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# A NEW MATRIX INVERSE

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ABSTRACT. We compute the inverse of a specific infinite-dimensional matrix, thus unifying a number of previous matrix inversions. Our inversion theorem is applied to derive a number of summation formulas of hypergeometric type.

## 1. INTRODUCTION

Let $F = (f_{nk})_{n,k\in\mathbb{Z}}$ ($\mathbb{Z}$ denotes the set of integers) be an infinite-dimensional lower-triangular matrix; i.e. $f_{nk} = 0$ unless $n \geq k$. The matrix $(f_{kl}^{-1})_{k,l\in\mathbb{Z}}$ is the inverse matrix of $F$ if and only if

$$\sum_{n\geq k\geq l} f_{nk} f_{kl}^{-1} = \delta_{nl}$$

for all $n, l \in \mathbb{Z}$. Such matrix inversions are very important in many fields of combinatorics and special functions. For example, when dealing with combinatorial sums, application of the so-called "inverse relations" (see (4.1) and (4.2)), which are based on matrix inversion, helps to simplify problems, or yields new identities. Riordan dedicated two chapters of his book [21] to inverse relations and its applications. Riordans inverse relations were classified and given a unified method of proof by Egorychev [7, Ch. 3]. Studying a specific class of inverse relations in a series of papers [12–15], Gould and Hsu [16] finally discovered a very general matrix inversion, which is equivalent to:

*If*

$$A_{nk} = \frac{\prod_{j=k}^{n-1}(a_j + kb_j)}{(n-k)!}, \tag{1.1)(1}$$

*then*

$$A_{kl}^{-1} = (-1)^{k-l}\frac{a_l + lb_l}{a_k + kb_k}\frac{\prod_{j=l+1}^{k}(a_j + kb_j)}{(k-l)!}\ . \tag{1.1)(2}$$

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(By convenience, products of the form $\prod_{j=u}^{u-1}$ are defined to be equal to 1, while for $u > v-1$ a product $\prod_{j=u}^{v-1}$ by definition is equal to 0.) The inverse pair (1.1) contains a lot of inverse relations of Gould–type (cf. [21, pp. 50–51]) and Abel–type (cf. [21, p. 99]). Carlitz [6] found a $q$-analogue of (1.1) which is equivalent to:

*If*

$$B_{nk} = q^{\binom{n-k}{2}} \frac{\prod_{j=k}^{n-1}(a_j + q^k b_j)}{(q;q)_{n-k}}, \tag{1.2)(1}$$

*then*

$$B_{kl}^{-1} = (-1)^{k-l} \frac{a_l + q^l b_l}{a_k + q^k b_k} \frac{\prod_{j=l+1}^{k}(a_j + q^k b_j)}{(q;q)_{k-l}}\ , \tag{1.2)(2}$$

*where* $(a;q)_m = (1-a)(1-aq)\cdots(1-aq^{m-1})$. Special cases of (1.2) are of great significance in $q$-series theory. Andrews [1] showed that the Bailey transform [2, 3], used to prove identities of Rogers–Ramanujan type, is equivalent to the case $a_j = 1$ and $b_j = -bq^j$ of (1.2). Gessel and Stanton [11] used several specializations of the $a_j = 1$, $b_j = -bp^j$ case of (1.2) to derive a number of basic hypergeometric summations and transformations, and, once again, identities of Rogers–Ramanujan type.

In [4] Bressoud considers finite forms of Rogers–Ramanujan identities. The transform which he uses to prove them is equivalent to the matrix inversion [5]:

*If*

$$C_{nk} = \frac{(1-aq^{2k})(b;q)_{n+k}\,(ba^{-1};q)_{n-k}(ba^{-1})^k}{(1-a)(aq;q)_{n+k}\,(q;q)_{n-k}}, \tag{1.3)(1}$$

*then*

$$C_{kl}^{-1} = \frac{(1-bq^{2l})(a;q)_{k+l}\,(ab^{-1};q)_{k-l}\,(ab^{-1})^l}{(1-b)(bq;q)_{k+l}\,(q;q)_{k-l}}\ . \tag{1.3)(2}$$

The purpose of this paper is to give the following generalization of (1.1)–(1.3), which will be proved in section 2:

**Theorem.** *Let $(a_i)_{i\in\mathbb{Z}}$, $(b_i)_{i\in\mathbb{Z}}$ and $(c_i)_{i\in\mathbb{Z}}$ be arbitrary sequences such that $c_i \neq c_j$ if $i \neq j$.*

*If*

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j + c_k b_j)}{\prod_{j=k+1}^{n}(c_j - c_k)}, \tag{1.4)(1}$$

*then*

$$f_{kl}^{-1} = \frac{a_l + c_l b_l}{a_k + c_k b_k} \frac{\prod_{j=l+1}^{k}(a_j + c_k b_j)}{\prod_{j=l}^{k-1}(c_j - c_k)}\ . \tag{1.4)(2}$$

In fact, (1.1) is the special case $c_k = k$, (1.2) is equivalent to the case $c_k = q^k$, and (1.3) is equivalent to the case $c_k = q^{-k} + aq^k$, $a_j = aq^{-j-1} + b^2q^{j-1}$, $b_j = -b/q$. Moreover, replacing $c_k$ by $(1+ac_k^2)/c_k$, $a_j$ by $(a+b_j^2)/c_{j+1}$ and $b_j$ by $-b_j/c_{j+1}$ in (1.4), we get after some simplification the following

**Corollary.** *With the assumption of the Theorem, $a \neq (c_j c_k)^{-1}$ for all $j, k \in \mathbb{Z}$, if*

$$g_{nk} = \frac{\prod_{j=k}^{n-1}(1 - c_k b_j)\prod_{j=k}^{n-1}(b_j - ac_k)}{\prod_{j=k+1}^{n}(1 - ac_k c_j)\prod_{j=k+1}^{n}(c_j - c_k)}, \tag{1.5)(1}$$

*then*

$$g_{kl}^{-1} = \frac{(1 - c_l b_l)(b_l - ac_l)}{(1 - c_k b_k)(b_k - ac_k)} \frac{\prod_{j=l+1}^{k}(1 - c_k b_j)\prod_{j=l+1}^{k}(b_j - ac_k)}{\prod_{j=l}^{k-1}(1 - ac_k c_j)\prod_{j=l}^{k-1}(c_j - c_k)}\ . \tag{1.5)(2}$$

For $c_k = q^k$ the cases $b_j = bq^j$ and $b_j = bq^{-j}$ are equivalent to (1.3). In this sense the form (1.5) of (1.4) is the natural generalization of Bressoud's matrix inverse (1.3), such as (1.4) is the natural generalization of Gould's and Hsu's (1.1) and Carlitz's (1.2).

A special case of (1.5), namely (4.3), which involves rising $q$-factorials with two different bases, has been crucial in papers by Gasper and Rahman [8, 19]. They used inversion together with an indefinite bibasic sum to derive numerous beautiful bibasic, cubic, and quartic summation formulas for basic hypergeometric series (see section 3 for "hypergeometric" definitions). (They also extended this method to obtain bibasic, cubic, and quartic transformation formulas [9; 20; 10, sec. 3.6].) In section 4 we add two more applications of this inversion. We derive a summation formula (identity (4.8)) for series of hypergeometric type, and as a by-product of the second application we obtain a basic hypergeometric transformation formula (identity (4.12)). The former contains an infinite family of summation formulas for very well-poised hypergeometric series. Besides, we use the opportunity to clearly demonstrate that what Gasper and Rahman do in [8, 19] is indeed inversion, though in disguise. This fact does not seem to be as accepted as it should be. (Their extension in [9; 20; 10, sec. 3.6] has an explanation in terms of "partial inversion".)

Finally, in section 5 we apply other special cases of (1.4) to obtain curious identities ((5.5), (5.9), (5.12)) which involve cubic, quartic, and quintic analogues of rising factorials, respectively.

## 2. Proof of the Theorem

In [18] the author gave a method for solving Lagrange inversion problems. These are closely connected with the problem of inverting lower-triangular matrices. For convenience, by a formal Laurent series (fLs) we always mean a series of the form $\sum_{i \geq m} \alpha_i z^i$, for some $m \in \mathbb{Z}$. Given the fLs's $a(z)$ and $b(z)$ we introduce the bilinear form $\langle\, , \rangle$ by

$$\langle a(z), b(z)\rangle = \langle z^0\rangle\ a(z)\cdot b(z)\ ,$$

where $\langle z^0\rangle c(z)$ denotes the coefficient of $z^0$ in $c(z)$. Given any linear operator $L$ acting on a fLs, $L^*$ denotes the adjoint of $L$ with respect to $\langle\, , \rangle$; i.e. $\langle La(z), b(z)\rangle = \langle a(z), L^* b(z)\rangle$ for all fLs $a(z)$ and $b(z)$. What we need is the following special case of [18, Theorem 1].

**Lemma.** *Let $F = (f_{nk})_{n,k\in\mathbb{Z}}$ be an infinite lower-triangular matrix with $f_{kk} \neq 0$ for all $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, the formal Laurent series $f_k(z)$ and $\tilde{f}_k(z)$ are defined by*

*$f_k(z)=\sum_{n\ge k} f_{nk}z^n$ and $\tilde f_k(z)=\sum_{l\le k} f_{kl}^{-1}z^{-l}$, where $(f_{kl}^{-1})_{k,l\in\mathbb{Z}}$ is the uniquely determined inverse matrix of $F$. Suppose that for $k\in\mathbb{Z}$*

$$Uf_k(z)=c_k\cdot Vf_k(z) \tag{2.1}$$

*holds, where $U,V$ are linear operators acting on formal Laurent series, $U$ being bijective, and $(c_k)_{k\in\mathbb{Z}}$ is a sequence of constants. If $h_k(z)$ is a solution of*

$$U^*h_k(z)=c_k\cdot V^*h_k(z)\ , \tag{2.2}$$

*with $h_k(z)\not\equiv 0$ for all $k\in\mathbb{Z}$, then*

$$\tilde f_k(z)=\frac{1}{\langle f_k(z),V^*h_k(z)\rangle}V^*h_k(z)\ . \tag{2.3}$$

In order to prove (1.4), we set $f_k(z)=\sum_{n\ge k} f_{nk}z^n$ with $f_{nk}$ of (1.4)(1). Obviously, for $n\ge k$

$$(c_n-c_k)f_{nk}=(a_{n-1}+c_kb_{n-1})f_{n-1,k}\ . \tag{2.4}$$

If we define the linear operators $\mathcal{A},\mathcal{B},\mathcal{C}$ by $\mathcal{A}z^k=a_kz^k,\mathcal{B}z^k=b_kz^k$ and $\mathcal{C}z^k=c_kz^k$, (2.4) may be rewritten in the form

$$(\mathcal{C}-z\mathcal{A})f_k(z)=c_k(1+z\mathcal{B})f_k(z)\ , \tag{2.5}$$

valid for all $k\in\mathbb{Z}$. Equation (2.5) is a system of equations of type (2.1) with $U=\mathcal{C}-z\mathcal{A}$ and $V=1+z\mathcal{B}$. The dual equations (2.2) for the auxiliary fLs $h_k(z)$ in this case read

$$(\mathcal{C}^*-\mathcal{A}^*z)h_k(z)=c_k(1+\mathcal{B}^*z)h_k(z)\ . \tag{2.6}$$

Because of $\mathcal{A}^*z^{-k}=a_kz^{-k}$ etc., by comparing the coefficients of $z^{-l}$ in (2.6) we obtain

$$(c_l-c_k)h_{kl}=(a_l+c_kb_l)h_{k,l+1}\ .$$

If we set $h_{kk}=1$, we get

$$h_{kl}=\frac{\prod_{j=l}^{k-1}(a_j+c_kb_j)}{\prod_{j=l}^{k-1}(c_j-c_k)}\ .$$

Taking into account (2.3), here we have $\tilde f_k(z)=(1+\mathcal{B}^*z)h_k(z)$. Hence again, comparing coefficients of $z^{-l}$ leads to

$$f_{kl}^{-1}=h_{kl}+b_lh_{k,l+1}=\frac{a_l+c_lb_l}{a_k+c_kb_k}\frac{\prod_{j=l+1}^{k}(a_j+c_kb_j)}{\prod_{j=l}^{k-1}(c_j-c_k)}\ ,$$

which is exactly (1.4)(2).

## 3. Hypergeometric and basic hypergeometric notation

All the notation and terminology is adopted from [10, pp. 1–6]. The (*generalized*) *hypergeometric series* is defined by

$$ {}_rF_s\left[\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{n!\,(b_1)_n \cdots (b_s)_n} z^n \ , $$

where the rising factorial $(a)_n$ is given by $(a)_n := a(a+1)\cdots(a+n-1)$, $n \ge 1$, $(a)_0 := 1$. The gamma function can be used to extend rising factorials by defining $(a)_\beta = \lim_{\gamma\to a} \Gamma(\gamma+\beta)/\Gamma(\gamma)$, $\beta$ arbitrary (cf. [17, p. 211f]). A hypergeometric series ${}_{r+1}F_r$ is called *very well-poised* if $a_i + b_i = 1 + a_0$ for $i = 1, 2, \dots, r$, and among the parameters $a_i$ occurs $1 + a_0/2$. We use the standard abbreviation for very well-poised hypergeometric series,

$$ \begin{aligned} &{}_{r+1}V_r(a_0; a_2, a_3, \dots, a_r; z) \\ &\quad := {}_{r+1}F_r\left[\begin{matrix} a_0, 1+a_0/2, a_2, a_3, \dots, a_r \\ a_0/2, 1+a_0-a_2, 1+a_0-a_3, \dots, 1+a_0-a_r \end{matrix}; z\right]. \end{aligned} $$

We shall also use the compact Gasper-Rahman notation

$$ (a_1, a_2, \dots, a_r)_n := (a_1)_n\,(a_2)_n \,\cdots\, (a_r)_n $$

and

$$ \Gamma\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}\right] := \frac{\Gamma(a_1)\,\Gamma(a_2)\,\cdots\,\Gamma(a_r)}{\Gamma(b_1)\,\Gamma(b_2)\,\cdots\,\Gamma(b_s)} \ . $$

Given a (fixed) complex number $q$ with $|q| < 1$, the *basic hypergeometric series* is defined by

$$ {}_r\phi_s\left[\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1} z^n \ , $$

where, as before, the rising $q$-factorial $(a;q)_n$ is given by

$$ (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n \ge 1, (a)_0 := 1. $$

The infinite $q$-factorial $(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$ can be used to extend (finite) $q$-factorials by defining $(a;q)_\beta := (a;q)_\infty/(aq^\beta;q)_\infty$, $\beta$ arbitrary. A basic hypergeometric series ${}_{r+1}\phi_r$ is called *very well-poised* if $a_i b_i = q a_0$ for $i = 1, 2, \dots, r$, and among the parameters $a_i$ occur both $q\sqrt{a_0}$ and $-q\sqrt{a_0}$. We use the standard abbreviation for very well-poised basic hypergeometric series,

$$ {}_{r+1}W_r(a_0; a_3, a_4, \dots, a_r; z) := {}_{r+1}\phi_r\left[\begin{matrix} a_0, q\sqrt{a_0}, -q\sqrt{a_0}, a_3, a_4, \dots, a_r \\ \sqrt{a_0}, -\sqrt{a_0}, qa_0/a_3, qa_0/a_4, \dots, qa_0/a_r \end{matrix}; z\right]. $$

We use short notations in the basic hypergeometric context which are analogous to the hypergeometric ones.

All identities in our paper are subject to suitable conditions on the parameters such that the involved hypergeometric or basic hypergeometric series converge. We shall not state these conditions for each identity. The reader should consult [10, pp. 4–5].

## 4. Hypergeometric summation formulas from inversion

There is a standard technique for deriving new summation formulas from known ones by using inverse matrices (cf. [1, 11, 21]). If $(f_{nk})$ and $(f^{-1}_{kl})$ are lower-triangular matrices that are inverses of each other, then of course the following is true:

$$\sum_{k=0}^{n} f_{nk} a_k = b_n \quad \text{if and only if} \quad \sum_{l=0}^{k} f^{-1}_{kl} b_l = a_k. \tag{4.1}$$

If one side in (4.1) is known, then the other produces another summation formula. There is also a less used dual version, which in Riordan's book [21] is called "rotated inversion". It reads

$$\sum_{n=k}^{\infty} f_{nk} B_n = A_k \quad \text{if and only if} \quad \sum_{k=l}^{\infty} f^{-1}_{kl} A_k = B_l, \tag{4.2}$$

subject to suitable convergence conditions. Again, if one side in (4.2) is known, then the other produces a possibly new identity.

What Gasper and Rahman do in [8, 19] is rotated inversion, though in disguise. Recall that the rotated inversion (4.2) bases on the orthogonal relation $\sum_{k=l}^{n} f_{nk} f^{-1}_{kl} = \delta_{nl}$. ($\delta_{nl}$ is the usual Kronecker delta.) Typically, Gasper and Rahman start with such an orthogonality relation, but with $l = 0$, i.e. $\sum_{k=0}^{n} f_{nk} f^{-1}_{k0} = \delta_{n0}$. (As inverse pair $(f_{nk})$, $(f^{-1}_{kl})$ they choose special cases of (4.3), with the roles of $f_{nk}$ and $f^{-1}_{kl}$ interchanged.) Then they multiply both sides by $B_n$, say, and sum over all $n$. This gives $\sum_{n=0}^{\infty} B_n \sum_{k=0}^{n} f_{nk} f^{-1}_{k0} = B_0$. Next the sums are interchanged to give $\sum_{k=0}^{\infty} f^{-1}_{k0} \sum_{n=k}^{\infty} f_{nk} B_n = B_0$. The inner sum in all cases is simplified by means of Bailey's nonterminating extension of Jackson's ${}_8\phi_7$-sum, to $A_k$ say. Thus the identity $\sum_{k=0}^{\infty} f^{-1}_{k0} A_k = B_0$ is proved. Of course, the reader will have observed that this is exactly the proof of the left-to-right implication of (4.2), for $l = 0$. Hence, why not start with $l$ generic? The answer is that there is no problem of doing the same with arbitrary $l$. But in the end it turns out that the resulting identity is not more general. In fact, a simple parameter substitution eliminates all occurrences of $l$, and one is left with the identity for $l = 0$. This $l = 0$ phenomenon will be exemplified in the following application of the inversion (1.5).

Before we start, another digression appears to be of interest. Gasper and Rahman's indefinite bibasic sum [10, (3.6.13)], which is so important in their derivations, comes from a telescoping argument which is based on a mysterious factorization [10, (3.6.10)] of a difference of two four-term products into a four-term product. This factorization loses some of its mystery if one observes that it is just equivalent to the $n = 1$ case of Jackson's ${}_8\phi_7$-sum [10, (2.6.2)] (replace $a$ by $adp^kq^{k-1}$, $b$ by $bp^k$, $c$ by $cq^k$, $d$ by $ap^k$ in Jackson's sum with $n = 1$).

In this section we use the inverse matrices of (1.5) with the choices $c_k = q^k$ and $b_j = bp^j$. This inversion can be written in the form:

*If*

$$f_{nk} = \frac{(bp^kq^k;p)_{n-k}\,(bp^kq^{-k}/a;p)_{n-k}}{(aq^{2k+1};q)_{n-k}\,(q;q)_{n-k}}\ , \tag{4.3)(1}$$

*then*

(4.3)(2)
$$f_{kl}^{-1} = p^{l-k}\frac{(1-bp^lq^l)(1-ap^{-l}q^l/b)}{(1-bp^kq^k)(1-ap^{-k}q^k/b)}\frac{(bp^kq^k;p^{-1})_{k-l}\,(bp^kq^{-k}/a;p^{-1})_{k-l}}{(aq^{2k-1};q^{-1})_{k-l}\,(q^{-1};q^{-1})_{k-l}}\ .$$

If in (4.3) we replace $a$ by $q^B$, $b$ by $q^{B-C}$, $q$ by $p^A$, and let $p \to 1$, we obtain the inverse pair:

*If*

$$f_{nk} = \frac{(AB-AC+(1+A)k)_{n-k}(-AC+(1-A)k)_{n-k}}{(B+2k+1)_{n-k}(n-k)!}\ , \tag{4.4)(1}$$

*then*

$$f_{kl}^{-1} = \frac{(-AC+(1-A)l)\,(AB-AC+(1+A)l)}{(-AC+(1-A)k)\,(AB-AC+(1+A)k)} \cdot \frac{(-AB+AC-(1+A)k)_{k-l}(AC-(1-A)k)_{k-l}}{(1-B-2k)_{k-l}\,k!}\ . \tag{4.4)(2}$$

For $B_n$ we choose

$$B_n = \frac{(D)_n}{(-B+AB-2AC+D)_n}. \tag{4.5}$$

Now we form the sum $A_k = \sum_{n=k}^{\infty} f_{nk}B_n$. In hypergeometric notation this is

$$\begin{aligned} A_k =& \frac{(D)_k}{(-B+AB-2AC+D)_k} \\ &\cdot {}_3F_2\left[\begin{matrix} D+k, -AC+k-Ak, AB-AC+k+Ak \\ -B+AB-2AC+D+k, 1+B+2k \end{matrix}; 1\right]\ . \end{aligned}$$

To the ${}_3F_2$ we apply a nonterminating extension of the Pfaff–Saalschütz summation [23, (2.4.4.4)]

(4.6)
$$\begin{aligned} {}_3F_2&\left[\begin{matrix} a, b, c \\ d, 1+a+b+c-d \end{matrix}; 1\right] \\ &= \Gamma\left[\begin{matrix} 1+a-d, 1+b-d, 1+c-d, 1+a+b+c-d \\ 1-d, 1+b+c-d, 1+a+c-d, 1+a+b-d \end{matrix}\right] \\ &\quad - \Gamma\left[\begin{matrix} d-1, 1+a-d, 1+b-d, 1+c-d, 1+a+b+c-d \\ 1-d, a, b, c, 2+a+b+c-2d \end{matrix}\right] \\ &\quad \cdot {}_3F_2\left[\begin{matrix} 1+a-d, 1+b-d, 1+c-d \\ 2-d, 2+a+b+c-2d \end{matrix}; 1\right], \end{aligned}$$

thus obtaining

(4.7)
$$A_k=\Gamma\left[\begin{matrix}1+B-AB+2AC,1+B-AB+AC-D-Ak,1+B+AC-D+Ak\\1+B-AB+2AC-D-k,1+B-D+k,1+B+AC+k+Ak\end{matrix}\right]$$
$$\cdot\Gamma\left[\begin{matrix}1+B+2k,-B+AB-2AC+D,D+k\\1+B-AB+AC+k-Ak,D,-B+AB-2AC+D+k\end{matrix}\right]$$
$$-\Gamma\left[\begin{matrix}-1-B+AB-2AC+D+k,1+B-AB+2AC,1+B-AB+AC-D-Ak\\1+B-AB+2AC-D-k,-AC+k-Ak,AB-AC+k+Ak\end{matrix}\right]$$
$$\cdot\Gamma\left[\begin{matrix}1+B+AC-D+Ak,1+B+2k,-B+AB-2AC+D\\2+2B-AB+2AC-D+k,D,-B+AB-2AC+D+k\end{matrix}\right]$$
$${}_3F_2\left[\begin{matrix}1+B-AB+2AC,1+B-AB+AC-D-Ak,1+B+AC-D+Ak\\2+B-AB+2AC-D-k,2+2B-AB+2AC-D+k\end{matrix};1\right].$$

By rotated inversion (4.2) the identity $\sum_{k=l}^{\infty} f_{kl}^{-1}A_k=B_l$, with $f_{kl}^{-1},A_k,B_l$ given by (4.4)(2), (4.7), (4.5), respectively, holds. The sum on the left-hand side splits into two sums, one of which is actually a double sum. In this double sum we interchange summations and simplify. In the resulting identity we replace $B$ by $B-2l$, $C$ by $C+(1-A)l/A$, and $D$ by $D-l$. The effect of this last step is the same as if we would have set $l=0$. This is the $l=0$ phenomenon which was described before. Thus we arrive at the summation

$$\sum_{k=0}^{\infty}\frac{(B)_k(1+\frac{B}{2})_k(AB-AC)_{(1+A)k}(1+B+AC-D)_{Ak}(-AC)_{(1-A)k}}{k!\,(\frac{B}{2})_k(1+B+AC)_{(1+A)k}(1+AB-AC)_{Ak}(1+B-AB+AC)_{(1-A)k}}$$
$$\frac{(1+B-AB+AC-D)_{-Ak}(D)_k}{(1-AC)_{-Ak}(1+B-D)_k}$$
$$-\Gamma\left[\begin{matrix}1+B-AB+AC,1+B+AC,1+B-D,-1-B+AB-2AC+D\\AB-AC,-AC,2+2B-AB+2AC-D,D\end{matrix}\right]$$
$$\sum_{j=0}^{\infty}\frac{(1+B-AB+2AC)_j(1+B+AC-D)_j(1+B-AB+AC-D)_j}{j!\,(2+B-AB+2AC-D)_j(2+2B-AB+2AC-D)_j}$$
$$\sum_{k=0}^{\infty}\frac{(B)_k(1+\frac{B}{2})_k(-1-B+AB-2AC+D-j)_k}{k!\,(\frac{B}{2})_k(2+2B-AB+2AC-D+j)_k}$$
$$\frac{(1+B+AC-D+j)_{Ak}(1+B-AB+AC-D+j)_{-Ak}}{(1+AB-AC)_{Ak}(1-AC)_{-Ak}}$$
$$=\Gamma\left[\begin{matrix}1+B-AB+AC,1+B+AC,1+B-D,1+B-AB+2AC-D\\1+B,1+B-AB+2AC,1+B-AB+AC-D,1+B+AC-D\end{matrix}\right]. \tag{4.8}$$

Unfortunately, the inner sum in the double sum cannot be evaluated in general. (It can for $A=1$. In this case even a $q$-analogue exists, see below.) If we terminate the first series by setting $D=-n$, $n$ a nonnegative integer, then the term with the double sum vanishes because of the occurrence of $\Gamma(D)$ in the denominator of this expression. The resulting summation can be rewritten as

$$\sum_{k=0}^{\infty}\frac{(B)_k(1+\frac{B}{2})_k(AB-AC)_{(A+1)k}(AC)_{Ak}(-B+AB-AC)_{(A-1)k}}{k!\,(\frac{B}{2})_k(1+B+AC)_{(A+1)k}(1+AB-AC)_{Ak}(1+AC)_{(A-1)k}}$$
$$\frac{(1+B+AC+n)_{Ak}(-n)_k}{(-B+AB-AC-n)_{Ak}(1+B+n)_k}=\frac{(1+B)_n(1+B-AB+2AC)_n}{(1+B+AC)_n(1+B-AB+AC)_n}. \tag{4.9}$$

If $A$ is a positive integer (4.8) and (4.9) give summation formulas for (nonterminating, respectively terminating) very well-poised ${}_{4A+3}F_{4A+2}$-series. The case $A=1$

of (4.9) is almost equivalent to a known ${}_7F_6$-summation [23, (2.4.1.5)]. In fact, both formulas can be derived from each other by contiguous relations. Besides, identity (4.8) for a fixed $A$ is equivalent to (4.8) with $A$ replaced by $-A$. (Hence, the same is true for (4.9).) In fact, replacing $A$ by $-A$ is the same as replacing $C$ by $B-C$.

Letting $n$ tend to $\infty$ in (4.9) leads to the summation

$$\sum_{k=0}^{\infty}\frac{(B)_k(1+\frac{B}{2})_k(AB-AC)_{(A+1)k}(AC)_{Ak}(-B+AB-AC)_{(A-1)k}}{k!\,(\frac{B}{2})_k(1+B+AC)_{(A+1)k}(1+AB-AC)_{Ak}(1+AC)_{(A-1)k}} = \Gamma\left[\begin{matrix}1+B,1+B-AB+2AC\\1+B+AC,1+B-AB+AC\end{matrix}\right]. \tag{4.10}$$

If $A$ is a positive integer (4.10) gives a summation formula for nonterminating very well-poised ${}_{3A+2}F_{3A+1}$-series.

It is only the case $A=1$, where we are able to give a $q$-analogue of (4.8) (and hence of (4.9) and (4.10)). This is due to the absence of a suitable bibasic analogue of the Pfaff–Saalschütz formula. (Recently a bibasic analogue of the Pfaff–Saalschütz formula has been discovered by Singer [22], but it is not suited for our purposes.) To obtain the $q$-analogue of (4.8) in case $A=1$, start with the inverse pair $(f_{nk})$, $(f_{kl}^{-1})$ from (4.3), with $p=q$, $a=B$, $b=B/C$. For $B_n$ choose $B_n=q^n(D;q)_n/(D/C^2;q)_n$. Then proceed as before. Instead of (4.6) use the nonterminating extension of the $q$-Pfaff–Saalschütz sum [10, (2.10.12)]. To evaluate the inner sum of the arising double sum, use the very well-poised ${}_6\phi_5$-summation [10, (2.7.1)]. What is finally obtained is the summation

$$\begin{aligned}&{}_{10}W_9(B;\frac{\sqrt{B}}{\sqrt{C}},-\frac{\sqrt{B}}{\sqrt{C}},\frac{\sqrt{Bq}}{\sqrt{C}},-\frac{\sqrt{Bq}}{\sqrt{C}},\frac{BCq}{D},C,D;q,q)\\&\quad-\frac{(Bq,1/C,D,B/C,D/C^2,BCq^2/D;q)_\infty}{(Bq/D,BCq,D/C^2q,D/C,q,Bq/C;q)_\infty}\\&\quad\cdot{}_3\phi_2\left[\begin{matrix}C^2q,C^2q/D,BCq/D\\C^2q^2/D,BCq^2/D\end{matrix};q,\frac{q}{C}\right]\\&=\frac{(C^2q,Cq/D,BCq/D,Bq;q)_\infty}{(Cq,C^2q/D,Bq/D,BCq;q)_\infty}.\end{aligned}\tag{4.11}$$

By comparing this identity with a special case of a transformation of Verma and Jain [24, (7.1); 10, Ex. 2.25, $a=B/C$, $b=qB/C^2$, $c=1$, $d=1$, $e=D$],

$$\begin{aligned}&{}_{10}W_9(B;\frac{\sqrt{B}}{\sqrt{C}},-\frac{\sqrt{B}}{\sqrt{C}},\frac{\sqrt{Bq}}{\sqrt{C}},-\frac{\sqrt{Bq}}{\sqrt{C}},C,D,\frac{BCq}{D};q,q)\\&\quad+\frac{(Bq,Cq/D,B/C,D,C,Bq^2/D,BC^3q^3/D^2;q)_\infty}{(Cq,Bq/D,BCq,C^2q/D,D/Cq,Bq/C,BC^2q^3/D^2;q)_\infty}\\&\quad\cdot{}_{10}W_9(\frac{BC^2q^2}{D^2};\frac{\sqrt{BC}q}{D},-\frac{\sqrt{BC}q}{D},\frac{\sqrt{BC}q^{\frac32}}{D},\\&\qquad\qquad-\frac{\sqrt{BC}q^{\frac32}}{D},Cq,\frac{BCq}{D},\frac{C^2q}{D};q,q)\\&=\frac{(Bq,Cq/D,C^2q,BCq/D;q)_\infty}{(Cq,Bq/D,BCq,C^2q/D;q)_\infty},\end{aligned}$$

we obtain a transformation formula between a very well-poised ${}_{10}\phi_9$ and a ${}_3\phi_2$ series,

$$\begin{aligned}{}_{10}W_9(&\frac{BC^2q^2}{D^2};\frac{\sqrt{BC}q}{D},-\frac{\sqrt{BC}q}{D},\frac{\sqrt{BC}q^{\frac{3}{2}}}{D},\\&-\frac{\sqrt{BC}q^{\frac{3}{2}}}{D},Cq,\frac{C^2q}{D},\frac{BCq}{D};q,q)\\&=\frac{(q/C,BCq^2/D,C^2q^2/D,BC^2q^3/D^2;q)_\infty}{(q,Cq^2/D,Bq^2/D,BC^3q^3/D^2;q)_\infty}\\&\cdot{}_3\phi_2\left[\begin{matrix}C^2q,C^2q/D,BCq/D\\C^2q^2/D,BCq^2/D\end{matrix};q,\frac{q}{C}\right],\end{aligned}$$

or after having replaced $B$ by $BD^2/C^2q^2$, $C$ by $B/C$, $D$ by $qD/C$, in that order,

$$\begin{aligned}{}_{10}W_9(&B;\sqrt{C},-\sqrt{C},\sqrt{Cq},-\sqrt{Cq},D,\frac{Bq}{C},\frac{B^2}{CD};q,q)\\&=\frac{(Cq/B,Dq,B^2q/CD,Bq;q)_\infty}{(q,Bq/D,CDq/B,B^2q/C;q)_\infty}{}_3\phi_2\left[\begin{matrix}B^2q/C^2,B^2/CD,D\\B^2q/CD,Dq\end{matrix};q,\frac{Cq}{B}\right].\end{aligned}\tag{4.12}$$

Mizan Rahman (private communication) showed me how to derive this transformation from standard basic hypergeometric transformation formulas by a sequence of series manipulations.

## 5. More identities

In this section we restrict ourselves to terminating series. Therefore here we use the "usual" inversion (4.1). We use special cases of (1.4) which are different from the one, (4.3), used in the previous section.

First, let us take $c_k=(k+c)^3$, $a_j=(j+a)^3$, $b_j=1$. The inverse pair (1.4) in this special case reads

$$f_{nk}=\frac{(n+a)^3+(n+c)^3}{(k+a)^3+(k+c)^3}\frac{\prod_{j=k}^{n-1}((j+a)^3+(k+c)^3)}{\prod_{j=k+1}^{n}((j+c)^3-(k+c)^3)}\tag{5.1)(1}$$

and

$$f_{kl}^{-1}=\frac{\prod_{j=l+1}^{k}((j+a)^3+(k+c)^3)}{\prod_{j=l}^{k-1}((j+c)^3-(k+c)^3)}\ .\tag{5.1)(2}$$

Now choose

$$b_l=\frac{(a+c,1+a/2+c/2,d,2a-c-d+1)_l}{(1,a/2+c/2,1+a+c-d,2c-a+d)_l}\ .\tag{5.2}$$

We form $a_k=\sum_{l=0}^{k}f_{kl}^{-1}b_l$, which can be written as

$$\begin{aligned}a_k=&\frac{\prod_{j=1}^{k}((j+a)^3+(k+c)^3)}{\prod_{j=0}^{k-1}((j+c)^3-(k+c)^3)}\\&{}_7V_6(a+c;1+2a-c-d,d,c-c\omega-k\omega,c-c\omega^2-k\omega^2,-k;1).\end{aligned}$$

Here, $\omega$ denotes a third root of unity. The ${}_7V_6$ can be evaluated by means of Dougall's sum [23, (2.3.4.4)]

$$
(5.3)\quad {}_7V_6(a;b,c,d,1+2a-b-c-d+n,-n;1) = \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-b-c-d)_n}\,,
$$

thus obtaining

$$
(5.4)\qquad a_k = \frac{(1+a+c)_{2k}\,(-a+2c)_k}{(1+a+c-d)_{2k}\,(-a+2c+d)_k}\,\frac{\prod_{j=1}^{k}((j+a-d)^3+(k+c)^3)}{\prod_{j=0}^{k-1}((j+c)^3-(k+c)^3)}
$$

after simplification. Now, by the inversion (4.1), the identity $\sum_{k=0}^{n} f_{nk}a_k = b_n$ holds, with $f_{nk}, a_k, b_n$ given in (5.1)(1), (5.4), (5.2), respectively. This identity reads

$$
(5.5)\quad \begin{aligned}
&\sum_{k=0}^{n}\frac{(n+a)^3+(n+c)^3}{(k+a)^3+(k+c)^3}\,\frac{\prod_{j=k}^{n-1}((j+a)^3+(k+c)^3)}{\prod_{j=k+1}^{n}((j+c)^3-(k+c)^3)}\\
&\qquad\cdot\frac{\prod_{j=1}^{k}((j+a-d)^3+(k+c)^3)}{\prod_{j=0}^{k-1}((j+c)^3-(k+c)^3)}\,\frac{(1+a+c)_{2k}\,(-a+2c)_k}{(1+a+c-d)_{2k}\,(-a+2c+d)_k}\\
&\quad=\frac{(a+c,1+a/2+c/2,d,2a-c-d+1)_n}{(1,a/2+c/2,1+a+c-d,2c-a+d)_n}.
\end{aligned}
$$

The products which occur in the sum can be considered as cubic analogues of the rising factorials. For example, $\prod_{j=k}^{n-1}((j+a)^3+(k+c)^3)$ is a cubic analogue of $(2k+a+c)_{n-k} = \prod_{j=k}^{n-1}((j+a)+(k+c))$.

Next we choose $c_k = (k+c)^4$, $a_j = (j+a)^4$, $b_j = -1$. The inverse pair (1.4) now reads

$$
(5.6)(1)\qquad f_{nk} = \frac{(n+a)^4-(n+c)^4}{(k+a)^4-(k+c)^4}\,\frac{\prod_{j=k}^{n-1}((j+a)^4-(k+c)^4)}{\prod_{j=k+1}^{n}((j+c)^4-(k+c)^4)}
$$

and

$$
(5.6)(2)\qquad f_{kl}^{-1} = \frac{\prod_{j=l+1}^{k}((j+a)^4-(k+c)^4)}{\prod_{j=l}^{k-1}((j+c)^4-(k+c)^4)}\,.
$$

For $b_l$ we take

$$
(5.7)\qquad b_l = \frac{(a+c,1+a/2+c/2,1-2c+2a)_l}{(1,a/2+c/2,3c-a)_l}\,.
$$

Then we form $a_k = \sum_{l=0}^{k} f_{kl}^{-1} b_l$. Again, the sum can be evaluated by means of Dougall's sum (5.3). Thus we get

$$
(5.8)\qquad a_k = \frac{(1+a+c)_{2k}}{(3c-a)_{2k}}\,\frac{\prod_{j=0}^{k-1}((j+2c-a)^4-(k+c)^4)}{\prod_{j=0}^{k-1}((j+c)^4-(k+c)^4)}\,.
$$

The inversion (4.1) then yields

$$\sum_{k=0}^{n} \frac{(n+a)^4-(n+c)^4}{(k+a)^4-(k+c)^4} \frac{\prod_{j=k}^{n-1}((j+a)^4-(k+c)^4)}{\prod_{j=k+1}^{n}((j+c)^4-(k+c)^4)} \cdot \frac{\prod_{j=0}^{k-1}((j+2c-a)^4-(k+c)^4)}{\prod_{j=0}^{k-1}((j+c)^4-(k+c)^4)} \frac{(1+a+c)_{2k}}{(3c-a)_{2k}} = \frac{(a+c,1+a/2+c/2,1-2c-2a)_n}{(1,a/2+c/2,1+a+c-d,3c-a)_n}. \tag{5.9}$$

In this identity there occur quartic analogues of rising factorials.

Finally, we take $c_k=(k+c)^5$, $a_j=(j+a)^5$, $b_j=1$. The inverse pair (1.4) then reads

$$f_{nk} = \frac{(n+a)^5+(n+c)^5}{(k+a)^5+(k+c)^5} \frac{\prod_{j=k}^{n-1}((j+a)^5+(k+c)^5)}{\prod_{j=k+1}^{n}((j+c)^5-(k+c)^5)} \tag{5.10)(1}$$

and

$$f_{kl}^{-1} = \frac{\prod_{j=l+1}^{k}((j+a)^5+(k+c)^5)}{\prod_{j=l}^{k-1}((j+c)^5-(k+c)^5)} . \tag{5.10)(2}$$

Now choose

$$b_l = \frac{(a+c,1+a/2+c/2)_l}{(1,a/2+c/2)_l} . \tag{5.11}$$

Then for $c$ choose $(2a+1)/3$ and subsequently replace $a$ by $3a+1$. If we do the same as in the previous two derivations, we finally obtain the following identity which contains quintic analogues of rising factorials,

$$\sum_{k=0}^{n} \frac{(n+3a+1)^5+(n+2a+1)^5}{(k+3a+1)^5+(k+2a+1)^5} \frac{\prod_{j=k}^{n-1}((j+3a+1)^5+(k+2a+1)^5)}{\prod_{j=k+1}^{n}((j+2a+1)^5-(k+2a+1)^5)} \cdot \frac{(5a+3)_{2k}\,(1+a+(1+2a+k)(\omega+\omega^2),1+a+(1+2a+k)(\omega+\omega^3),}{\prod_{j=0}^{k-1}((j+2a+1)^5-(k+2a+1)^5)} \cdot \frac{1+a+(1+2a+k)(\omega+\omega^4))_k}{} = \frac{(5a+2,5a/2+2)_n}{(1,5a/2+1)_n}. \tag{5.12}$$

Here, $\omega$ denotes a fifth root of unity.

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## References

1. G. E. Andrews, *Connection coefficient problems and partitions*, Proc. Sympos. Pure Math. (D. Ray-Chaudhuri, ed.), vol. 34, Amer. Math. Soc., Providence, RI, 1979, pp. 1–24. MR **80c:**33004
2. W. N. Bailey, *Some identities in combinatory analysis*, Proc. London Math. Soc. (2) **49** (1947), 421–435. MR **9:**263d
3. ______, *Identities of the Roger–Ramanujan type*, Proc. London Math. Soc. (2) **50** (1949), 1–10. MR **9:**585b
4. D. M. Bressoud, *Some identities for terminating q–series*, Math. Proc. Cambridge Philos. Soc. **89** (1981), 211–223. MR **82d:**05019
5. ______, *A matrix inverse*, Proc. Amer. Math. Soc. **88** (1983), 446–448. MR **84g:**33003
6. L. Carlitz, *Some inverse relations*, Duke Math. J. **40** (1973), 893–901. MR **49:**2420
7. G. P. Egorychev, *Integral representation and the computation of combinatorial sums*, "Nauka" Sibirsk. Otdel., Novosibirsk, 1977; English transl., Transl. of Math. Monographs, vol. 59, Amer. Math. Soc., Providence, RI, 1984. MR **58:**10474
8. G. Gasper, *Summation, transformation, and expansion formulas for bibasic series*, Trans. Amer. Math. Soc. **312** (1989), 257–278. MR **89g:**33012
9. G. Gasper and M. Rahman, *An indefinite bibasic summation formula and some quadratic, cubic and quartic summation and transformation formulae*, Canad. J. Math. **42** (1990), 1–27. MR **90k:**33032
10. ______, *Basic hypergeometric series*, Encyclopedia Math. Appl., vol. 35, Cambridge Univ. Press, Cambridge, 1990. MR **91d:**33034
11. I. Gessel and D. Stanton, *Application of q–Lagrange inversion to basic hypergeometric series*, Trans. Amer. Math. Soc. **277** (1983), 173–203. MR **84f:**33009
12. H. W. Gould, *A series transformation for finding convolution identities*, Duke Math. J. **28** (1961), 193–202. MR **23:**A1216
13. ______, *A new convolution formula and some new orthogonal relations for inversion of series*, Duke Math. J. **29** (1962), 393–404. MR **25:**3333
14. ______, *A new series transform with application to Bessel, Legrende, and Tchebychev polynomials*, Duke Math. J. **31** (1964), 325–334. MR **28:**4272
15. ______, *Inverse series relations and other expansions involving Humbert polynomials*, Duke Math. J. **32** (1965), 691–711. MR **32:**5948
16. H. W. Gould and L. C. Hsu, *Some new inverse series relations*, Duke Math. J. **40** (1973), 885–891. MR **49:**2421
17. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete mathematics*, Addison-Wesley, Reading, MA, 1989. MR **91f:**00001
18. C. Krattenthaler, *Operator methods and Lagrange inversion, a unified approach to Lagrange formulas*, Trans. Amer. Math. Soc. **305** (1988), 431–465. MR **89d:**05017
19. M. Rahman, *Some quadratic and cubic summation formulas for basic hypergeometric series*, Canad. J. Math. **45** (1993), 394–411. MR **94d:**33015
20. ______, *Some cubic summation formulas for basic hypergeometric series*, Utilitas Math. **36** (1989), 161–172. MR **91c:**33030
21. J. Riordan, *Combinatorial identities*, J. Wiley, New York, 1968. MR **38:**53
22. D. Singer, *q-analogues of Lagrange inversion*, Ph.D. Thesis, Univ. of California, San Diego, CA, 1992.
23. L. J. Slater, *Generalized hypergeometric functions*, Cambridge Univ. Press, Cambridge, 1966. MR **34:**1570
24. A. Verma and V. K. Jain, *Transformations of nonterminating basic hypergeometric series, their contour integrals and applications to Rogers–Ramanujan identities*, J. Math. Anal. Appl. **87** (1982), 9–44. MR **83e:**33002

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