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GENERATING FUNCTIONS OF JACOBI POLYNOMIALS

IZUMI KUBO*

Abstract. Multiplicative renormalization method (MRM) for deriving generating functions of orthogonal polynomials is introduced by Asai–Kubo–Kuo. They and Namli gave complete lists of MRM-applicable measures for MRM-factors $h(x) = e^x$ and $(1-x)^{-\kappa}$. In this paper, MRM-factors $h(x)$ for which the beta distribution $B(p,q)$ over [0, 1] is MRM-applicable are determined. In other words, all generating functions of Boas-Buck type of Jacobi polynomials over [0, 1] are obtained. There are only two types ${}_2F_1\left(\frac{p+q}{2}, \frac{p+q\pm 1}{2}; p; 4x\right)$ up to scaling. For the proofs, a general framework will be given together with an example.

1. Multiplicative Renormalization Method

A probability measure μ on $\mathbb{R}$ with density $f_\mu(x)$ is said to be *applicable to the multiplicative renormalization method* for $h(x)$ (or simply, *MRM-applicable*), if there exists a suitable analytic function $\rho(t)$ around $t = 0$ with $\rho(0) = 0$, $r_1 = \rho'(0) \neq 0$ such that

$$\psi(t,x) = \frac{h(\rho(t)x)}{\theta(\rho(t))} \quad \text{with} \quad \theta(t) = \int h(tx)\, d\mu(x) \tag{1.1}$$

is a generating function of the orthogonal polynomials $\{P_n(x)\}$ in $L^2(\mu)$ with leading coefficient of one. Then there exist *Jacobi-Szegö parameters* $\{\alpha_n, \omega_n\}$ satisfying the recursive relation

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \omega_n P_{n-1}(x) \tag{1.2}$$

with $\omega_0 = 1, P_{-1}(x) = 0$.

Let us suppose that *an MRM-factor* $h(x)$ is expanded as

$$h(x) = \sum_{n=0}^{\infty} h_n x^n, \quad h_0 = 1,\ h_n \neq 0, \quad n \geq 1.$$

Then we have the expansion

$$\psi(t,x) = \sum_{n=0}^{\infty} r_1^n h_n P_n(x) t^n. \tag{1.3}$$

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Proposition 1.1. ([3]) $\psi(t,x) = \dfrac{h(\rho(t)x)}{\varphi(t)}$ *with* $\varphi(t) = \theta(\rho(t))$ *is a generating function of orthogonal polynomials if and only if*

$$\Theta_\rho(t,s) = \frac{\widetilde{\theta}(\rho(t),\rho(s))}{\varphi(t)\varphi(s)}$$

is a function $J(ts)$ *depending only on* ts*, where*

$$\widetilde{\theta}(t,s) = \int h(tx)h(sx)\,d\mu(x).$$

In the previous papers, we have given complete lists of MRM-applicable measures for MRM-factors $h(x) = e^x$ and $h(x) = \dfrac{1}{(1-x)^\kappa}$ (see [9] [11] [12] [13]) by using the proposition. Typical MRM-applicable measures for $h(x) = e^x$ are Gaussian, Poisson, gamma, beta, negative binomial and Meixner distributions. In the second case $h(x) = \dfrac{1}{(1-x)^\kappa}$, we can show that $\kappa > -\dfrac{1}{2}$ and $\kappa \neq 0$ must hold. The case $\kappa = 1$ is extremal as seen [11]. Possible ρ and φ-functions are

$$\rho(t) = \frac{2t}{\alpha + 2\beta t + \gamma t^2}, \quad \varphi(t) = \frac{\alpha + 2\beta t + \gamma t^2}{\alpha + 2(\beta - a)t + (\gamma - 2b)t^2}. \tag{1.4}$$

By affine transform, the density f_μ of MRM-applicable measure μ for $h(x) = (1-x)^{-1}$ can be represented as

$$f_\mu(x) = \frac{b\sqrt{1-x^2}}{\pi(a^2 + b^2 - 2a(1-b)x + (1-2a)x^2)} \quad (|a| \leq 1 - b). \tag{1.5}$$

Then ρ and φ are standardized as $\rho(t) = \dfrac{2t}{1+t^2}$ and $\varphi(t) = \dfrac{1+t^2}{1 - 2at + (1-2b)t^2}$.

For $\kappa > -\dfrac{1}{2}, \kappa \neq 1$, κ's are classified in two groups. For $\dfrac{1}{2} \geq \kappa > -\dfrac{1}{2}$, there is only one typical MRM-applicable measure. For $\kappa > \dfrac{1}{2}, \kappa \neq 1$, there are only three typical MRM-applicable measures. (see [6] [13]).

Typical MRM-applicable measures for $h(x) = \frac{1}{(1-x)^\kappa}$, $\kappa \neq 0, 1$				
	μ	$\rho(t)$	$\varphi(t)$	$\psi(t,x)$
(i)	$\widetilde{B}(\kappa + \frac{1}{2}, \kappa + \frac{1}{2})$ $\kappa > -\frac{1}{2}$	$\frac{2t}{1+t^2}$	$(1+t^2)^\kappa$	$\frac{1}{(1-2tx+t^2)^\kappa}$
(ii)	$\widetilde{B}(\kappa - \frac{1}{2}, \kappa - \frac{1}{2})$ $\kappa > \frac{1}{2}$	$\frac{2t}{1+t^2}$	$\frac{(1+t^2)^\kappa}{1-t^2}$	$\frac{1-t^2}{(1-2tx+t^2)^\kappa}$
(iii)	$\widetilde{B}(\kappa + \frac{1}{2}, \kappa - \frac{1}{2})$ $\kappa > \frac{1}{2}$	$\frac{2t}{1+t^2}$	$\frac{(1+t^2)^\kappa}{1-t}$	$\frac{1-t}{(1-2tx+t^2)^\kappa}$

Here $\widetilde{B}(p,q)$ is the beta distribution over $[-1,\ 1]$ with the density

$$\widetilde{f}_{p,q}(x) = \frac{\Gamma(p+q)}{2^{p+q-1}\Gamma(p)\Gamma(q)}(1+x)^{p-1}(1-x)^{q-1}.$$

Their orthogonal polynomials are given by Gegenbauer polynomials respectively:

(i) $$\frac{n!\Gamma(\kappa)}{2^n\Gamma(n+\kappa)}C_n^{\kappa}(x),$$

(ii) $$\frac{n!\Gamma(\kappa-1)}{2^n\Gamma(n+\kappa-1)}C_n^{\kappa-1}(x),$$

(iii) $$\frac{n!\Gamma(\kappa)}{2^n\Gamma(n+\kappa)}\big(C_n^{\kappa}(x)-C_{n-1}^{\kappa}(x)\big).$$

They can also be expressed by using Jacobi polynomials as follows: (see Eq. (4) in §132 of [15]):

$$P^{(\alpha,\beta)}(x) = \frac{n!(\alpha+\beta+1)_n}{(\alpha+\beta+1)_{2n}}\sum_{k=0}^{n}\frac{(-1)^{n-k}(\alpha+\beta+1)_{n+k}}{k!(n-k)!(\alpha+\beta+1)_k}\left(\frac{x+1}{2}\right)^k.$$

In the footnote of [5], Boas and Buck wrote *"Jacobi polynomials $P^{(\alpha,\beta)}(x)$ have a generating relation of the form (1.1) only $\alpha-\beta=-1,\ 0$ or 1 (Smith* [18]*), but $P^{(\alpha,\beta)}(x+1)$ do have a generating relation (1.1) (Erdélyi* [8] *III, p. 264)."* However, on p. 264 of Erdélyi [8] III, only the formula

$$(1-t)^{-1-\alpha-\beta}\,{}_2F_1\left(\frac{1+\alpha+\beta}{2},\frac{2+\alpha+\beta}{2};1+\alpha;2t(x-1)(1-t)^{-2}\right) \tag{9}$$

is shown without any comment. Here hypergeometric functions are defined by

$${}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;x) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n n!}x^n,$$

where

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Let us observe the affine transform $x \mapsto \dfrac{1+x}{2}$. Then we have the following table corresponding to $B(p,q)$, also. Here $B(p,q)$ for $p,q>0$ means the beta distribution over $[0,\ 1]$ with the density

$$f_{p,q}(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}x^{p-1}(1-x)^{q-1} \quad \text{for } 0 \le x \le 1. \tag{1.6}$$

MRM-applicable beta distributions $B(p,q)$ over $[0,1]$ for $h(x)=(1-x)^{-\kappa}$			
μ	κ	$\rho(t)$	$\varphi(t)$
$B(p,p)$ $p>0$	$p-\dfrac{1}{2}$	$\dfrac{4t}{(1+t)^2}$	$(1+t)^{2p-1}$
$B(p,p)$ $p>0$	$p+\dfrac{1}{2}$	$\dfrac{4t}{(1+t)^2}$	$\dfrac{(1+t)^{2p+1}}{1-t^2}$
$B(p,p-1)$ $p>1$	$p-\dfrac{1}{2}$	$\dfrac{4t}{(1+t)^2}$	$\dfrac{(1+t)^{2p-1}}{1-t}$
$B(p,p+1)$ $p>0$	$p+\dfrac{1}{2}$	$\dfrac{4t}{(1+t)^2}$	$\dfrac{(1+t)^{2p+1}}{1-t}$

Here $\psi(t,x)$ is given by $\dfrac{1-t}{(1+2(1-2x)t+t^2)^{\kappa}}$. Orthogonal polynomials of $B(p,q)$ are given by

$$\begin{aligned} P_n^{p,q}(x) &= \frac{n!(p)_n}{(p+q-1)_{2n}}\sum_{k=0}^{n}\frac{(-1)^{n-k}(p+q-1)_{n+k}}{k!(n-k)!(p)_k}x^k \\ &= \frac{(p)_n n!}{(p+q-1)_{2n}}P_n^{(q-1,p-1)}(2x-1). \end{aligned} \tag{1.7}$$

Bateman's generating function is given by

$${}_0F_1(-;p;tx)\,{}_0F_1(-;q;t(1-x))=\sum_{n=0}^{\infty}\frac{(p+q-1+n)_n}{n!(p)_n(q)_n}P_n^{p,q}(x)t^n \tag{1.8}$$

(see §133 of [15]). A generating function of the type of Eq. (1.1) can be given by Boas and Buck's footnote mentioned above. It is remarkable that Proposition 1.1 is applicable to show the orthogonality for the case.

Theorem 1.2. *The beta distribution $\mu_{p,q}$ over $[0,\ 1]$ is MRM-applicable for*

$$h^{p,q}(x)={}_2F_1\left(\frac{p+q}{2},\frac{p+q-1}{2};p;4x\right) \tag{1.9}$$

with $\rho(t)=\dfrac{t}{(1+t)^2}$ and $\varphi^{p,q}(t)=(1+t)^{p+q-1}$ for $p+q>1$. Moreover, the generating function is given by

$$\begin{aligned} \psi^{p,q}(t,x) &= \frac{1}{(1+t)^{p+q-1}}\,{}_2F_1\left(\frac{p+q}{2},\frac{p+q-1}{2};p;\frac{4t}{(1+t)^2}x\right) \\ &= \sum_{n=0}^{\infty}\frac{(p+q-1)_{2n}}{(p)_n n!}P_n^{p,q}(x)t^n. \end{aligned}$$

2. Another Generating Function for Beta Distribution

As seen in Section 1, the set of orthogonal polynomials for the beta distribution $B(p,p)$ over $[0,\ 1]$ has only two generating functions of type (1.1) with the same $\rho(t) = \dfrac{4t}{(1+t)^2}$. This fact shows the possibility of another generating function for $B(p,q)$ different from Eq. (1.9). We first calculate Jacobi-Szegö parameters.

Proposition 2.1. *Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ of the orthogonal polynomials $\{P_n^{p,q}(x)\}$ of Eq. (1.7) are given by*

$$\alpha_n = \frac{2n^2 + 2(p+q-1)n + p(p+q-2)}{(p+q+2n-2)(p+q+2n)} \quad \textit{for } n \geq 0, \tag{2.1}$$

$$\omega_n = \frac{n(p+n-1)(q+n-1)(p+q+n-2)}{(p+q+2n-3)(p+q+2n-2)^2(p+q+2n-1)} \quad \textit{for } n \geq 1 \tag{2.2}$$

and $\omega_0 = 1$.

Proof. By Eq. (1.9), $\rho(t) = \dfrac{t}{(1+t)^2}$ and $B(t) = \dfrac{1}{\varphi^{p,q}(t)} = \dfrac{1}{(1+t)^{p+q-1}}$ in Theorem 1.2, we have

$$\begin{aligned}
h_n &= \frac{\left(\frac{p+q}{2}\right)_n \left(\frac{p+q-1}{2}\right)_n}{n!(p)_n}, \\
b_1 &= B'(0) = -p-q+1, \quad b_2 = \frac{1}{2}B''(0) = \frac{1}{2}(p+q-1)(p+q), \\
r_2 &= \frac{1}{2}\rho''(0) = -2, \quad r_3 = \frac{1}{6}\rho'''(0) = 3.
\end{aligned}$$

Obviously, we see that

$$\begin{aligned}
\frac{h_n}{h_{n+1}} &= \frac{(n+1)(p+n)}{(p+q+2n-1)(p+q+2n)}, \\
\frac{h_{n-1}}{h_n} &= \frac{n(p+n-1)}{(p+q+2n-3)(p+q+2n-2)}, \\
\frac{h_{n-1}}{h_{n+1}} &= \frac{n(n+1)(p+n)(p+n-1)}{(p+q+2n-3)(p+q+2n-2)(p+q+2n-1)(p+q+2n)}, \\
\frac{h_{n-2}}{h_n} &= \frac{n(n-1)(p+n-2)(p+n-1)}{(p+q+2n-5)(p+q+2n-4)(p+q+2n-3)(p+q+2n-2)}.
\end{aligned}$$

Applying Lemma 3.4, we have Eq. (2.1) for $n \geq 1$ and Eq. (2.2) for $n \geq 2$. It is easily seen that α_0 and ω_1 satisfy Eq. (2.1) and Eq. (2.2), respectively. Thus we have the assertion. □

Theorem 2.2. *The beta distribution $B(p,q)$ over $[0,\ 1]$ is MRM-applicable for*

$$h_+^{p,q}(x) = {}_2F_1\left(\frac{p+q}{2}, \frac{p+q+1}{2}; p; 4x\right) \tag{2.3}$$

with $\rho(t) = \dfrac{t}{(1+t)^2}$ and $\varphi_+^{p,q}(t) = \dfrac{(1+t)^{p+q}}{1-t}$. Moreover, we have the generating function

$$\psi_+^{p,q}(t,x) = \frac{1-t}{(1+t)^{p+q}}\, {}_2F_1\left(\frac{p+q}{2}, \frac{p+q+1}{2}; p; \frac{4t}{(1+t)^2}x\right).$$

Proof. By Theorem 1.2, $\psi_+^{p,q}(t,x)$ can be rewritten as

$$\begin{aligned}
\psi_+^{p,q}(t,x) &= \frac{1-t}{(1+t)^{p+q}}\, {}_2F_1\left(\frac{p+q}{2}, \frac{p+q+1}{2}; p; \frac{4t}{(1+t)^2}x\right) \\
&= (1-t)\psi^{p,q+1}(t,x) \\
&= (1-t)\sum_{n=0}^{\infty} \frac{(p+q)_{2n}}{(p)_n n!} P_n^{p,q+1}(x) t^n \\
&= 1 + \sum_{n=1}^{\infty} \left(\frac{(p+q)_{2n}}{(p)_n n!} P_n^{p,q+1}(x) - \frac{(p+q)_{2n-2}}{(p)_{n-1}(n-1)!} P_{n-1}^{p,q+1}(x) \right) t^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{(p+q)_{2n-2}}{(p)_n n!} \Big((p+q+2n-2)(p+q+2n-1) P_n^{p,q+1}(x) \\
&\qquad\qquad -n(p+n-1) P_{n-1}^{p,q+1}(x) \Big) t^n.
\end{aligned}$$

Here we see by Eq. (1.7)

$$\begin{aligned}
&(p+q+2n-2)(p+q+2n-1)P_n^{p,q+1}(x) - n(p+n-1)P_{n-1}^{p,q+1}(x) \\
&= \frac{n!(p)_n}{(p+q)_{2n-2}} \sum_{k=0}^{n} \frac{(-1)^{n-k}(p+q)_{n+k}}{k!(n-k)!(p)_k} x^k \\
&\qquad - \frac{n!(p)_n}{(p+q)_{2n-2}} \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}(p+q)_{n+k-1}}{k!(n-k-1)!(p)_k} x^k \\
&= (p+q+2n-2)(p+q+2n-1)x^n \\
&\quad + \frac{n!(p)_n}{(p+q)_{2n-2}} \sum_{k=0}^{n-1} \frac{(-1)^{n-k}(p+q)_{n+k-1}}{k!(n-k)!(p)_k} (p+q+2n-1)x^k \\
&= (p+q+2n-2)(p+q+2n-1)P_n^{p,q}(x).
\end{aligned}$$

Thus $\psi_+^{p,q}(t,x)$ is a generating function of $\{P_n^{p,q}(x)\}$ as

$$\psi_+^{p,q}(t,x) = \sum_{n=0}^{\infty} \frac{(p+q)_{2n}}{(p)_n n!} P_n^{p,q}(x) t^n.$$

□

Theorem 2.3. *For the beta distribution $B(p,q)$ over $[0,\ 1]$, there are only two MRM-factors up to scaling given by*

$$h^{p,q}(x) = {}_2F_1\left(\frac{p+q}{2}, \frac{p+q-1}{2}; p; 4x\right), \quad \varphi^{p,q}(t) = (1+t)^{p+q-1}$$

and

$$h_+^{p,q}(x) = {}_2F_1\left(\frac{p+q}{2}, \frac{p+q+1}{2}; p; 4x\right), \quad \varphi_+^{p,q}(t) = \frac{(1+t)^{p+q}}{1-t}$$

with the common ρ-function $\rho(t) = \dfrac{t}{(1+t)^2}$.

Proof. We will apply a general framework to be proved in Section 3 later. Suppose that $\mu^{p,q}$ is the measure of the beta distribution $B(p,q)$ over $[0,\ 1]$ and that

$$h(x)=\sum_{n=0}^{\infty}h_nx^n,\quad \rho(t)=\sum_{n=0}^{\infty}r_nt^n\quad\text{and}\quad B(t)=\sum_{n=0}^{\infty}b_nt^n. \tag{2.4}$$

We may normalize as $h(0)=h_0=1,\ B(0)=b_0=1$ and $\rho(0)=r_0=0,\rho'(0)=r_1=1$ as seen in Remark 3.5 (ii). Suppose that $\psi(t,x)=B(t)h(\rho(t)x)$ is a generating function of $\{P_n\}$ with Jacobi-Szegö parameters $\{\alpha_n,\omega_n\}$ given by Proposition 2.1.

Define $\{W_n(x)\}$ and $\{W_{n,m}; n-1\ge m\ge 0\}$ by Eq. (3.9) and Eq. (3.10), respectively. Since $\{\alpha_n,\omega_n\}$ are given by Proposition 2.1, recursion formulas in Lemma 3.3 become

$$\begin{aligned}
h_1 &= -\frac{(p+q)b_1}{p},\\
h_{n+1} &= \frac{(b_1+r_2n)(p+q+2n-2)(p+q+2n)h_n^2}{\begin{array}{l}-\big(2n^2+2(p+q-1)n+p(p+q-2)\big)h_n\\ \quad +(b_1+r_2(n-1))(p+q+2n-2)(p+q+2n)h_{n-1}\end{array}}
\end{aligned}\tag{2.5}$$

$$\begin{aligned}
b_{n+1} &= -\frac{\big(2n^2+2(p+q-1)n+p(p+q-2)\big)h_{n+1}}{(p+q+2n-2)(p+q+2n)h_n}b_n\\
&\quad -\frac{n(p+n-1)(q+n-1)(p+q+n-2)h_{n+1}}{(p+q+2n-3)(p+q+2n-2)^2(p+q+2n-1)h_{n-1}}b_{n-1},
\end{aligned}\tag{2.6}$$

$$\begin{aligned}
r_{n+1} &= \frac{b_nh_{n+1}}{h_1h_n}-\sum_{m=1}^{n}b_{n+1-m}r_m\\
&\quad -\frac{2n^2+2(p+q-1)n+p(p+q-2)}{(p+q+2n-2)(p+q+2n)}\,\frac{h_{n+1}}{h_n}\sum_{m=1}^{n}b_{n-m}r_m\\
&\quad -\frac{n(p+n-1)(q+n-1)(p+q+n-2)}{(p+q+2n-3)(p+q+2n-2)^2(p+q+2n-1)}\\
&\qquad \times\frac{h_{n+1}}{h_{n-1}}\sum_{m=1}^{n-1}b_{n-1-m}r_m.
\end{aligned}\tag{2.7}$$

Put $n=1,\ n=2$, and $n=3$ to get

$$h_2=\frac{b_1(b_1+r_2)(p+q)(p+q+2)}{2p(p+1)},\quad b_2=\frac{b_1(b_1+r_2)(p+q)}{2(p+q+1)},$$

$$\begin{aligned}
h_3 &= -\frac{b_1(b_1+r_2)(b_1+2r_2)(p+q)(p+q+2)(p+q+4)}{6p(p+1)(p+2)},\\
b_3 &= \frac{b_1(b_1+r_2)(b_1+2r_2)(p+q)}{6(p+q+3)},\\
r_3 &= \frac{b_1^2-(p+q)b_1r_2+(p+q+1)(p+q+2)r_2^2}{(p+q+1)(p+q+3)}
\end{aligned}$$

and

$$
\begin{aligned}
h_4 &= \frac{b_1(b_1+r_2)(b_1+2r_2)(b_1+3r_2)(p+q)(p+q+2)(p+q+4)(p+q+6)}{24p(p+1)(p+2)(p+3)},\\
b_4 &= \frac{b_1(b_1+r_2)(b_1+2r_2)(b_1+3r_2)(p+q)(p+q+2)}{24(p+q+3)(p+q+5)},\\
r_4 &= \frac{\begin{array}{l}\big(-b_1+(p+q+2)r_2\big)\\ \quad\times\big(4b_1^2-2(p+q-1)b_1r_2+(p+q+1)(p+q+3)r_2^2\big)\end{array}}{(p+q+1)(p+q+3)(p+q+5)}.
\end{aligned}
$$

Then $W_{4,2}$ becomes

$$
W_{4,2} = -\frac{12(2b_1-r_2(p+q-1))(2b_1-r_2(p+q+1))(p+2)(p+3)}{(b_1+2r_2)(b_1+3r_2)(p+q+1)_6}.
$$

Solving $W_{4,3}=0$ in r_2, we have

$$
r_2 = \frac{2b_1}{p+q-1} \quad \text{or} \quad r_2 = \frac{2b_1}{p+q+1}.
$$

By Eqs. (2.5), (2.6), (2.7) and Remark 3.5, $\{h_n, r_n, b_n\}$ are uniquely determined up to scaling for a given value of b_1/r_2. Thus we have only two triples $\{h(x), \rho(t), B(t)\}$'s. From Theorem 1.2 and Proposition 2.1, we know these two triples. In both cases, $r_2 = \rho''(0)/2 = -2$. Since $(\varphi^{p,q})'(0) = p+q-1$ and $(\varphi^{p,q}_+)'(0) = p+q+1$ hold, r_2 satisfies $r_2 = \dfrac{2b_1}{p+q-1}$ for the case of Theorem 1.2 and $r_2 = \dfrac{2b_1}{p+q+1}$ for the case of Theorem 2.2. Thus we conclude the assertion. In the following, we will prove more directly using lemmas in Section 3.

(i) The case of $r_2 = \dfrac{2b_1}{p+q-1}$:

We see that

$$
\begin{aligned}
h_2 &= \frac{b_1^2(p+q)_3}{2(p+q-1)(p)_2}, & b_2 &= \frac{b_1^2(p+q)_1}{2(p+q-1)}, & r_2 &= \frac{2b_1}{p+q-1},\\
h_3 &= -\frac{b_1^3(p+q)_5}{6(p+q-1)^2(p)_3}, & b_3 &= \frac{b_1^3(p+q)_2}{6(p+q-1)^2}, & r_3 &= \frac{3b_1^2}{(p+q-1)^2},\\
h_4 &= \frac{b_1^4(p+q)_7}{24(p+q-1)^3(p)_4}, & b_4 &= \frac{b_1^4(p+q)_3}{24(p+q-1)^3}, & r_4 &= \frac{4b_1^3}{(p+q-1)^3}
\end{aligned}
$$

by Lemma 3.3. Up to scaling, we may assume that

$$
b_1 = -(p+q-1).
$$

Therefore,

$$
\begin{aligned}
h_2 &= \frac{(p+q-1)_4}{2!(p)_2}, & b_2 &= \frac{1}{2!}(p+q-1)_2, & r_2 &= -2,\\
h_3 &= \frac{(p+q-1)_6}{3!(p)_3}, & b_3 &= -\frac{1}{3!}(p+q-1)_3, & r_3 &= 3,\\
h_4 &= \frac{(p+q-1)_8}{4!(p)_4}, & b_4 &= \frac{1}{4!}(p+q-1)_4, & r_4 &= -4.
\end{aligned}
$$

Recursively, we can obtain $(h_{n+1}, b_{n+1}, r_{n+1})$ uniquely by Lemma 3.3. Suppose that

$$h_k = \frac{(p+q-1)_{2k}}{k!(p)_k}, \quad b_k = \frac{(-1)^k (p+q-1)_k}{k!}, \quad r_k = -(-1)^k k \tag{2.8}$$

hold for $n \geq k \geq 1$. The first recursion formula in Lemma 3.3 can be rewritten as

$$h_{n+1} = \frac{(p+q+2n-1)h_n}{(p+q+2n-3)\dfrac{h_{n-1}}{h_n} + \alpha_n}. \tag{2.9}$$

Therefore,

$$\begin{aligned} h_{n+1} &= \frac{(p+q+2n-1)h_n}{\dfrac{n(p+n-1)}{p+q+2n-2} + \dfrac{2n^2+2(p+q-1)n+p(p+q-2)}{(p+q+2n-2)(p+q+2n)}} \\ &= \frac{(p+q+2n-1)(p+q+2n)}{(n+1)(p+n)} \frac{(p+q-1)_{2n}}{n!(p)_n} = \frac{(p+q-1)_{2n+2}}{(n+1)!(p)_{n+1}}. \end{aligned}$$

The second formula in Lemma 3.3 is rewritten as

$$b_{n+1} = -\frac{h_{n+1}}{h_n}\left(\alpha_n + \omega_n \frac{b_{n-1}}{b_n}\frac{h_n}{h_{n-1}}\right) b_n. \tag{2.10}$$

Since

$$\begin{aligned} &\alpha_n + \omega_n \frac{b_{n-1}}{b_n}\frac{h_n}{h_{n-1}} = \alpha_n - \omega_n \frac{(p+q+2n-3)(p+q+2n-2)}{(p+n-1)(p+q+n-2)} \\ =& \left(\frac{2n^2+2(p+q-1)n+p(p+q-2)}{(p+q+2n-2)(p+q+2n)} - \frac{n(q+n-1)}{(p+q+2n-2)(p+q+2n-1)}\right) \\ =& \frac{(p+n)(p+q+n-1)}{(p+q+2n-1)(p+q+2n)}, \end{aligned}$$

we have

$$\begin{aligned} b_{n+1} &= -\frac{(p+q+2n-1)(p+q+2n)}{(n+1)(p+n)} \frac{(p+n)(p+q+n-1)}{(p+q+2n-1)(p+q+2n)} b_n \\ &= -(-1)^n \frac{(p+q+n-1)}{n+1} \frac{(p+q-1)_n}{n!} = (-1)^{n+1} \frac{(p+q-1)_{n+1}}{(n+1)!} \end{aligned}$$

by Eq. (2.10). Since the equations in (2.8) hold for $n \geq k \geq 0$, we have

$$\sum_{k=0}^{m} b_{m-k} r_k = -(-1)^m \sum_{k=0}^{m} k \frac{(p+q-1)_{m-k}}{(m-k)!} = (-1)^{m-1} \frac{(p+q+1)_{m-1}}{(m-1)!}$$

for $n \geq m \geq 1$ by the formula $\displaystyle\sum_{k=0}^{m} \frac{(a)_k}{k!} = \frac{(a+1)_m}{m!}$. By the same reason,

$$\sum_{k=0}^{n} b_{n+1-k} r_k = (-1)^n \left(\frac{(p+q+1)_n}{n!} - (n+1)\right).$$

Since

$$\frac{h_{n+1}}{h_n} = \frac{(p+q+2n-1)(p+q+2n)}{(n+1)(p+n)},$$

$$\frac{h_{n+1}}{h_{n-1}} = \frac{(p+q+2n-3)(p+q+2n-2)(p+q+2n-1)(p+q+2n)}{n(n+1)(p+n-1)(p+n)},$$

we have

$$\begin{aligned}
r_{n+1} &= \frac{h_{n+1}b_n}{h_n h_1} - (-1)^n\left(\frac{(p+q+1)_n}{n!} - (n+1)\right) \\
&\quad -(-1)^{n-1}\alpha_n \frac{h_{n+1}}{h_n}\frac{(p+q+1)_{n-1}}{(n-1)!} - (-1)^{n-2}\omega_n \frac{h_{n+1}}{h_{n-1}}\frac{(p+q+1)_{n-2}}{(n-2)!} \\
&= -(-1)^{n+1}(n+1) + (-1)^n \frac{(p+q+1)_{n-2}}{(n+1)!(p+n)(p+q+2n-2)} \times \\
&\quad \Big(p(p+q+2n-2)(p+q+2n-1)(p+q+2n) \\
&\quad -(n+1)(p+n)(p+q+n-1)(p+q+n)(p+q+2n-2) \\
&\quad +n(p+q+n-1)(p+q+2n-1)\big(2n^2+p(p+q-2)+2n(p+q-1)\big) \\
&\quad -n(n-1)(q+n-1)(p+q+n-2)(p+q+2n)\Big) \\
&= -(-1)^{n+1}(n+1).
\end{aligned}$$

By induction, we conclude that equalities in (2.8) hold for any n. Thus we have

$$\begin{aligned}
h(x) &= \sum_{n=0}^{\infty}\frac{(p+q-1)_{2n}}{(p)_n n!}x^n = {}_2F_1\left(\frac{p+q-1}{2}, \frac{p+q}{2}; p; 4x\right), \\
r(t) &= \sum_{n=0}^{\infty}(-1)^{n+1}nt^n = \frac{t}{(1+t)^2}, \\
B(t) &= \sum_{n=0}^{\infty}(-1)^k(p+q-1)_n t^n = \frac{1}{(1+t)^{p+q-1}}.
\end{aligned}$$

Hence $\psi(t,x) = \frac{1}{(1+t)^{p+q-1}}\,{}_2F_1\left(\frac{p+q-1}{2}, \frac{p+q}{2}; p; \frac{4t}{(1+t)^2}x\right)$ and it is a generating function of the beta distribution $B(p,q)$.

(ii) The case of $r_2 = \frac{2b_1}{p+q+1}$:

Similar to the case (i), we may assume that

$$b_1 = -(p+q+1).$$

Then

$$\begin{aligned}
h_2 &= \frac{(p+q)_4}{2!(p)_2}, & b_2 &= \frac{1}{2!}(p+q)(p+q+3), & r_2 &= -2, \\
h_3 &= \frac{(p+q)_6}{3!(p)_3}, & b_3 &= -\frac{1}{3!}(p+q)_2(p+q+5), & r_3 &= 3, \\
h_4 &= \frac{(p+q)_8}{4!(p)_4}, & b_4 &= \frac{1}{4!}(p+q)_3(p+q+7), & r_4 &= -4.
\end{aligned}$$

Suppose that

$$h_k = \frac{(p+q)_{2k}}{k!(p)_k},\ b_k = \frac{(-1)^k(p+q)_{k-1}(p+q+2k-1)}{k!},\ r_k = -(-1)^k k \tag{2.11}$$

hold for $n \geq k \geq 1$. Then by Eqs.(2.1) and (2.5),

$$\begin{aligned} h_{n+1} &= \frac{(p+q+2n+1)h_n}{(p+q+2n-1)\frac{h_{n-1}}{h_n}+\alpha_n} \\ &= \frac{(p+q+2n-1)(p+q+2n)}{(n+1)(p+n)}h_n = \frac{(p+q)_{2n+2}}{(n+1)!(p)_{n+1}}. \end{aligned}$$

By Eqs.(2.1), (2.2), (2.6) and (2.10),

$$\begin{aligned} b_{n+1} &= -\frac{(p+q+2n)(p+q+2n+1)b_n}{(n+1)(p+n)}\frac{(p+n)(p+q+n-1)}{(p+q+2n-1)(p+q+2n)} \\ &= -\frac{(p+q+n-1)(p+q+2n+1)}{(n+1)(p+q+2n-1)}b_n \\ &= \frac{(-1)^{n+1}(p+q)_n(p+q+2n+1)}{(n+1)!}. \end{aligned}$$

By Eq. (2.11) and $b_k = (-1)^k\big((p+q)_k + k(p+q)_{k-1}\big)$, we have

$$\begin{aligned} \sum_{k=1}^{m} b_{m-k}r_k &= -(-1)^m\sum_{k=1}^{m}\frac{k(p+q)_{m-k}}{(m-k)!} - (-1)^m\sum_{k=1}^{m-1}\frac{k(p+q)_{m-k-1}}{(m-k-1)!} \\ &= \frac{(-1)^{m-1}(p+q+2m-1)(p+q+2)_{m-2}}{(m-1)!} \end{aligned}$$

for $n \geq m \geq 1$ and

$$\sum_{k=1}^{n} b_{n+1-k}r_k = (-1)^{n+1}(n+1) + \frac{(-1)^n(p+q+2n+1)(p+q+2)_{n-1}}{n!}.$$

By Eqs.(2.1), (2.2) and (2.7),

$$\begin{aligned} r_{n+1} &= (-1)^n(n+1) - \frac{(-1)^n(p+q+2n+1)(p+q+2)_{n-1}}{n!} \\ &\quad + \frac{h_{n+1}b_n}{h_1h_n} + (-1)^n\frac{\alpha_n h_{n+1}}{h_n}\frac{(p+q+2n-1)(p+q+2)_{n-2}}{(n-1)!} \\ &\quad - (-1)^n\frac{\omega_n h_{n+1}}{h_{n-1}}\frac{(p+q+2n-3)(p+q+2)_{n-3}}{(n-2)!} \\ &= -(-1)^{n+1}(n+1). \end{aligned}$$

By induction, we can conclude that the equalities in Eq. (2.11) hold for any n. Then we see that

$$\begin{aligned} h(x) &= \sum_{n=0}^{\infty}\frac{(p+q)_{2n}}{(p)_n n!}x^n = {}_2F_1\left(\frac{p+q}{2}, \frac{p+q+1}{2}; p; 4x\right), \\ r(t) &= \sum_{n=0}^{\infty}(-1)^{n+1}nt^n = \frac{t}{(1+t)^2}, \\ B(t) &= \sum_{n=0}^{\infty}(-1)^n(p+q)_{n-1}(p+q+2n-1)t^n = \frac{1-t}{(1+t)^{p+q}}. \end{aligned}$$

It follows that $\psi(t,x) = \dfrac{1-t}{(1+t)^{p+q}}\, {}_2F_1\left(\dfrac{p+q}{2}, \dfrac{p+q+1}{2}; p; \dfrac{4t}{(1+t)^2}x\right)$, which is a generating function of the beta distribution $B(p,q)$ in Theorem 2.2. $\square$

3. A General Framework

By Shohat's theorem [16], a set of polynomials $\{P_n\}$ with leading coefficient 1 satisfies the recursion relation (1.2) with Jcobi-Szegö parameters $\{\alpha_n, \omega_n\}$ satisfying $P_{-1}(x) = 0, \alpha_{-1} = 0, \omega_0 = 1$, if and only if they are orthogonal polynomials with respect to a singed measure μ of bounded variation.

We now assume that $\mu(\mathbb{R}) = 1$ and that

$$h(x) = \sum_{n=0}^{\infty} h_n x^n, \quad \rho(t) = \sum_{n=0}^{\infty} r_n t^n \quad \text{and} \quad B(t) = \sum_{n=0}^{\infty} b_n t^n. \tag{3.1}$$

We may normalize as $h(0) = h_0 = 1,\ B(0) = b_0 = 1$ and $\rho(0) = r_0 = 0, \rho'(0) = r_1 = 1$. For convenience, we put $b_{-1} = 0$. Suppose that $\psi(t,x) = B(t)h(\rho(t)x)$ is a generating function of $\{P_n\}$;

$$\psi(t,x) = B(t)h(\rho(t)x) = \sum_{n=0}^{\infty} h_n P_n(x) t^n. \tag{3.2}$$

Then

$$B(t) = \frac{1}{\varphi(t)}, \quad \theta(t) = \int_{\mathbb{R}} h(tx)\, d\mu(x) \quad \text{and} \quad \varphi(t) = \theta(\rho(t)). \tag{3.3}$$

Some terms of Eq. (3.2) (equivalently of Eq. (1.3) with $r_1 = 1$) are

$$\psi(t,x) = \sum_{n=0}^{5} h_n P_n(x) t^n + O(t^6)$$

with

$$\begin{aligned}
P_0(x) &= 1, \quad P_1(x) = x + \frac{b_1}{h_1}, \quad P_2(x) = x^2 + \frac{h_1(b_1 + r_2)}{h_2}x + \frac{b_2}{h_2}, \\
P_3(x) &= x^3 + \frac{h_2(b_1 + 2r_2)}{h_3}x^2 + \frac{h_1(b_2 + b_1 r_2 + r_3)}{h_3}x + \frac{b_3}{h_3}, \\
P_4(x) &= x^4 + \frac{h_3(b_1 + 3r_2)}{h_4}x^3 + \frac{h_2\left(b_2 + 2b_1 r_2 + r_2^2 + 2r_3\right)}{h_4}x^2 \\
&\quad + \frac{h_1(b_3 + b_2 r_2 + b_1 r_3 + r_4)}{h_4}x + \frac{b_4}{h_4}, \\
P_5(x) &= x^5 + \frac{h_4(b_1 + 4r_2)}{h_5}x^4 + \frac{h_3\left(b_2 + 3b_1 r_2 + 3r_2^2 + 3r_3\right)}{h_5}x^3 \\
&\quad + \frac{h_2\left(b_3 + 2b_2 r_2 + b_1 r_2^2 + 2b_1 r_3 + 2r_2 r_3 + 2r_4\right)}{h_5}x^2 \\
&\quad + \frac{h_1(b_4 + b_3 r_2 + b_2 r_3 + b_1 r_4 + r_5)}{h_5}x + \frac{b_5}{h_5}.
\end{aligned}$$

Lemma 3.1. *Let* $P_n(x) = \sum_{m=0}^{n} c_{n,m} x^m$ *and* $B(t)\rho^m(t) = \sum_{n=m}^{\infty} B_{m,n} t^n,\ m \geq 0.$ *Then*

$$P_n(x) = \frac{b_n}{h_n} + \sum_{m=1}^{n} \frac{h_m}{h_n} B_{m,n} x^m \tag{3.5}$$

with

$$B_{m,n} = \sum_{k=m}^{n} b_{n-k} \sum_{k_1+\cdots+k_m=k} r_{k_1} \cdots r_{k_m} \tag{3.6}$$

and

$$c_{n,m} = \begin{cases} \dfrac{b_n}{h_n} & \textit{for} \ \ m = 0, \\ \dfrac{h_m}{h_n} B_{m,n} & \textit{for} \ \ m \geq 1. \end{cases} \tag{3.7}$$

In particular,

$$\begin{aligned} c_{n,0} &= \frac{b_n}{h_n},\ c_{n,1} = \frac{h_1}{h_n} \sum_{k=1}^{n} b_{n-k} r_k,\ c_{n,n-1} = \frac{h_{n-1}}{h_n} \left(b_1 + (n-1) r_2\right), \\ c_{n,n-2} &= \frac{h_{n-2}}{2h_n} \big(2b_2 + (n-2)\big(2b_1 r_2 + 2r_3 + (n-3) r_2^2\big)\big). \end{aligned}$$

Proof. It is easily seen that

$$\rho(t)^m = \sum_{k=m}^{\infty} r_{m,k} t^k, \quad r_{m,k} = \sum_{k_1+\cdots+k_m=k} r_{k_1} \cdots r_{k_m} \quad \text{for} \ \ k \geq m,$$

$$B_{m,n} = \sum_{k=m}^{n} b_{n-k} r_{m,k} = \sum_{k=m}^{n} b_{n-k} \sum_{k_1+\cdots+k_m=k} r_{k_1} \cdots r_{k_m}. \tag{3.8}$$

By convention, let $B_{m,n} = 0 \ \ (m > n)$ and $r_{m,k} = 0 \ \ (m > k)$. Then

$$\begin{aligned} h(\rho(t)x) &= \sum_{m=0}^{\infty} h_m \rho(t)^m x^m = \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} h_m r_{m,k} t^k x^m \\ &= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} h_m r_{m,k} x^m \right) t^k \end{aligned}$$

and

$$\begin{aligned} B(t)h(\rho(t)x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_{n-k} \left(\sum_{m=0}^{k} h_m r_{m,k} x^m \right) t^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{k=m}^{n} b_{n-k} h_m r_{m,k} x^m \right) t^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} h_m \left(\sum_{k=m}^{n} b_{n-k} r_{m,k} \right) x^m \right) t^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} h_m B_{m,n} x^m \right) t^n. \end{aligned}$$

Since $h_m B_{m,n} = h_n$ holds for $m = n$, we have

$$P_n(x) = \sum_{m=0}^{n} \left(\sum_{k=m}^{n} \frac{h_m}{h_n} b_{n-k} r_{m,k} \right) x^m.$$

Thus we have Eq. (3.5) and Eq. (3.6). Therefore,

$$\begin{aligned}
c_{n,0} &= b_n, \\
c_{n,1} &= \frac{h_1}{h_n} \sum_{k=1}^{n} b_{n-k} r_k, \\
c_{n,m} &= \frac{h_m}{h_n} B_{m,n} \quad (n-2 \geq m \geq 2), \\
c_{n,n-1} &= \frac{h_{n-1}}{h_n} \sum_{k=n-1}^{n} b_{n-k} \sum_{k_1+\cdots+k_{n-1}=k} r_{k_1} \cdots r_{k_{n-1}} \\
&= \frac{h_{n-1}}{h_n} \left(b_1 + (n-1) r_2 \right), \\
c_{n,n-2} &= \frac{h_{n-2}}{h_n} \sum_{k=n-2}^{n} b_{n-k} \sum_{k_1+\cdots+k_{n-2}=k} r_{k_1} \cdots r_{k_m} \\
&= \frac{h_{n-2}}{h_n} \left(b_2 \sum_{k_1+\cdots+k_{n-2}=n-2} r_{k_1} \cdots r_{k_{n-2}} \right. \\
&\qquad \left. + b_1 \sum_{k_1+\cdots+k_{n-2}=n-1} r_{k_1} \cdots r_{k_{n-1}} + \sum_{k_1+\cdots+k_{n-2}=n} r_{k_1} \cdots r_{k_n} \right) \\
&= \frac{h_{n-2}}{h_n} \left(b_2 + b_1 (n-2) r_2 + \left((n-2) r_3 + \frac{(n-2)(n-3) r_2^2}{2} \right) \right) \\
&= \frac{h_{n-2}}{2h_n} \left(2b_2 + 2(n-2)(r_2 b_1 + r_3) + (n-2)(n-3) r_2^2 \right).
\end{aligned}$$

□

Define

$$W_n(x) = P_n(x) - (x - \alpha_{n-1}) P_{n-1}(x) + \omega_{n-1} P_{n-2}(x) \tag{3.9}$$

and let $W_{n,m}$ be the m-th coefficient of $W_n(x)$ for $n > m \geq 0$. Then

$$W_{n+1,m} = c_{n+1,m} - c_{n,m-1} + \alpha_n c_{n,m} + \omega_n c_{n-1,m} \tag{3.10}$$

for $n \geq m \geq 0$. Since $W_n(x) = 0$ must hold by the recursive relation (1.2), all $W_{n,m}$ must vanish. Thus by Lemma 3.1, we have the next lemma.

Lemma 3.2.

$$\begin{aligned}
W_{n+1,0} &= \frac{b_{n+1}}{h_{n+1}} + \alpha_n \frac{b_n}{h_n} + \omega_n \frac{b_{n-1}}{h_{n-1}}, \\
W_{n+1,1} &= \frac{h_1}{h_{n+1}} r_{n+1} + \frac{h_1}{h_{n+1}} \sum_{k=1}^{n} b_{n+1-k} r_k - \frac{b_n}{h_n} \\
&\qquad + \alpha_n \frac{h_1}{h_n} \sum_{k=1}^{n} b_{n-k} r_k + \omega_n \frac{h_1}{h_{n-1}} \sum_{k=1}^{n-1} b_{n-1-k} r_k,
\end{aligned}$$

$$
\begin{aligned}
W_{n+1,m} &= \frac{h_m}{h_{n+1}}B_{m,n+1} - \frac{h_{m-1}}{h_n}B_{m-1,n-1} + \alpha_n\frac{h_m}{h_n}B_{m,n-1} + \omega_n\frac{h_m}{h_{n-1}}B_{m,n-2},\\
W_{n+1,n-1} &= \frac{h_{n-1}}{2h_{n+1}}(2b_2 + 2(n-1)(b_1r_2 + r_3) + (n-1)(n-2)r_2^2)\\
&\quad -\frac{h_{n-2}}{2h_n}\big(2b_2 + 2(n-2)(b_1r_2 + r_3) + (n-2)(n-3)r_2^2\big)\\
&\quad +\frac{h_{n-1}}{h_n}(b_1 + (n-1)r_2)\alpha_n + \omega_n,\\
W_{n+1,n} &= \frac{h_n}{h_{n+1}}(b_1 + nr_2) - \frac{h_{n-1}}{h_n}(b_1 + (n-1)r_2) + \alpha_n.
\end{aligned}
$$

If μ is MRM-applicable for $h(x)$, then $W_{n,m} = 0$ for $n-1 \ge m \ge 0$.

Lemma 3.2 yields the following lemma for given Jacobi-Szegö parameters.

Lemma 3.3. *For given $\{\alpha_n, \omega_n\}$ and $\{b_1, r_2, r_3\}$, we have the recursion formulas:*

$$
\begin{aligned}
h_{n+1} &= \frac{h_n^2(b_1 + nr_2)}{h_{n-1}\big(b_1 + (n-1)r_2\big) - h_n\alpha_n},\\
b_{n+1} &= -\frac{h_{n+1}}{h_n}\alpha_n b_n - \frac{h_{n+1}}{h_{n-1}}\omega_n b_{n-1},\\
r_{n+1} &= \frac{b_n h_{n+1}}{h_1 h_n} - \sum_{m=1}^{n} b_{n+1-m}r_m - \frac{h_{n+1}}{h_n}\alpha_n\sum_{m=1}^{n} b_{n-m}r_m\\
&\quad -\frac{h_{n+1}}{h_{n-1}}\omega_n\sum_{m=1}^{n-1} b_{n-1-m}r_m,
\end{aligned}
$$

for $n \ge 1$ if $h_{n-1}\big(b_1 + (n-1)r_2\big) - h_n\alpha_n \neq 0$. $h_1 = -\dfrac{b_1}{\alpha_0}$, if $\alpha_0 \neq 0$.

In this paper, we have discussed the problem of determining all possible MRM-factors for which given special measures are MRM-applicable. For this purpose, we apply Lemma 3.3 for given Jacobi-Szegö parameters. In the previous papers ([9] [13]), we have determined MRM-applicable measures for given MRM-factors $h(x) = e^x$ and $(1-x)^{-\kappa}$. For such a problem, the following lemma is very useful and will be used in a forthcoming paper. Actually, we will see that Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ are determined from $h(x)$ and fixed constants $\{b_1, b_2, r_2, r_3\}$. Furthermore, b_n and r_n are obtained recursively by the next lemma.

Lemma 3.4. *For given $h(x)$ and fixed constants $\{b_1, b_2, r_2, r_3\}$, Jacobi-Szegö parameters are uniquely determined by*

$$
\begin{aligned}
\alpha_n &= -\frac{h_n}{h_{n+1}}(b_1 + nr_2) + \frac{h_{n-1}}{h_n}(b_1 + (n-1)r_2) \quad \textit{for} \ \ n \ge 1,\\
\omega_n &= \frac{h_{n-1}}{2h_{n+1}}\big(2(b_1^2 - b_2 + b_1r_2) + 2(n-1)(b_1r_2 - r_3) + (n-1)(n+2)r_2^2\big)\\
&\quad +\frac{h_{n-2}}{2h_n}\left(2b_2 + 2(n-2)(b_1r_2 + r_3) + (n-2)(n-3)r_2^2\right)\\
&\quad -\left(\frac{h_{n-1}}{h_n}(b_1 + (n-1)r_2)\right)^2 \quad \textit{for} \ \ n \ge 2,
\end{aligned}
$$

$\alpha_0 = -\dfrac{b_1}{h_1}$, $\omega_0 = 1$, $\omega_1 = \dfrac{b_1^2 - b_2 + b_1 r_2}{h_2} - \dfrac{b_1^2}{h_1^2}$. *Furthermore, recursion formulas are given as*

$$\begin{aligned} b_{n+1} &= -\frac{h_{n+1}}{h_n}\alpha_n b_n - \frac{h_{n+1}}{h_{n-1}}\omega_n b_{n-1} \quad \textit{for} \quad n \geq 2, \\ r_{n+1} &= \frac{b_n h_{n+1}}{h_1 h_n} - \sum_{m=1}^{n-1} b_{n-m} r_m - \frac{h_{n+1}}{h_n}\alpha_n \sum_{m=1}^{n} b_{n-m} r_m \\ &\quad - \frac{h_{n+1}}{h_{n-1}}\omega_n \sum_{m=1}^{n-1} b_{n-1-m} r_m \quad \textit{for} \quad n \geq 3. \end{aligned}$$

Proof. Let $W_{n+1,m}$ be as in Lemma 3.2. Solving $W_{n+1,0} = 0$ in h_{+1} and $W_{n+1,1} = 0$ in r_{+1}, we obtain the last two formulas. Solving $W_{n+1,n} = 0$ and $W_{n+1,n-1} = 0$ in (α_n, ω_n), we get the first two formulas. $\square$

Lemma 3.3 (or Lemma 3.4) does not mean the existence of $(h(x),\ B(t),\ \rho(t))$ (or $(B(t), \rho(t), \{\alpha_n, \omega_n\}$, respectively,) satisfying Eq. (1.2) and Eq. (3.2) for all $\{b_1, r_2, r_3\}$ (or $\{b_1, b_2, r_1, r_3\}$, respectively). Classification of possible parameters are our problem.

Remark 3.5. (i) We have normalized the quantities so that

$$h_0 = 1, \quad b_0 = 1, \quad r_1 = 1. \tag{3.11}$$

This does not give any essential restriction. Since $\theta(0) = h(0)$, we have $B(0)h(0) = b_0 h_0 = 1$. Suppose that $\psi(t,x) = B(t)h(\rho(t)x)$ is a generating function of orthogonal polynomials. If $h_0 \neq 1$, put $\widehat{h}(x) = \dfrac{h(x)}{h(0)}$ and $\widehat{B}(t) = h(0)B(t)$. Then $\widehat{\psi}(t,x) = \widehat{B}(t)\widehat{h}(\rho(t)x) = \psi(t,x)$ is a generating function satisfying Eq. (3.11).

If $r_1 \neq 1$, then put $\widehat{\rho}(t) = \rho(\dfrac{t}{r_1})$ and $\widehat{b}(t) = B(\dfrac{t}{r_1})$. Then $\widehat{\psi}(t,x) = \widehat{B}(t)h(\widehat{\rho}(t)x)$ is a generating function satisfying Eq. (3.11).

(ii) Suppose that $\psi(t,x) = B(t)h(\rho(t)x)$ is a generating function of $\{P_n\}$. For given $\widetilde{h}_1,\ \widetilde{r}_1 \neq 0$, scale transforms

$$\widetilde{h}(x) = h\Big(\frac{\widetilde{h}_1}{h_1}x\Big), \quad \widetilde{\rho}(t) = \frac{h_1}{\widetilde{h}_1}\rho\Big(\frac{\widetilde{h}_1\widetilde{r}_1}{h_1 r_1}t\Big) \quad \text{and} \quad \widetilde{B}(t) = B\Big(\frac{\widetilde{h}_1\widetilde{r}_1}{h_1 r_1}t\Big) \tag{3.12}$$

give a modified generating function

$$\widetilde{\psi}(t,x) = \widetilde{B}(t)\widetilde{h}(\widetilde{\rho}(t)x) = \psi\Big(\frac{\widetilde{h}_1\widetilde{r}_1}{h_1 r_1}t, x\Big)$$

satisfying $\widetilde{h}'(0) = \widetilde{h}_1,\ \widetilde{r}'(0) = \widetilde{r}_1$.

4. An Application

In this section, we apply Lemma 3.4 to an example in order to better understand the lemma. We first make the following remark.

Remark 4.1. For given $h(x)$, we can discuss as follows in generic cases. By Lemma 3.4, we can obtain $\{\alpha_n, \omega_n : n \geq 0\}$, $\{b_n : n \geq 3\}$ and $\{r_n : n \geq 4\}$, recursively. From the condition $W_{5,2} = 0$, we get r_3. Thus we determine all parameters except for b_1, b_2 and r_2. Those parameters satisfy

$$W_{n,m} = 0 \text{ for } 5 \geq n \geq m \geq 0, \quad W_{6,m} = 0 \text{ for } m = 0, 1, 2, 4, 5,$$
$$W_{n,m} = 0 \text{ for } m = 0, 1, n-2, n-1.$$

The parameters must satisfy the over determining equations $W_{n,m} = 0$ for any $n > m \geq 0$. It is not simple to determine b_1, b_2 and r_2, because we must solve non-linear equations.

The case $h(x) = (1-x)^{-1}$ is extremal and is not included by Remark 4.1. However, it is a very nice example for determining Jacobi-Szegö parameters, $\rho(t)$ and $B(t)$ by Lemmas 3.2 and 3.4. Since

$$h(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

we see that $h_n = 1$. By Lemma 3.4, we see that $\alpha_0 = -b_1$, $\omega_0 = 1$ and

$$\begin{aligned}
\alpha_n &= -(b_1 + nr_2) + (b_1 + (n-1)r_2) = -r_2 \quad (n \geq 1), \\
\omega_n &= \frac{1}{2}\big(2(b_1^2 - b_2 + b_1 r_2) + 2(n-1)(b_1 r_2 - r_3) + (n-1)(n+2)r_2^2\big) \\
&\quad +\big(2b_2 + 2(n-2)(b_1 r_2 + r_3) + (n-2)(n-3)r_2^2\big) - \big(b_1 + (n-1)r_2\big)^2 \\
&= r_2^2 - r_3 \quad (n \geq 1).
\end{aligned}$$

By Lemma 3.2,

$$W_{n,m} = B_{m,n} - B_{m-1,n-1} - r_2 B_{m,n-1} + (r_2^2 - r_3) B_{m,n-2}$$

vanish for $n \geq 2$ and so

$$\begin{array}{lllllllll}
B_{0,0} & = & 1, & B_{0,1} & = & b_1, & B_{0,2} & = & b_2, \\
B_{1,0} & = & 0, & B_{1,1} & = & 1, & B_{1,2} & = & b_1 + r_2, \\
B_{2,0} & = & 0, & B_{2,1} & = & 0, & B_{2,2} & = & 1.
\end{array}$$

Therefore,

$$\sum_{n=3}^{\infty} \frac{B_{m,n}}{n!} t^n = \begin{cases} B(t)\rho^m(t) - \big(1 + b_1 t + b_2 t^2\big) & \text{if } m = 0, \\ B(t)\rho^m(t) - t - (b_1 + r_2)t^2 & \text{if } m = 1, \\ B(t)\rho^m(t) - t^2 & \text{if } m = 2, \\ B(t)\rho^m(t) & \text{if } m \geq 3. \end{cases}$$

Since

$$\begin{aligned}
\sum_{n=3}^{\infty} W_{n,m} t^n &= \sum_{n=3}^{\infty} B_{m,n} t^n - \sum_{n=3}^{\infty} B_{m-1,n-1} t^n - r_2 \sum_{n=3}^{\infty} B_{m,n-1} t^n \\
&\quad +(r_2^2 - r_3) \sum_{n=3}^{\infty} B_{m,n-2} t^n \\
&= \sum_{n=3}^{\infty} B_{m,n} t^n - t \sum_{n=2}^{\infty} B_{m-1,n} t^n - r_2 t \sum_{n=2}^{\infty} B_{m,n} t^n
\end{aligned}$$

$$+(r_2^2 - r_3)t^2 \sum_{n=1}^{\infty} B_{m,n} t^n = 0,$$

we must have

$$\begin{aligned} & B(t) - r_2 t B(t) + (r_2^2 - r_3)t^2 B(t) \\ = \quad & (1 + b_1 t + b_2 t^2) - r_2 t(1 + b_1 t) + (r_2^2 - r_3)t^2. \end{aligned}$$

Hence

$$B(t) = \frac{1 + (b_1 - r_2)t + (b_2 - b_1 r_2 + r_2^2 - r_3)t^2}{1 - r_2 t + (r_2^2 - r_3)t^2}. \tag{4.1}$$

For $m \geq 3$, we have

$$B(t)\rho(t)^m - r_2 t B(t)\rho(t)^m + (r_2^2 - r_3)t^2 B(t)\rho(t)^m = tB(t)\rho(t)^{m-1}.$$

This implies that

$$\rho(t) = \frac{t}{1 - r_2 t + (r_2^2 - r_3)t^2}. \tag{4.2}$$

For these $B(t)$ and $\rho(t)$ with $m = 1$ and $m = 2$, we get

$$\begin{aligned} & B(t)\rho(t) - tB(t) - r_2 t B(t)\rho(t) + (r_2^2 - 2r_3)t^2 B(t)\rho(t) \\ = \quad & t + (b_1 + r_2)t^2 - t(1 + b_1 t) - r_2 t^2 = 0 \end{aligned}$$

and

$$B(t)\rho(t)^2 - tB(t)\rho(t) - r_2 t B(t)\rho(t)^2 + (r_2^2 - r_3)t^2 B(t)\rho(t)^2 = t^2 - t^2 = 0.$$

The results in Eq. (4.1) and Eq. (4.2) are the same as that in Eq. (1.4). However, it is not easy to determine all corresponding probability measures. We need hard calculus to obtain the explicit forms (1.5) of the corresponding densities as in [11].

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Izumi Kubo: Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan

E-mail address: izumi_kubo@ccv.ne.jp