_{i_1} \otimes \cdots \otimes v_{i_r}) \otimes (v_{j_1} \otimes \cdots \otimes v_{j_s}) \mapsto v_{k_1} \otimes \cdots \otimes v_{k_{r+s}} \in V^{I \cup J},$$

where $I = \{i_1, \dots, i_r\}$ with $i_1 < \cdots < i_r$, $J = \{j_1, \dots, j_s\}$ with $j_1 < \cdots < j_s$, and $I \cup J = \{k_1, \dots, k_{r+s}\}$ with $k_1 < \cdots < k_{r+s}$.

For $\alpha \in \mathbb{Z}_{>0}$, we define the isomorphism $(\cdot)^\alpha : V \rightarrow V^{\{\alpha\}}$ by $(v)^\alpha := v$. We omit the tensor product \otimes when there is no fear of confusion.

Now, we put

$$\begin{aligned} V_1 &:= \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n, & e_1 &:= (e_1, \dots, e_n), \\ V_2 &:= \mathbb{C}e_{n+1} \oplus \cdots \oplus \mathbb{C}e_{2n}, & e_2 &:= (e_{n+1}, \dots, e_{2n}). \end{aligned}$$

Let $\operatorname{alt}^l(V_1)$ (resp. $\operatorname{alt}^l(V_2)$) be the l -th alternating tensor product of V_1 (resp. V_2), i.e.,

$$\operatorname{alt}^l(V_j) := \operatorname{span} \left\{ \sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma) (e_{j^t a_1})^{\sigma(1)} \cdots (e_{j^t a_l})^{\sigma(l)} \mid a_1, \dots, a_l \in \mathbb{C}^n \right\} \quad (j = 1, 2),$$

where \mathfrak{S}_l is the l -th symmetric group. For each $g \in GL(n, \mathbb{C})$, $\rho_j(g) := \det^k \otimes \text{alt}^l(g)$ acts on $\text{alt}^l(V_j)$ ($j = 1, 2$) by

$$\begin{aligned} \rho_j(g) & \sum_{\sigma \in \mathfrak{S}_l} \text{sgn}(\sigma) (e_j^t a_1)^{\sigma(1)} \cdots (e_j^t a_l)^{\sigma(l)} \\ & := \det(g)^k \sum_{\sigma \in \mathfrak{S}_l} \text{sgn}(\sigma) (e_j g^t a_1)^{\sigma(1)} \cdots (e_j g^t a_l)^{\sigma(l)}. \end{aligned}$$

Let ι be the isomorphism from V_1 to V_2 defined by $\iota(e_j) = e_{n+j}$ ($1 \leq j \leq n$). It induces the isomorphism (also denoted by ι) from $\text{alt}^l(V_1)$ to $\text{alt}^l(V_2)$. For an $\text{alt}^l(V_1)$ -valued function f on \mathfrak{H}_n and for $Z \in \mathfrak{H}_n$, we define $\iota(f)$ by

$$(\iota(f))(Z) := \iota(f(Z)).$$

For a symmetric matrix A of size $2n$ and $\alpha, \beta \in \mathbb{Z}_{>0}$, we define

$$\begin{aligned} A^{\alpha\beta} & := ((e_1)^\alpha, \dots, (e_n)^\alpha, 0, \dots, 0) A^t ((e_1)^\beta, \dots, (e_n)^\beta, 0, \dots, 0), \\ A_\beta^\alpha & := ((e_1)^\alpha, \dots, (e_n)^\alpha, 0, \dots, 0) A^t (0, \dots, 0, (e_{n+1})^\beta, \dots, (e_{2n})^\beta), \\ A_{\alpha\beta} & := (0, \dots, 0, (e_{n+1})^\alpha, \dots, (e_{2n})^\alpha) A^t (0, \dots, 0, (e_{n+1})^\beta, \dots, (e_{2n})^\beta). \end{aligned}$$

Let $\mathfrak{Z} = (z_{\mu\nu})$ be a variable on \mathfrak{H}_{2n} . We put

$$\frac{\partial}{\partial \mathfrak{Z}} := \left(\frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial z_{\mu\nu}} \right)_{1 \leq \mu, \nu \leq 2n},$$

where $\delta_{\mu\nu}$ is the Kronecker's delta, and for C^∞ -functions, we define the differential operator \mathcal{D} by

$$\mathcal{D} := \sum_{\sigma \in \mathfrak{S}_l} \text{sgn}(\sigma) \left(\frac{\partial}{\partial \mathfrak{Z}} \right)_{\sigma(1)}^1 \cdots \left(\frac{\partial}{\partial \mathfrak{Z}} \right)_{\sigma(l)}^l.$$

Then we have:

Proposition 1. *Let $n, k \in \mathbb{Z}_{>0}$ and $2k \geq n$.*

(i) *Let F be any \mathbb{C} -valued C^∞ -function on \mathfrak{H}_{2n} . Then for each $g_1, g_2 \in \Gamma^n$ and $\mathfrak{Z}_0 = \begin{pmatrix} Z^{(n)} & 0 \\ 0 & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, we get the following commutation relation:*

$$((\mathcal{D}F)|_{\rho_1}(g_1)_Z |_{\rho_2}(g_2)_W)(\mathfrak{Z}_0) = (\mathcal{D}(F|_k(g_1^\uparrow g_2^\downarrow)))(\mathfrak{Z}_0),$$

where $(\)_Z$ (resp. $(\)_W$) denotes the action on Z (resp. W) and for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in$

Γ^n , we put

$$M^\dagger := \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1_n & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1_n \end{pmatrix}, \quad M^\downarrow := \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1_n & 0 \\ 0 & C & 0 & D \end{pmatrix}.$$

(ii) The operator \mathcal{D} sends modular forms to modular forms:

$$\mathcal{D} : M_k^{2n\infty} \longrightarrow M^n(\text{alt}^l(V_1))^\infty \otimes M^n(\text{alt}^l(V_2))^\infty.$$

Moreover, \mathcal{D} is a holomorphic operator and it satisfies

$$\mathcal{D} : M_k^{2n} \longrightarrow M^n(\text{alt}^l(V_1)) \otimes M^n(\text{alt}^l(V_2)).$$

Proof. Let $X_j = (x_{\mu\nu}^{(j)})$ ($j = 1, 2$) be variables on $\mathbb{C}^{(n,2k)}$. We put $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, and

$$Q(X^t X) := \sum_{\sigma \in \mathfrak{S}_l} \text{sgn}(\sigma) (X^t X)_{\sigma(1)}^1 \cdots (X^t X)_{\sigma(l)}^l.$$

Then the polynomial Q is pluri-harmonic for X_1, X_2 , i.e., for each $1 \leq \mu, \nu \leq n$,

$$\sum_{\kappa=1}^{2k} \frac{\partial}{\partial x_{\mu\kappa}^{(j)}} \frac{\partial}{\partial x_{\nu\kappa}^{(j)}} Q = 0 \quad (j = 1, 2).$$

Therefore, by Ibukiyama [7] (see also [16]), we get Proposition 1. □

Now we prove Theorem 1 according to Böcherer’s method in [2]. For this, we prove the following:

Proposition 2. Let $n \in \mathbb{Z}_{>0}$, $k \in 2\mathbb{Z}_{>0}$, $s \in \mathbb{C}$ and $k + 2\text{Re}(s) > 2n + 1$. Suppose that $2k \geq n > 2$. For $\mathfrak{Z}_0 = \begin{pmatrix} Z^{(n)} & 0 \\ 0 & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, we get

$$(\mathcal{DE}_k^{2n})(\mathfrak{Z}_0, s) = \prod_{j=1}^l \left(-k - s + \frac{j-1}{2} \right) \sum_{r=0}^n \sum_{T \in \mathbb{T}^{(r)}} \mathcal{P}_r(Z, W, T, s),$$

where

$$\begin{aligned} \mathcal{P}_r(Z, W, T, s) := & \sum_{g_2 \in P_{n,r} \backslash \Gamma^n} \sum_{g'_2 \in P_{n,r} \backslash \Gamma^n} \sum_{g_1 \in G_{n,r}} \sum_{g'_1 \in \Gamma^r(T) \backslash G_{n,r}} \\ & \cdot \left\{ \det(\text{Im}(Z))^s \det(\text{Im}(W))^s |\det(1_n - \tilde{T} W \tilde{T} Z)|^{-2s} \right\} \end{aligned}$$

$$\begin{aligned} & \cdot \rho_1((1_n - \tilde{T}W\tilde{T}Z)^{-1}) \sum_{\sigma \in \mathfrak{S}_l} \text{sgn}(\sigma) \left(\begin{smallmatrix} * & \tilde{T} \\ \tilde{T} & * \end{smallmatrix} \right)_{\sigma(1)}^1 \cdots \left(\begin{smallmatrix} * & \tilde{T} \\ \tilde{T} & * \end{smallmatrix} \right)_{\sigma(l)}^l \} \\ & |(\tilde{g}'_1)_W|(\tilde{g}_1)_Z|(g'_2)_W|(g_2)_Z, \\ G_{n,r} & := \left\{ \begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & A^{(r)} & 0 & B^{(r)} \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & C^{(r)} & 0 & D^{(r)} \end{pmatrix} \in \Gamma^n \right\}, \end{aligned}$$

and for $T \in \mathbb{T}^{(r)}$,

$$\Gamma^r(T) := \left\{ g \in \Gamma^r \mid \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} g \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} \in \Gamma^r \right\} \quad \text{and} \quad \tilde{T}^{(n)} = \begin{pmatrix} 0 & 0 \\ 0 & T^{(r)} \end{pmatrix}.$$

Proof. By Garrett [5], the left coset $P_{2n,0} \backslash \Gamma^{2n}$ has a complete system of representatives $g_{\tilde{T}} \tilde{g}_1^{\uparrow} g_2^{\uparrow} \tilde{g}'_1{}^{\downarrow} g'_2{}^{\downarrow}$ with

$$\begin{aligned} g_{\tilde{T}} &= \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & \tilde{T} & 1_n & 0 \\ \tilde{T} & 0 & 0 & 1_n \end{pmatrix}, \quad T \in \mathbb{T}^{(r)} \quad (0 \leq r \leq n), \\ \tilde{g}_1 &\in G_{n,r}, \quad g_2 \in P_{n,r} \backslash \Gamma^n, \quad \tilde{g}'_1 \in \Gamma^r(T) \backslash G_{n,r}, \quad g'_2 \in P_{n,r} \backslash \Gamma^n. \end{aligned}$$

Therefore, it follows from Proposition 1 that

$$\begin{aligned} (\mathcal{D}E_k^{2n})(\mathfrak{Z}_0, s) &= \sum_{r=0}^n \sum_{T \in \mathbb{T}^{(r)}} \sum_{g_2 \in P_{n,r} \backslash \Gamma^n} \sum_{g'_2 \in P_{n,r} \backslash \Gamma^n} \sum_{\tilde{g}_1 \in G_{n,r}} \sum_{\tilde{g}'_1 \in \Gamma^r(T) \backslash G_{n,r}} \\ & \{ \mathcal{D}(\det(\text{Im}(\mathfrak{Z}))^s |_{k g_{\tilde{T}}})|_{\mathfrak{Z}=\mathfrak{Z}_0} \} |(\tilde{g}'_1)_W|(\tilde{g}_1)_Z|(g'_2)_W|(g_2)_Z. \end{aligned}$$

If for each \tilde{T} we put $g_{\tilde{T}} = \begin{pmatrix} * & * \\ \mathfrak{C}^{(2n)} & \mathfrak{D}^{(2n)} \end{pmatrix}$, we get

$$\mathcal{D}(\det(\text{Im}(\mathfrak{Z}))^s |_{k g_{\tilde{T}}})|_{\mathfrak{Z}=\mathfrak{Z}_0} = \det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-s} \mathcal{D}(\det(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-k-s} \det(\text{Im}(\mathfrak{Z}))^s)|_{\mathfrak{Z}=\mathfrak{Z}_0},$$

by the form of \mathcal{D} and $((1/2i)(\text{Im}(\mathfrak{Z}_0))^{-1})_{\sigma(j)}^j = 0$,

$$= \det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-s} \det(\text{Im}(\mathfrak{Z}_0))^s \mathcal{D}(\det(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-k-s})|_{\mathfrak{Z}=\mathfrak{Z}_0}.$$

To compute $\mathcal{D}(\det(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-k-s})$, we prove the following lemma.

Lemma 1. We put $\delta := \det(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})$ and $\Delta := (\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}\mathfrak{C}$. For $\lambda \in \mathbb{C}$,

$$\sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma) \left(\frac{\partial}{\partial \bar{\mathfrak{Z}}}\right)_{\sigma(1)}^1 \cdots \left(\frac{\partial}{\partial \bar{\mathfrak{Z}}}\right)_{\sigma(l)}^l \delta^\lambda = \delta^\lambda \prod_{j=1}^l \left(\lambda + \frac{j-1}{2}\right) \sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma) \Delta_{\sigma(1)}^1 \cdots \Delta_{\sigma(l)}^l.$$

Proof of Lemma 1. We use induction on l . Since

$$\frac{\partial}{\partial \bar{\mathfrak{Z}}} \delta^\lambda = \delta^\lambda \lambda \Delta,$$

for $l = 1$, the lemma holds. Let $l > 1$.

$$\begin{aligned} (3.1) \quad & \sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma) \left(\frac{\partial}{\partial \bar{\mathfrak{Z}}}\right)_{\sigma(1)}^1 \cdots \left(\frac{\partial}{\partial \bar{\mathfrak{Z}}}\right)_{\sigma(l)}^l \delta^\lambda \\ &= \sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma) \left(\frac{\partial}{\partial \bar{\mathfrak{Z}}}\right)_{\sigma(1)}^1 \left\{ \delta^\lambda \prod_{j=1}^{l-1} \left(\lambda + \frac{j-1}{2}\right) \Delta_{\sigma(2)}^2 \cdots \Delta_{\sigma(l)}^l \right\} \\ &= \delta^\lambda \prod_{j=1}^{l-1} \left(\lambda + \frac{j-1}{2}\right) \left\{ \lambda \sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma) \Delta_{\sigma(1)}^1 \cdots \Delta_{\sigma(l)}^l \right. \\ &\quad \left. - \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma) \sum_{\kappa=2}^l \Delta_{\sigma(2)}^2 \cdots \widehat{\Delta_{\sigma(\kappa)}^\kappa} \cdots \Delta_{\sigma(l)}^l (\Delta^{1\kappa} \Delta_{\sigma(1)\sigma(\kappa)} + \Delta_{\sigma(\kappa)}^1 \Delta_{\sigma(1)}^\kappa) \right\}. \end{aligned}$$

We note

$$(3.2) \quad \sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma) \Delta_{\sigma(2)}^2 \cdots \widehat{\Delta_{\sigma(\kappa)}^\kappa} \cdots \Delta_{\sigma(l)}^l \Delta^{1\kappa} \Delta_{\sigma(1)\sigma(\kappa)} = 0$$

and

$$(3.3) \quad \sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma) \Delta_{\sigma(2)}^2 \cdots \widehat{\Delta_{\sigma(\kappa)}^\kappa} \cdots \Delta_{\sigma(l)}^l \Delta_{\sigma(\kappa)}^1 \Delta_{\sigma(1)}^\kappa = - \sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma) \Delta_{\sigma(1)}^1 \cdots \Delta_{\sigma(l)}^l.$$

Thus by (3.1), (3.2) and (3.3), the lemma holds. □

Using Lemma 1, we obtain

$$\begin{aligned} \mathcal{D}(\det(\operatorname{Im}(\mathfrak{Z}))^s |{}_k g_{\tilde{T}})|_{\mathfrak{Z}=\mathfrak{Z}_0}) &= \prod_{j=1}^l \left(-k - s + \frac{j-1}{2}\right) \det(\operatorname{Im}(\mathfrak{Z}_0))^s |\det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})|^{-2s} \\ &\quad \cdot \det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-k} \sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma) ((\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}\mathfrak{C})_{\sigma(1)}^1 \cdots ((\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}\mathfrak{C})_{\sigma(l)}^l. \end{aligned}$$

Since $\det(\operatorname{Im}(\mathfrak{Z}_0)) = \det(\operatorname{Im}(Z)) \det(\operatorname{Im}(W))$, $\det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D}) = \det(1_n - \tilde{T}W\tilde{T}Z)$, and $(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}\mathfrak{C} = \begin{pmatrix} * & (1_n - \tilde{T}W\tilde{T}Z)^{-1}\tilde{T} \\ * & * \end{pmatrix}$, Proposition 2 is proved. □

4. Proof of Theorem 1

Theorem 4. *Let $n \in \mathbb{Z}_{>0}$, $k, l \in 2\mathbb{Z}_{>0}$, and $2k \geq n > 2$. If $f \in S^n(\text{alt}^l(V_2))$ is an eigenform,*

$$\begin{aligned} & \left(f, (\mathcal{D}\mathbb{E}_k^{2n}) \left(\begin{pmatrix} -\bar{Z}^{(n)} & 0 \\ 0 & * \end{pmatrix}, \frac{\bar{s} + n}{2} \right) \right) \\ &= 2^{1-l} \pi^{(-n^2+\varepsilon)/2} (\pi i)^{nk+l} \gamma(s) \Lambda(s, f, \text{St})(\iota^{-1}(f))(Z), \end{aligned}$$

where

$$\gamma(s) := \frac{\Gamma_n((s+n)/2)}{\Gamma_{n-1}((s-1)/2)\Gamma((s+\varepsilon)/2)} = \gamma(1-s).$$

Proof of Theorem 4. It follows from Theorem 3 that

$$\left(f, (\mathcal{D}E_k^{2n}) \left(\begin{pmatrix} -\bar{Z}^{(n)} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right)$$

converges absolutely and locally uniformly for $k + 2 \text{Re}(s) > 2n + 1$. We consider that the Petersson inner product $(f, \mathcal{P}_r(-\bar{Z}, *, T, \bar{s}))$. For $r < n$, by the same reason as that Klingens [10, Satz 2],

$$(f, \mathcal{P}_r(-\bar{Z}, *, T, \bar{s})) = 0.$$

Therefore we only consider that $(f, \mathcal{P}_n(-\bar{Z}, *, T, \bar{s}))$.

Now, we have

$$\mathcal{P}_n(Z, W, T, s) = \det(T)^{-k-2s} \cdot \mathcal{P}(Z, W, s) \Big| \left(\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n \right)_W,$$

where

$$\begin{aligned} \mathcal{P}(Z, W, s) = & \sum_{\tilde{g}_1 \in \Gamma^n} \left\{ \det(\text{Im}(Z))^s \det(\text{Im}(W))^s |\det(Z+W)|^{-2s} \right. \\ & \left. \cdot \rho_1((Z+W)^{-1}) \sum_{\sigma \in \mathfrak{S}_l} \text{sgn}(\sigma) \begin{pmatrix} * & 1_n \\ 1_n & * \end{pmatrix}_{\sigma(1)}^1 \cdots \begin{pmatrix} * & 1_n \\ 1_n & * \end{pmatrix}_{\sigma(l)}^l \right\} \Big|_{(\tilde{g}_1)_Z}. \end{aligned}$$

Since the Hecke operators are Hermitian operators and f is an eigenform, we have

$$(f, \mathcal{P}_n(-\bar{Z}, *, T, \bar{s})) = \lambda(f, T) \det(T)^{-k-2s} (f, \mathcal{P}(-\bar{Z}, *, \bar{s})).$$

If we compute the integral $(f, \mathcal{P}(-\bar{Z}, *, \bar{s}))$ according to Klingens [9, §1] (see also [2],

[4], [15]), we obtain

$$(f, \mathcal{P}(-\bar{Z}, *, \bar{s})) = 2^{n(n+1-2s)+1} (2^{-1}i)^{nk+l} c(s-n-1, \rho_1)(\iota^{-1}(f))(Z)$$

and

$$c(s-n-1, \rho_1) \text{id} = \int_{S^n} \det(1_n - S\bar{S})^{s-n-1} \rho_1(1_n - S\bar{S}) dS,$$

where $S^n := \{S \in \mathbb{C}^{(n)} \mid S = {}^t S, 1_n - S\bar{S} > 0\}$. Thus, by Proposition 2, (2.1) and (2.2),

$$\begin{aligned} (4.1) \quad & \left(f, (\mathcal{D}E_k^{2n}) \left(\begin{pmatrix} -\bar{Z}^{(n)} & 0 \\ 0 & * \end{pmatrix}, \frac{\bar{s} + n - k}{2} \right) \right) \\ &= 2^{n(1-s+k)+1} (2^{-1}i)^{nk+l} c \left(\frac{s+n-k}{2} - n - 1, \rho_1 \right) (-2^{-1})^l \frac{\Gamma(s+n+k+1)}{\Gamma(s+n+k+1-l)} \\ & \cdot \zeta(s+n)^{-1} \prod_{j=1}^n \zeta(2s+2n-2j)^{-1} L(s, f, \underline{S}t)(\iota^{-1}(f))(Z). \end{aligned}$$

To compute $c((s+n-k)/2 - n - 1, \rho_1)$, we prove the following lemma.

Lemma 2. *Let (ρ, V_ρ) be an irreducible rational representation of $GL(n, \mathbb{C})$ whose signature is $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. For $s \in \mathbb{C}$ such that $\text{Re}(s) > -\lambda_n - 1$, we put*

$$\psi(s, \rho) := \int_{S^n} \det(1_n - S\bar{S})^s \rho(1_n - S\bar{S}) dS.$$

Then there exists a constant $c(s, \rho)$ satisfying $\psi(s, \rho) = c(s, \rho) \text{id}$, and

$$(4.2) \quad c(s, \rho) = \frac{2^n \pi^{n(n+1)/2}}{\prod_{1 \leq \mu \leq \nu \leq n} (\lambda_\mu + \lambda_\nu + 2s + 2n + 2 - \mu - \nu)}.$$

Proof of Lemma 2. The lemma is proved in the same way as that by Hua [6, §2.3] (see also [2], [4], [9], [15]).

For any unitary matrix $U \in U(n)$, we have $\psi(s, \rho) = \rho(U^{-1})\psi(s, \rho)\rho(U)$. Since ρ is an irreducible representation of $U(n)$, $\psi(s, \rho)$ is a homothety by Schur's lemma, i.e., there exists a constant $c(s, \rho)$ satisfying $\psi(s, \rho) = c(s, \rho) \text{id}$.

We compute $c(s, \rho)$. Let $\mathbf{v} \in V_\rho$ be the highest weight vector with $\langle \mathbf{v}, \mathbf{v} \rangle = 1$. Then,

$$\begin{aligned} c(s, \rho) &= \langle \psi(s, \rho) \mathbf{v}, \mathbf{v} \rangle \\ &= \int_{S^n} \det(1_n - S\bar{S})^s \langle \rho(1_n - S\bar{S}) \mathbf{v}, \mathbf{v} \rangle dS. \end{aligned}$$

Let ρ_0 be an irreducible representation of $GL(n, \mathbb{C})$ such that $\rho = \det^{\lambda_n} \otimes \rho_0$ and the signature of ρ_0 is $(\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n, 0)$. Then,

$$c(s, \rho) = \int_{S^n} \det(1_n - S\bar{S})^{s+\lambda_n} \langle \rho_0(1_n - S\bar{S})\mathbf{v}, \mathbf{v} \rangle dS.$$

We set $S = \begin{pmatrix} S_1^{(n-1)} & {}^t v \\ v & z \end{pmatrix}$ and $\rho'_0(g^{(n-1)}) = \rho_0\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right)$, by Hua [6, Theorem 2.3.2],

$$\begin{aligned} c(s, \rho) &= \frac{\pi}{s + \lambda_n + 1} \int_{1_{n-1} - S_1 \bar{S}_1 - {}^t v \bar{v} > 0} \frac{\det(1_{n-1} - S_1 \bar{S}_1 - {}^t v \bar{v})^{s+\lambda_n}}{(1 + \bar{v}(1_{n-1} - S_1 \bar{S}_1 - {}^t v \bar{v})^{-1} {}^t v)^{s+\lambda_n+2}} \\ &\quad \cdot \langle \rho'_0(1_{n-1} - S_1 \bar{S}_1 - {}^t v \bar{v})\mathbf{v}, \mathbf{v} \rangle dS_1 dv \\ &= \frac{\pi}{s + \lambda_n + 1} \int_{1_{n-1} - S_1 \bar{S}_1 > 0} \det(1_{n-1} - S_1 \bar{S}_1)^{s+\lambda_n+1} \\ &\quad \cdot \int_{1 - \bar{u}_0 {}^t u_0 > 0} (1 - \bar{u}_0 {}^t u_0)^{2(s+\lambda_n+1)} \langle \rho'_0(\Gamma(1_{n-1} - {}^t u_0 \bar{u}_0) {}^t \bar{\Gamma})\mathbf{v}, \mathbf{v} \rangle du_0 dS_1, \end{aligned}$$

where $u_0 = (u_1, \dots, u_{n-1})$ and $\Gamma {}^t \bar{\Gamma} = 1_{n-1} - S_1 \bar{S}_1$. We put

$$\varphi(s, \rho) := \int_{1 - \bar{u} {}^t u > 0} (1 - \bar{u} {}^t u)^s \rho(1_n - {}^t u \bar{u}) du, \quad u = (u_1, \dots, u_n).$$

Using Schur's lemma again, there exists a constant $d(s, \rho)$ satisfying $\varphi(s, \rho) = d(s, \rho) \text{id}$. Therefore,

$$\begin{aligned} (4.3) \quad c(s, \rho) &= \frac{\pi}{s + \lambda_n + 1} \int_{1_{n-1} - S_1 \bar{S}_1 > 0} \det(1_{n-1} - S_1 \bar{S}_1)^{s+\lambda_n+1} \\ &\quad \cdot \varphi(2(s + \lambda_n + 1), \rho'_0) \langle \rho'_0(1_{n-1} - S_1 \bar{S}_1)\mathbf{v}, \mathbf{v} \rangle dS_1 \\ &= \frac{\pi}{s + \lambda_n + 1} c(s + \lambda_n + 1, \rho'_0) d(2(s + \lambda_n + 1), \rho'_0). \end{aligned}$$

We compute $d(s, \rho)$.

$$\begin{aligned} d(s, \rho) &= \int_{1 - \bar{u} {}^t u > 0} (1 - \bar{u} {}^t u)^{s+\lambda_n} \langle \rho_0(1_n - {}^t u \bar{u})\mathbf{v}, \mathbf{v} \rangle du \\ &= \int_{\substack{1 - \bar{u}_0 {}^t u_0 > 0 \\ |u_n| < 1 - \bar{u}_0 {}^t u_0}} ((1 - \bar{u} {}^t u)^{s+\lambda_n} du_n) \langle \rho'_0(1_{n-1} - {}^t u_0 \bar{u}_0)\mathbf{v}, \mathbf{v} \rangle du_0, \end{aligned}$$

where $u = (u_1, \dots, u_n)$ and $u_0 = (u_1, \dots, u_{n-1})$. Since

$$\begin{aligned} \int_{|u_n| < 1 - \bar{u}_0 {}^t u_0} (1 - \bar{u} {}^t u)^{s+\lambda_n} du_n &= \frac{\pi}{s + \lambda_n + 1} (1 - \bar{u}_0 {}^t u_0)^{s+\lambda_n+1}, \\ d(s, \rho) &= \frac{\pi}{s + \lambda_n + 1} \int_{1 - \bar{u}_0 {}^t u_0 > 0} (1 - \bar{u}_0 {}^t u_0)^{s+\lambda_n+1} \langle \rho'_0(1_{n-1} - {}^t u_0 \bar{u}_0)\mathbf{v}, \mathbf{v} \rangle du_0 \end{aligned}$$

$$= \frac{\pi}{s + \lambda_n + 1} d(s + \lambda_n + 1, \rho'_0).$$

Therefore

$$(4.4) \quad d(s, \rho) = \frac{\pi^n}{\prod_{j=1}^n (s + \lambda_{n-j+1} + j)}.$$

By (4.3) and (4.4), we get (4.2). \square

Since the signature of ρ_1 is $(\underbrace{k+1, \dots, k+1}_l, \underbrace{k, \dots, k}_{n-l})$, it follows from Lemma 2 that

$$\begin{aligned} c\left(\frac{s+n-k}{2} - n - 1, \rho_1\right) &= \frac{2^n \pi^{n(n+1)/2}}{\prod_{1 \leq \mu < \nu \leq n} (\lambda_\mu + \lambda_\nu + s + n - k - \mu - \nu)} \\ &= 2^n \pi^{n(n+1)/2} \frac{\Gamma(s+n+k+1-l)}{\Gamma(s+n+k+1-2l)} \\ &\quad \cdot \prod_{j=1}^l \frac{\Gamma(s+k+1-j)}{\Gamma(s+n+k+3-2j)} \prod_{j=l+1}^n \frac{\Gamma(s+k-j)}{\Gamma(s+n+k+1-2j)}. \end{aligned}$$

Then, by (4.1), we obtain Theorem 4. \square

It follows from Theorem 2 and Theorem 4 that the functional equation of $\Lambda(s, f, \text{St})$. Moreover using the same way that by Mizumoto [12] (see also [15]), the holomorphy is proved. Hence Theorem 1 holds.

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