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A NEW APPROACH TO q-GENOCCHI NUMBERS AND POLYNOMIALS

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Abstract. In this paper, new q-analogs of Genocchi numbers and polynomials are defined. Some important arithmetic and combinatoric relations are given, in particular, connections with q-Bernoulli numbers and polynomials are obtained.

1. Introduction, definitions and notation

Carlitz has introduced the q-Bernoulli numbers and polynomials in [1]. Srivastava and Pinter proved some relations and theorems between the Bernoulli polynomials and Euler polynomials in [22]. They also gave some generalizations of these polynomials. In [6, 7, 10, 11, 12, 14, 15, 16, 17], Kim et al. investigated some properties of the q-Euler polynomials and Genocchi polynomials. They gave some recurrence relations. In [2], Cenkci et al. gave the q-extension of Genocchi numbers in a different manner. In [13], Kim gave a new concept for the q-Genocchi numbers and polynomials. In [20], Simsek et al. investigated the q-Genocchi zeta function and l-function by using generating functions and Mellin transformation.

By using exponential function $e_q(x)$, Hegazi and Mansour [5] defined q-Bernoulli polynomials by means of

$$\sum_{n=0}^{\infty} B_n(x,q)\frac{z^n}{[n]_q!} = \frac{z}{(1-q)\left(e^{\frac{z}{1-q}}-1\right)}e_q(zx).$$

They proved some distribution relations as well. In [9], Kim gave q-Euler polynomials with the help of the exponential function $e_q(x)$ as

$$\sum_{n=0}^{\infty} E_{n,q}(x)\frac{z^n}{[n]_q!} = \frac{[2]_q}{e_q(z)+1}e_q(zx).$$

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In this paper we define new q-Genocchi numbers and polynomials by employing quantum calculus identities. We also give some properties such as recurrence relations and show the connections with q-Bernoulli numbers and polynomials. These relations, in fact, exhibit the connections between other papers on related subjects.

Let $q \in (0,1)$ and define the q-shifted factorials by (cf. [4])

$$(a;q)_0 = 1,\ (a;q)_\infty = \prod_{i=0}^{\infty} \left(1 - aq^i\right).$$

Two q-exponential functions are defined by the following relations (cf. [3, 8, 19, 21]):

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty} \quad \text{and}$$
$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{(q;q)_n} = (-z;q)_\infty,$$

where $z \in \mathbb{C}$. These functions satisfy the following basic equalities (cf. [18]):

$$e_q(z) E_q(-z) = 1,\ e_q(qz) = (1-z) e_q(z),\ E_q(z) = (1+z) E_q(qz),$$

in particular, for the q-commuting variables x and y such that $xy = qyx$,

$$e_q(x+y) = e_q(x) e_q(y),\ \ E_q(x+y) = E_q(x) E_q(y). \tag{1.1}$$

Note that $\lim_{q\to 1} e_q((1-q)z) = e^z = \lim_{q\to 1} E_q((1-q)z)$.

Consider an arbitrary function $f(x)$. Its q-differential is

$$d_q f(x) = f(qx) - f(x).$$

The q-derivative operator D_q is defined by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(x) - f(qx)}{x - qx},$$

where $x \neq 0$. Note that $\lim_{q\to 1} D_q f(x) = \frac{df(x)}{dx}$. Suppose $0 < a < b$. The definite q-integral (also known as Jackson integral) is defined as

$$\int_0^b f(x)\, d_q x = (1-q)\, b \sum_{j=0}^{\infty} q^j f\left(q^j b\right) \quad \text{and}$$
$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x.$$

An important concept of the q-integration theory is the Fundamental Theorem of q-integration:

Theorem 1.1. *If $f'(x)$ exists in a neighborhood of $x=0$ and is continuous at $x=0$, where $f'(x)$ denotes the ordinary derivative of $f(x)$, we have*

$$\int_a^b D_q f(x)\, d_q x = f(b) - f(a).$$

The q-analogue of the factorial is defined by

$$[n]_q! = \begin{cases} 1, & \text{if } n=0; \\ [n]_q [n-1]_q \cdots [2]_q [1]_q, & \text{if } n=1,2,3,\ldots, \end{cases}$$

where $[n]_q$ is the quantum number which is defined by

$$[n]_q = \frac{1-q^n}{1-q}.$$

The q-binomial coefficient $\binom{n}{k}_q$ is defined by

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \frac{[n]_q!}{[k]_q! [n-k]_q!}$$

for $k=0,1,2,\ldots$.

2. q-Genocchi numbers and polynomials

The classical Genocchi numbers G_n and polynomials $G_n(x)$ are defined by means of the generating functions

$$\sum_{n=0}^{\infty} G_n \frac{z^n}{n!} = \frac{2z}{e^z+1} \quad \text{and} \quad \sum_{n=0}^{\infty} G_n(x) \frac{z^n}{n!} = \frac{2z}{e^z+1} e^{zx}$$

for $|z| < \pi$, respectively. Note that the following relation between Genocchi polynomials and numbers can directly be obtained from the definition above:

$$G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}.$$

These numbers and polynomials are closely related to other special numbers and polynomials such as

$$G_n = 2(1-2^n) B_n,\ G_{n+1}(x) = (n+1) E_n(x),$$

where

$$\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{e^z-1} \quad \text{and} \quad \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{z}{e^z-1} e^{zx}$$

are Bernoulli numbers and polynomials, and

$$\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2}{e^z+1} e^{zx}$$

are Euler polynomials.

By using q-exponential function $e_q(z)$, we define new q-Genocchi numbers and polynomials as follows:

Definition. We define q-Genocchi polynomials $G_n(x;q)$ as

$$\sum_{n=0}^{\infty} G_n(x;q)\frac{z^n}{[n]_q!} = \frac{2z}{(1-q)\left(e^{\frac{z}{1-q}}+1\right)}e_q(zx). \tag{2.1}$$

For $x=0$, $G_n(0;q) = G_n(q)$ are q-Genocchi numbers, thus

$$\sum_{n=0}^{\infty} G_n(q)\frac{z^n}{[n]_q!} = \frac{2z}{(1-q)\left(e^{\frac{z}{1-q}}+1\right)}.$$

Note that $\lim_{q\to 1} G_n(x;q) = G_n(x)$ and $\lim_{q\to 1} G_n(q) = G_n$. This definition is motivated from Hegazi and Mansour [5]. In that work, q-Bernoulli polynomials $B_n(x;q)$ are defined by

$$\sum_{n=0}^{\infty} B_n(x;q)\frac{z^n}{[n]_q!} = \frac{z}{(1-q)\left(e^{\frac{z}{1-q}}-1\right)}e_q(zx). \tag{2.2}$$

The values of $B_n(x;q)$ at $x=0$ are called q-Bernoulli numbers, that is,

$$\sum_{n=0}^{\infty} B_n(q)\frac{z^n}{[n]_q!} = \frac{z}{(1-q)\left(e^{\frac{z}{1-q}}-1\right)}.$$

We note that $\lim_{q\to 1} B_n(x;q) = B_n(x)$ and $\lim_{q\to 1} B_n(q) = B_n$.

Alternative definitions of special polynomials also arise in the literature. For instance, by using q-exponential function $e_q(z)$, Kim [9] defined q-Euler polynomials $E_{n,q}(x)$ by

$$\sum_{n=0}^{\infty} E_{n,q}(x)\frac{z^n}{n!} = \frac{[2]_q}{e_q(z)+1}e_q(zx).$$

In the sequel, we list some properties of q-Genocchi numbers and polynomials as well as recurrence relations and identities involving q-Bernoulli numbers and polynomials.

Proposition 2.1. *We have*

$$D_q G_n(x;q) = [n]_q G_{n-1}(x;q).$$

Proof. From (2.1) we can write

$$\sum_{n=0}^{\infty} D_q G_n(x;q)\frac{z^n}{[n]_q!} = \frac{2z}{(1-q)\left(e^{\frac{z}{1-q}}+1\right)}D_q e_q(zx)$$

$$\begin{aligned}
&= \frac{2z}{(1-q)\left(e^{\frac{z}{1-q}}+1\right)} z e_q(zx) \\
&= z \sum_{n=0}^{\infty} G_n(x;q) \frac{z^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} [n]_q G_{n-1}(x;q) \frac{z^n}{[n]_q!} \, .
\end{aligned}$$

Comparing coefficients on both sides yields the result. □

Theorem 2.2. *For q-commuting variables x and y such that $xy = qyx$, we have*

$$G_n(x+y;q) = \sum_{j=0}^{n} \binom{n}{j}_q G_j(x;q) \, y^{n-j}.$$

Proof. From (2.1) and (1.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n(x+y;q) \frac{z^n}{[n]_q!} &= \frac{2z}{(1-q)\left(e^{\frac{z}{1-q}}+1\right)} e_q(z(x+y)) \\
&= e_q(zy) \frac{2z}{(1-q)\left(e^{\frac{z}{1-q}}+1\right)} e_q(zx) \\
&= e_q(zy) \sum_{n=0}^{\infty} G_n(x;q) \frac{z^n}{[n]_q!} \, .
\end{aligned} \tag{2.3}$$

On the other hand,

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j}_q G_j(x;q) \, y^{n-j} \frac{z^n}{[n]_q!} &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(yz)^{n-j}}{[n-j]_q!} \frac{G_j(x;q) \, z^j}{[j]_q!} \\
&= \sum_{n=0}^{\infty} \frac{y^n z^n}{[n]_q!} \sum_{n=0}^{\infty} G_n(x;q) \frac{z^n}{[n]_q!} \\
&= e_q(zy) \sum_{n=0}^{\infty} G_n(x;q) \frac{z^n}{[n]_q!} \, .
\end{aligned} \tag{2.4}$$

(2.3) and (2.4) entail the result. □

Theorem 2.3. *We have*

$$G_n(x;q) = 2\left(B_n(x;q) - 2^n B_n\left(\frac{x}{2};q\right)\right).$$

Proof. From (2.1) and (2.2), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_n(x;q)\frac{z^n}{[n]_q!} \\
&= \frac{2z}{(1-q)\left(e^{\frac{z}{1-q}}+1\right)}e_q(zx) = \frac{2z\left(e^{\frac{z}{1-q}}-1\right)}{(1-q)\left(e^{\frac{2z}{1-q}}-1\right)}e_q(zx) \\
&= 2\frac{z}{(1-q)\left(e^{\frac{z}{1-q}}-1\right)}e_q(zx) - 2\frac{2z}{(1-q)\left(e^{\frac{2z}{1-q}}-1\right)}e_q\left(2z\frac{x}{2}\right) \\
&= 2\sum_{n=0}^{\infty} B_n(x;q)\frac{z^n}{[n]_q!} - 2\sum_{n=0}^{\infty} B_n\left(\frac{x}{2};q\right)\frac{2^n z^n}{[n]_q!}\ .
\end{aligned}$$

Comparing power series gives the result. □

Taking $x = 0$ in Theorem 2.3, we obtain

$$G_n(q) = 2(1-2^n)B_n(q)\,. \tag{2.5}$$

Note that as $q \to 1$, this identity reduces to the well known relation between classical Bernoulli and Genocchi numbers.

Next relation is the representation of q-Genocchi numbers as a finite sum of q-Bernoulli numbers.

Theorem 2.4. *For $n \geqslant 1$, we have*

$$G_n(q) = \sum_{k=1}^{n}\binom{n}{k}_q \frac{1}{(1-q)^k}\frac{[k]_q!}{k!}2^{n-k}B_{n-k}(q)\,.$$

Proof. From (2.1) and (2.2), we write

$$\sum_{n=0}^{\infty} B_n(q)\frac{2^n z^n}{[n]_q!} = \frac{2z}{(1-q)\left(e^{\frac{2z}{1-q}}-1\right)} = \frac{1}{e^{\frac{z}{1-q}}-1}\sum_{n=0}^{\infty} G_n(q)\frac{z^n}{[n]_q!}. \tag{2.6}$$

Multiplying both sides of (2.6) by $e^{\frac{z}{1-q}} - 1$, expanding the resulting power series, arranging the limits of the summations and simplifying, we get

$$\sum_{n=0}^{\infty} G_n(q)\frac{z^n}{[n]_q!} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{(1-q)^k}\frac{1}{k!}2^{n-k}\frac{B_{n-k}(q)}{[n-k]_q!}\right)z^n.$$

Comparing coefficients of z^n gives the desired result. □

Utilizing (2.5) in Theorem 2.4, we obtain a recurrence relation for q-Genocchi numbers.

Theorem 2.5. *For $n \geqslant 1$, q-Genocchi numbers satisfy the recurrence relation*

$$G_n(q) = \sum_{k=1}^{n}\binom{n}{k}_q \frac{1}{(1-q)^{n-k}}\frac{[n-k]_q!}{(n-k)!}\frac{2^{k-1}}{1-2^k}G_k(q)\,.$$

By the same method proceeded in the proof of Theorem 2.4, we find similar relations for q-Genocchi and q-Bernoulli polynomials.

Theorem 2.6. *For $n \geqslant 1$, we have*

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(1-q)^k k!}{[k]_q!} \{G_k(x;q) - 2B_k(x;q)\} = 2,$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ is the binomial coefficient.

Proof. Comparing defining equations of q-Genocchi and q-Bernoulli polynomials and equating common terms, we get

$$\frac{1}{2}\left(e^{\frac{z}{1-q}}+1\right)\sum_{n=0}^{\infty} G_n(x;q)\frac{z^n}{[n]_q!} = \left(e^{\frac{z}{1-q}}-1\right)\sum_{n=0}^{\infty} B_n(x;q)\frac{z^n}{[n]_q!}\,.$$

Arranging this equality yields

$$G_0(x;q)+\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{(1-q)^k}\frac{1}{k!}\frac{1}{[n-k]_q!}\left\{\frac{G_{n-k}(x;q)}{2} - B_{n-k}(x;q)\right\}\right)z^n = 0.$$

Thus $G_0(x;q) = 0$ and

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(1-q)^k k!}{[k]_q!}\{G_k(x;q) - 2B_k(x;q)\} = 2B_0(x;q)\,.$$

Since $B_0(x;q) = 1$, the proof is completed. □

Theorem 2.7. *q-Genocchi polynomials and q-Bernoulli polynomials satisfy the following relation*

$$\frac{2q-1}{2(1-q)^n n!} + \sum_{k=1}^{n}\frac{1}{(1-q)^k}\frac{1}{k!}\frac{1}{[n-k]_q!}\{G_{n-k}(x;q) - B_{n-k}(x;q)\} = 0,$$

where $n \geqslant 1$.

Higher order generalizations of the q-Genocchi polynomials can be defined in a natural way:

Definition. For $\alpha \in \mathbb{Z}$, $\alpha > 1$, we define q-Genocchi polynomials of order α as

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x;q)\frac{z^n}{[n]_q!} = \left(\frac{2z}{(1-q)\left(e^{\frac{z}{1-q}}+1\right)}\right)^{\alpha} e_q(zx)\,.$$

For $\alpha = 1$, $G_n^{(1)}(x;q) = G_n(x;q)$ and for $x = 0$, $G_n^{(\alpha)}(0;q) = G_n^{(\alpha)}(q)$ are q-Genocchi numbers of order α.

The higher order q-Genocchi polynomials satisfy the following relations.

Theorem 2.8. *For the q-Genocchi polynomials of order α, we have*

$$G_n^{(\alpha)}(x;q)=\sum_{j=0}^{n}\binom{n}{j}_q G_j^{(\alpha)}(q)\,x^{n-j}.$$

Theorem 2.9. *For the q-commuting variables x and y such that $xy=qyx$ and $\alpha,\ \beta\in\mathbb{Z},\ \alpha>1,\ \beta>1$, we have*

$$G_n^{(\alpha+\beta)}(x+y;q)=\sum_{j=0}^{n}\binom{n}{j}_q G_j^{(\alpha)}(x;q)\,G_{n-j}^{(\beta)}(y;q)\,.$$

Theorem 2.10. *For the q-Genocchi polynomials of order α, we have*

$$G_n^{(\alpha)}(x;q)=\sum_{j=0}^{n}\binom{n}{j}_q G_j^{(\alpha)}(q)\,x^{n-j}.$$

All these results can be proved by the methods presented in this paper, so we omit the proofs.

References

[1] L. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math. J. **15** (1948), 987–1000.

[2] M. Cenkci, M. Can, and V. Kurt, *q-extensions of Genocchi numbers*, J. Korean Math. Soc. **43** (2006), no. 1, 183–198.

[3] G. Gasper, *Lecture notes for an introductory minicourse on q-series*, arXiv.math. CA/9509223.

[4] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[5] A. S. Hegazi and M. Mansour, *A note on q-Bernoulli numbers and polynomials*, J. Nonlinear Math. Phys. **13** (2006), no. 1, 9–18.

[6] L. C. Jang and T. Kim, *q-Genocchi numbers and polynomials associated with fermionic p-adic invariant integrals on $\mathbb{Z}_p$*, Abstr. Appl. Anal. **2008** (2008), Art. ID 232187, 8 pp. doi:10.1155/2008/232187.

[7] L. C. Jang, T. Kim, D. H. Lee, and D. W. Park, *An application of polylogarithms in the analogue of Genocchi numbers*, NNTDM **7** (2000), 66–70.

[8] V. Kac and P. Cheung, *Quantum Calculus*, Springer Verlag, New York, 2002.

[9] T. Kim, *q-generalized Euler numbers and polynomials*, Russ. J. Math. Phys. **13** (2006), no. 3, 293–298.

[10] ———, *On the q-extension of Euler and Genocchi numbers*, J. Math. Anal. Appl. **326** (2007), no. 2, 1458–1465.

[11] ———, *A note on p-adic q-integral on $\mathbb{Z}_p$ associated with q-Euler numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **15** (2007), no. 2, 133–137.

[12] ———, *q-Euler numbers and polynomials associated with p-adic q-integrals*, J. Nonlinear Math. Phys. **14** (2007), no. 1, 15–27.

[13] ———, *A note on the q-Genocchi numbers and polynomials*, J. Inequal. Appl. **2007** (2007), Art. ID 71452, 8 pp. doi:10.1155/2007/71452.

[14] ———, *On the multiple q-Genocchi and Euler numbers*, Russ. J. Math. Phys. **15** (2008), no. 4, 481–486.

[15] ———, *Note on q-Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **17** (2008), no. 1, 9–15.

[16] T. Kim, L.-C. Jang, and H.-K. Pak, *A note on q-Euler and Genocchi numbers*, Proc. Japan Acad. Ser. A Math. Sci. **77** (2001), no. 8, 139–141.

[17] T. Kim, M.-S. Kim, L.C. Jang, and S.-H. Rim, *New q-Euler numbers and polynomials associated with p-adic q-integrals*, Adv. Stud. Contemp. Math. (Kyungshang) **15** (2007), no. 2, 243–252.

[18] T. H. Koornwinder, *Special functions and q-commuting variables*, Special functions, q-series and related topics (Toronto, ON, 1995), 131–166, Fields Inst. Commun., 14, Amer. Math. Soc., Providence, RI, 1997.

[19] B. A. Kupershmidt, *Reflection symmetries of q-Bernoulli polynomials*, J. Nonlinear Math. Phys. **12** (2005), suppl. 1, 412–422.

[20] Y. Simsek, I. N. Cangul, V. Kurt, and D. Kim, *q-Genocchi numbers and polynomials associated with q-Genocchi-type l-functions*, Adv. Difference Equ. **2008** (2008), Art. ID 815750, 12 pp. doi:10.11555.2008/85750.

[21] A. De Sole and V. Kac, *On integral representations of q-gamma and q-beta functions*, arXiv:math QA/0302032.

[22] H. M. Srivastava and A. Pinter, *Remarks on some relationships between the Bernoulli and Euler polynomials*, Appl. Math. Lett. **17** (2004), no. 4, 375–380.

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