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CLASSIFICATION OF CLASSICAL ORTHOGONAL POLYNOMIALS

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ABSTRACT. We reconsider the problem of classifying all classical orthogonal polynomial sequences which are solutions to a second-order differential equation of the form

$$\ell_2(x)y''(x) + \ell_1(x)y'(x) = \lambda_n y(x).$$

We first obtain new (algebraic) necessary and sufficient conditions on the coefficients $\ell_1(x)$ and $\ell_2(x)$ for the above differential equation to have orthogonal polynomial solutions. Using this result, we then obtain a complete classification of all classical orthogonal polynomials : up to a *real* linear change of variable, there are the six distinct orthogonal polynomial sets of Jacobi, Bessel, Laguerre, Hermite, twisted Hermite, and twisted Jacobi.

1. Introduction

All polynomials in this work are assumed to be real polynomials in the real variable x and we let $\mathcal{P}$ be the space of all these real polynomials. We denote the degree of a polynomial $\pi(x)$ by $\deg(\pi)$ with the convention that $\deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ with $\deg(\phi_n) = n$, $n \geq 0$. Note that a PS forms a basis for $\mathcal{P}$.

A PS $\{\phi_n(x)\}_{n=0}^{\infty}$ is called orthogonal if there is a function $\mu : \mathbb{R} \to \mathbb{R}$ of bounded variation on the real line $\mathbb{R}$ such that

$$\int_{\mathbb{R}} x^n \, d\mu(x) \tag{1.1}$$

is finite for all $n = 0, 1, \ldots$ and

$$\int_{\mathbb{R}} \phi_m(x)\phi_n(x)\, d\mu(x) = K_n\delta_{mn} \quad (m \text{ and } n \geq 0), \tag{1.2}$$

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where K_n are non-zero real constants and δ_{mn} is the Kronecker delta function. Furthermore, we shall say that $\{\phi_n(x)\}_{n=0}^{\infty}$ is *classical* if each $\phi_n(x)$ $(n \geq 0)$ satisfies a fixed second-order differential equation of the form

$$L[y](x) = \ell_2(x)y''(x) + \ell_1(x)y'(x) = \lambda_n y(x), \tag{1.3}$$

where $\ell_2(x)$ and $\ell_1(x)$ are real-valued functions independent of n and λ_n is a real constant depending only on n. The classification of classical orthogonal polynomials is generally attributed to Bochner [3]. In fact, Bochner [3] considered a general second-order Sturm-Liouville differential equation of the form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) + \lambda y(x) = 0, \tag{1.4}$$

where $a_i(x)$ $(i = 0, 1, 2)$ are real- or complex-valued functions and λ is a constant. He then raised and solved the problem : determine all cases such that for each integer $n \geq 0$, there is an eigenvalue $\lambda = \lambda_n$ for which there is a corresponding polynomial solution of degree n. He first observed that if the differential equation (1.4) has polynomial solutions of degree 0, 1, and 2, then $a_i(x)$ must be a polynomial of degree $\leq i$, $i = 0, 1, 2$. He then considered cases according to the degree of $a_2(x)$ and, in each case, reduced the differential equation into a normal form by a suitable complex linear change of variable. Then, through a detailed analysis of each case, Bochner showed that up to a complex linear change of variable, the only PS's that arise as eigensolutions of the differential equation (1.4) are the following (apart from non-zero constant factors) :

(a) Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ $(\alpha,\ \beta,\ \alpha + \beta + 1 \notin \{-1, -2, \dots\})$;
(b) Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ $(\alpha \notin \{-1, -2, \dots\})$;
(c) Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$;
(d) $\{x^n\}_{n=0}^{\infty}$;
(e) Bessel polynomials $\{B_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ $(\alpha \notin \{0, -1, -2, \dots\}$ and $\beta \neq 0)$.

The orthogonality of the Jacobi polynomials for α and $\beta > -1$, Laguerre polynomials for $\alpha > -1$, and Hermite polynomials was known long before Bochner's work. In fact, Bochner [3] did not mention the orthogonality of the PS's that he found. The problem of classifying all classical orthogonal polynomials was handled by many authors thereafter : see, for example, [1], [5], [6], [11], [27], and [31]. The problem was settled by Lesky [27] in 1962 at least for classical orthogonal polynomials satisfying the orthogonality relation (1.2) in which the function $\mu(x)$ is non-decreasing. Lesky [27] showed that the only such orthogonal polynomials are Jacobi polynomials with α and $\beta > -1$, Laguerre polynomials with $\alpha > -1$, and Hermite polynomials.

It is easy to see that the PS $\{x^n\}_{n=0}^{\infty}$ in case (d) above cannot be orthogonal. The orthogonality of the Bessel polynomials was first observed by H.L. Krall [18] and later investigated in depth by Krall and Frink [19]. Bochner [3] observed the relation between the PS in case (e) above and the half-integer Bessel functions and it is this relation which motivates the name Bessel polynomials in [19]. The orthogonality of the Jacobi polynomials for α or $\beta < -1$ and Laguerre polynomials for $\alpha < -1$ was recently treated by Morton and Krall [32].

A natural question arises : are these four PS's of Jacobi, Laguerre, Hermite, and Bessel the only classical orthogonal polynomials? Of course, if we allow for a complex linear change of variable, as Bochner does in [3], the answer is yes. However, if we restrict our attention to a *real* linear change of variable, as we shall do in this paper, are there any more classical orthogonal polynomials? As far as the authors know, no previous work on this classification problem really exhausts all possibilities.

After obtaining necessary and sufficient conditions (see Theorem 2.9) for the differential equation (1.3) to have orthogonal polynomials of solutions in section two, we give a complete classification of classical orthogonal polynomials in section three. Finally, in section four, we will discuss the integral or distributional representation of orthogonality for each classical orthogonal polynomial system found in section three.

2. Necessary and sufficient conditions

We call any linear functional σ on $\mathcal{P}$ a moment functional and denote its action on a polynomial $\pi(x)$ by $\langle\sigma, \pi\rangle$. We define the nth moment of σ by $\langle\sigma, x^n\rangle$ $(n = 0, 1, \ldots)$.

We shall remind the reader in section four below that any moment functional σ has a representation of the form

$$\langle\sigma, \pi\rangle = \int_{\mathbb{R}} \pi(x)\, d\mu(x) \qquad (\pi \in \mathcal{P}),$$

or

$$\langle\sigma, \pi\rangle = \int_{\mathbb{R}} \pi(x)\phi(x)\,dx \qquad (\pi \in \mathcal{P}),$$

where $\mu(x)$ is, in general, a function of bounded variation on $\mathbb{R}$ and where $\phi(x)$ is a C^{∞}-function of the Schwartz class. Hence, the orthogonality relation in (1.2) can be expressed in terms of moment functionals. As we shall see, it is very convenient and advantageous to use moment functionals instead of using their integral representations in discussing orthogonal polynomials.

We say that a moment functional σ is quasi-definite (respectively, positive-definite) if its moments $\{\sigma_n\}_{n=0}^{\infty}$ satisfy the Hamburger condition

$$\Delta_n(\sigma) = \det[\sigma_{i+j}]_{i,j=0}^{n} \neq 0 \quad (\text{respectively, } \Delta_n(\sigma) > 0) \tag{2.1}$$

for every $n \geq 0$.

Any PS $\{P_n(x)\}_{n=0}^{\infty}$ determines a moment functional σ (uniquely up to a non-zero constant multiple), called a canonical moment functional for $\{P_n(x)\}_{n=0}^{\infty}$, by the conditions

$$\langle\sigma, P_0\rangle \neq 0 \quad \text{and} \quad \langle\sigma, P_n\rangle = 0, \qquad n \geq 1. \tag{2.2}$$

DEFINITION 2.1. A PS $\{P_n(x)\}_{n=0}^{\infty}$ is called a weak orthogonal polynomial system (WOPS) if there is a non-trivial moment functional σ such that

$$\langle\sigma, P_m P_n\rangle = 0 \quad \text{if} \quad m \neq n \quad (m \text{ and } n \geq 0). \tag{2.3}$$

If we further have

$$\langle\sigma, P_m P_n\rangle = K_n \delta_{mn}, \tag{2.4}$$

where K_n are non-zero real constants, then we call $\{P_n(x)\}_{n=0}^{\infty}$ an orthogonal polynomial system (OPS). If each $K_n > 0$, then we call $\{P_n(x)\}_{n=0}^{\infty}$ a positive-definite OPS. In either case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is a WOPS or an OPS relative to σ and call σ an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

It is immediate from the orthogonality (2.3) that for any WOPS $\{P_n(x)\}_{n=0}^{\infty}$, its orthogonalizing moment functional must be a canonical moment functional for $\{P_n(x)\}_{n=0}^{\infty}$ so that it is unique up to a non-zero constant multiple.

It is well known (for example, see [4, Chapter 1]) that a moment functional σ is quasi-definite (respectively, positive-definite) if and only if there is an OPS (respectively, a positive-definite OPS) relative to σ. It is clear that if $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ, then so is $\{C_n P_n(x)\}_{n=0}^{\infty}$ for every sequence of non-zero constants C_n. Conversely if σ is any quasi-definite moment functional and $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ, then each $P_n(x)$ is uniquely determined up to an arbitrary non-zero factor. In particular, for any quasi-definite moment functional σ, there is a unique monic OPS relative to σ given by

$$P_n(x) = \frac{1}{\Delta_{n-1}(\sigma)} \det \begin{bmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_n \\ \sigma_1 & \sigma_2 & \dots & \sigma_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n-1} & \sigma_n & \dots & \sigma_{2n-1} \\ 1 & x & \dots & x^n \end{bmatrix} \qquad (n \geq 0), \tag{2.5}$$

where $\Delta_{-1}(\sigma) = 1$ (see [4, Chapter 1]).

We shall call an OPS $\{P_n(x)\}_{n=0}^{\infty}$ a *classical OPS* if for each $n \geq 0$, $P_n(x)$ satisfies the differential equation (1.3) for some eigenparameter λ_n. As mentioned in the introduction, if the differential equation (1.3) has a PS of solutions, then it is necessary that the coefficients $\ell_2(x)$, $\ell_1(x)$, and λ_n be given by

$$\begin{aligned} \ell_i(x) &= \sum_{j=0}^{i} \ell_{ij} x^j \qquad (i = 1, 2), \\ \lambda_n &= n(n-1)\ell_{22} + n\ell_{11} \qquad (n \geq 0), \end{aligned} \tag{2.6}$$

where $\ell_{11}^2 + \ell_{22}^2 \neq 0$.

From here on, we shall assume that the differential equation (1.3) has coefficients given by (2.6).

In 1938, H.L. Krall [17] obtained necessary and sufficient conditions for an OPS to satisfy a Sturm-Liouville type differential equation of any order. In case of the second-order differential equation (1.3), Krall's result can be stated as :

THEOREM 2.1. *A PS* $\{P_n(x)\}_{n=0}^{\infty}$ *is an OPS* (*respectively, a positive-definite OPS*) *satisfying the differential equation* (1.3) *if and only if its canonical* *moment functional* σ *is quasi-definite* (*respectively, positive-definite*) *and the moments* $\{\sigma_n\}_{n=0}^{\infty}$ *of* σ *satisfy the recurrence relation*

(2.7)
$$(n\ell_{22}+\ell_{11})\sigma_{n+1}+(n\ell_{21}+\ell_{10})\sigma_n+n\ell_{20}\sigma_{n-1}=0 \qquad (n\geq 0\ ;\ \sigma_{-1}=0).$$

For a new and somewhat simpler proof of Theorem 2.1, see [23] ; for another proof of the general Krall characterization theorem, see [20] and [25].

We call the recurrence relation (2.7) the *moment equation* for the differential equation (1.3). We may use Theorem 2.1 to classify all possible classical OPS's. However it is very difficult, in general, to solve the moment equation (2.7) and to see whether the corresponding moment functional is quasi-definite or not. The disadvantage of the conditions in Theorem 2.1 is that the equation (2.7) contains not only the coefficients of (1.3) but also the moments of a canonical moment functional of a classical OPS of which the existence is not known apriori.

Below, we shall first obtain a necessary condition (see Theorem 2.5) and then necessary and sufficient conditions (see Theorem 2.9) for the differential equation (1.3) to have an OPS of solutions. Unlike those in Theorem 2.1, these conditions involve only the coefficients of the differential equation (1.3).

We begin with introducing some formal calculus on moment functionals. For a moment functional σ and $\pi\in\mathcal{P}$, we let σ', the derivative of σ and $\pi\sigma$, multiplication of σ by a polynomial, be those moment functionals defined by

(2.8)
$$\langle\sigma',\,p\rangle=-\langle\sigma,\,p'\rangle \qquad (p\in\mathcal{P})$$

and

(2.9)
$$\langle\pi\sigma,\,p\rangle=\langle\sigma,\pi p\rangle \qquad (p\in\mathcal{P}).$$

It is then easy to obtain the following Leibnitz rule for any moment functional σ and polynomial $\pi(x)$:

(2.10)
$$(\pi\sigma)'=\pi'\sigma+\pi\sigma'.$$

LEMMA 2.2. *Let* σ *be a moment functional and* $\pi(x)$ *a polynomial.*

(i) *Then* $\sigma=0$ *if and only if* $\sigma'=0$.
(ii) *If* σ *is quasi-definite, then* $\pi(x)\sigma=0$ *if and only if* $\pi(x)=0$.

Proof. (i) If $\sigma' = 0$, then

$$\langle \sigma, x^n \rangle = \langle \sigma, \frac{1}{n+1}(x^{n+1})' \rangle = \frac{-1}{n+1}\langle \sigma, x^{n+1} \rangle = 0$$

for every $n \geq 0$ so that $\sigma = 0$. The converse is trivial.

(ii) Assume σ is quasi-definite and $\pi(x)\sigma = 0$. Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to σ. Suppose $\pi(x) \not\equiv 0$ so that $\deg(\pi) = N \geq 0$ and write $\pi(x) = \sum_{k=0}^{N} C_k P_k(x)$ with $C_N \neq 0$. Then we have

$$0 = \langle \pi\sigma, P_N \rangle = \sum_{k=0}^{K} C_k \langle \sigma, P_k P_N \rangle = C_N \langle \sigma, P_N^2 \rangle$$

so that $C_N = 0$ since $\langle \sigma, P_N^2 \rangle \neq 0$, contradicting our assumption. The converse is trivial.

LEMMA 2.3. *If the differential equation* (1.3) *has a PS of solutions, then any canonical moment functional* σ *of this PS satisfies the functional equation*

$$(\ell_2(x)\sigma)' - \ell_1(x)\sigma = 0. \tag{2.11}$$

Proof. Suppose that $\{P_n(x)\}_{n=0}^{\infty}$ is a PS of solutions of the differential equation (1.3). Let σ be a canonical moment functional for this PS. Then we have for each integer $n \geq 1$,

$$0 = \lambda_n \langle \sigma, P_n \rangle = \langle \sigma, \lambda_n P_n \rangle = \langle \sigma, \ell_2 P_n'' + \ell_1 P_n' \rangle = \langle \ell_1 \sigma - (\ell_2 \sigma)', P_n' \rangle,$$

which implies (2.11) since $\{P_n'(x)\}_{n=1}^{\infty}$ is also a PS.

Note that the zero in the right hand side of the equation (2.11) means the zero moment functional. In other words, the equation (2.11) means

$$\langle (\ell_2\sigma)' - \ell_1\sigma, x^n \rangle = 0 \qquad (n \geq 0),$$

which is exactly the moment equation (2.7) when it is expressed in terms of the moments $\{\sigma_n\}_{n=0}^{\infty}$ of σ.

We call the equation (2.11) the weight equation for the differential equation (1.3).

REMARK 2.1. If we view the equation (2.11) as a classical differential equation:

$$(\ell_2(x)s)' - \ell_1(x)s = 0, \tag{2.12}$$

then any non-trivial solution $s(x)$ of (2.12) is a *symmetry factor* (see [29]) of the differential expression $L[\cdot]$ in (1.3). In this sense, we call the equation (2.12) the *symmetry equation* of the differential expression $L[\cdot]$. For more details on symmetry factors, symmetry equations, and their applications to orthogonal polynomials, see [16], [25], [28], and [29].

It is natural to ask if the differential equation (1.3) always has a PS of solutions. By direct calculation, it is easy to see that (1.3) has a unique monic polynomial solution of degree n for each integer $n \geq 0$ except possibly for a finite number of values of n and, for those exceptional cases of n (if there is any), there may be no polynomial solution of degree n or there will be infinitely many monic polynomial solutions of degree n.

EXAMPLE. Consider the following second-order differential equation :

$$L[y](x) = (1 + x^2)y''(x) + [(1 - k)x + b]y'(x) = n(n - k)y(x), \tag{2.13}$$

where $k \geq 1$ is an integer and b is a real constant. Now it is easy to see that the equation (2.13) has a PS of solutions if and only if k is odd and $b = 0$. Moreover when $k = 2j+1$, $j \geq 0$ and $b = 0$, the equation (2.13) has a unique monic polynomial solution of degree n for $n \notin \{j + 1, j + 2, \ldots, 2j + 1\}$. For $n \in \{j + 1, j + 2, \ldots, 2j + 1\}$, it has infinitely many monic polynomial solutions of degree n.

DEFINITION 2.2 (Krall and Sheffer [21]). The differential expression $L[\cdot]$ in (1.3) (or the differential equation (1.3) itself) is called admissible if

$$\lambda_m \neq \lambda_n \quad \text{for} \quad m \neq n \quad (m \text{ and } n \geq 0). \tag{2.13}$$

LEMMA 2.4. *For the differential expression* $L[\cdot]$ *in* (1.3)*, the following are equivalent :*

(i) $L[\cdot]$ *is admissible ;*
(ii) $\lambda_n = n(n - 1)\ell_{22} + n\ell_{11} \neq 0$ $(n \geq 1)$ *;*
(iii) $\ell_{11} \notin \{-n\ell_{22} \mid n = 0, 1, 2, \ldots\}$ *;*
(iv) *The moment equation* (2.7) (*or equivalently the weight equation* (2.11)) *has only one linearly independent solution ;*
(v) *For each* $n \geq 0$*, the differential equation* (1.3) *has a unique monic polynomial solution of degree* n*.*

Proof. The proofs of (i)⇒(ii) and (ii)⇔(iii)⇔(iv) are trivial.

(ii)⇒(i) : This follows from the identity

$$(n+m)(\lambda_n-\lambda_m)=(n-m)(n+m)(\ell_{22}(n+m-1)+\ell_{11})=(n-m)\lambda_{n+m}.$$

(i)⇒(v) : For any integer $n\geq 1$, let

$$P_n(x)=\sum_{k=0}^{n}C_k^n x^k \qquad (C_n^n=1)$$

be a monic polynomial of degree n. Then $P_n(x)$ satisfies (1.3) if and only if

$$\ell_{20}(k+2)(k+1)C_{k+2}^n+(k+1)(\ell_{21}k+\ell_{10})C_{k+1}^n+(\lambda_k-\lambda_n)C_k^n=0 \tag{2.14}$$

$(k=0,1,\ldots,n-1)$, where $C_{n+1}^n=0$. If $L[\cdot]$ is admissible, then all C_k^n $(k=0,1,\ldots,n-1)$ are uniquely and successively determined by the equation (2.14) and our assumption that $C_n^n=1$.

(v)⇒(i) : Assume that the differential equation (1.3) has a unique monic PS $\{P_n(x)\}_{n=0}^{\infty}$ of solutions but $L[\cdot]$ is not admissible. Hence from (ii), we have $\lambda_N=\lambda_0=0$ for some integer $N\geq 1$. But then

$$L[P_N+kP_0]=\lambda_N P_N+k\lambda_0 P_0=0=\lambda_N(P_N+kP_0)$$

for any constant k. Hence $L[y]=\lambda_N y$ has infinitely many monic polynomial solutions of degree N, which contradicts our assumption.

REMARK 2.2. Let $N\geq 0$ be the largest integer such that $\lambda_N=0$. Then for any $n>N$ the differential equation (1.3) can have only one linearly independent polynomial solution of degree n.

REMARK 2.3. When $\ell_2(x)\equiv 0$, the differential equation (1.3) reduces to the first-order equation

$$(\ell_{11}x+\ell_{10})y'(x)=n\ell_{11}y(x),$$

which is admissible if and only if $\ell_{11}\neq 0$. In this case, the corresponding weight equation is

$$(\ell_{11}x+\ell_{10})\sigma=0,$$

of which the general solution is

$$\sigma = c\delta(\ell_{11}x + \ell_{10}),$$

where c is an arbitrary constant and $\delta(\ell_{11}x + \ell_{10})$ is the Dirac delta moment functional defined by

$$\langle \delta(\ell_{11}x + \ell_{10}), \pi(x)\rangle = \pi(-\ell_{10}/\ell_{11}) \qquad (\pi \in \mathcal{P}).$$

Since σ is not quasi-definite, we can conclude that the above first-order differential equation can never have an OPS of solutions (see Theorem 2.1).

By Remark 2.3, we may assume $\ell_2(x) \not\equiv 0$ in the differential equation (1.3). Now we are ready to give a necessary condition for the differential equation (1.3) to have an OPS of solutions, which will be very useful in our classification in the next section.

THEOREM 2.5. *If the differential equation* (1.3) *has an* OPS $\{P_n(x)\}_{n=0}^{\infty}$ *of solutions, then* $L[\cdot]$ *is admissible.*

Proof. Assume that (1.3) has an OPS $\{P_n(x)\}_{n=0}^{\infty}$ of solutions and let σ be an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Then σ is a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$ and, by Lemma 2.3, σ satisfies the weight equation (2.11). If $L[\cdot]$ is not admissible, then, by Lemma 2.4 (ii), there is an integer $N \geq 1$ such that $\lambda_N = 0$. Consequently, we have

$$\begin{aligned} 0 &= \lambda_N P_N \sigma = (\ell_2 P_N'' + \ell_1 P_N')\sigma \\ &= (\ell_2 P_N' \sigma)' - P_N'(\ell_2\sigma)' + P_N'(\ell_1\sigma) = (\ell_2 P_N' \sigma)'. \end{aligned}$$

Hence, by Lemma 2.2, we have $\ell_2 P_N' \equiv 0$. However, $\ell_2(x) \not\equiv 0$ (see Remark 2.3) so that $P_N'(x) \equiv 0$, which implies $N = 0$ contradicting the fact that $N \geq 1$.

Theorem 2.5 was first proved by Lesky [27] only for positive-definite classical OPS's. However his method of proof cannot be extended to general classical OPS's since he used the following fact which holds only for positive-definite OPS's : for any positive-definite OPS $\{P_n(x)\}_{n=0}^{\infty}$, the zeros of $P_n(x)$, $n \geq 1$, are real and distinct and no two polynomials from $\{P_n(x)\}_{n=0}^{\infty}$ can have common zeros (see Chihara [4]).

The converse of Theorem 2.5 does not hold in general. For example, the PS $\{x^n\}_{n=0}^\infty$ satisfies the admissible differential equation

$$x^2 y''(x) + xy'(x) = n^2 y(x)$$

but $\{x^n\}_{n=0}^\infty$ is not an OPS. However, we have the following partial converse of Theorem 2.5.

THEOREM 2.6. *If the differential operator* $L[\cdot]$ *in* (1.3) *is admissible, then any PS* $\{P_n(x)\}_{n=0}^\infty$ *of solutions to the differential equation* (1.3) *is a WOPS.*

Proof. By Lemma 2.4, we may assume that $\{P_n(x)\}_{n=0}^\infty$ is the unique monic PS of solutions to (1.3). Let σ be a canonical moment functional of $\{P_n(x)\}_{n=0}^\infty$. Then $\sigma \neq 0$ by definition and, by Lemma 2.3, σ satisfies the weight equation (2.11). Then we have for m and $n \geq 0$

$$\begin{aligned}(\lambda_m - \lambda_n)P_m P_n &= \ell_2(P_m' P_n - P_m P_n')' + \ell_1(P_m' P_n - P_m P_n')\\ &= \ell_2 W_{m,n}' + \ell_1 W_{m,n},\end{aligned}$$

where $W_{m,n} = P_m' P_n - P_m P_n'$ is the Wronskian of P_m and P_n. Hence, by (2.11), we have

$$\begin{aligned}(\lambda_m - \lambda_n)\langle\sigma, P_m P_n\rangle &= \langle\sigma, \ell_2 W_{m,n}' + \ell_1 W_{m,n}\rangle\\ &= \langle \ell_1\sigma - (\ell_2\sigma)', W_{m,n}\rangle = 0.\end{aligned}$$

Consequently, $\langle\sigma, P_m P_n\rangle = 0$ for $m \neq n$ if $L[\cdot]$ is admissible.

REMARK 2.4. In fact we can prove, by the same reasoning as in the proof of Theorem 2.6, something more than Theorem 2.6. If $L[p] = \lambda p$ and $L[q] = \mu q$ for some polynomials $p(x)$ and $q(x)$ and $\lambda \neq \mu$, then $\langle\sigma, pq\rangle = 0$ for any moment functional solution σ of the weight equation (2.11). Here we do not need to assume $L[\cdot]$ is admissible.

We now seek a criterion for when a WOPS $\{P_n(x)\}_{n=0}^\infty$ is an OPS, which does not involve a canonical moment functional of $\{P_n(x)\}_{n=0}^\infty$.

For any monic PS $\{P_n(x)\}_{n=0}^\infty$, there are constants $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ such that

$$P_{n+1}(x) - (x - \alpha_n)P_n(x) + \beta_n P_{n-1}(x) \qquad (n \geq 1) \tag{2.15}$$

is a polynomial of degree $\leq n-2$. In fact if $P_n(x) = \sum_{k=0}^{n} C_k^n x^k$ ($C_n^n = 1$; $n \geq 1$), then

(2.16)
$$\alpha_n = C_{n-1}^n - C_n^{n+1}$$

and

(2.17)
$$\beta_n = C_{n-2}^n - (C_{n-1}^n - C_n^{n+1})C_{n-1}^n - C_{n-1}^{n+1} \qquad (C_{-1}^1 = 0).$$

At this moment, we need to recall Favard's theorem (see [10]) which asserts that a monic PS $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS (respectively, a positive-definite OPS) if and only if $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a three term recurrence relation

(2.18)
$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n P_{n-1}(x) \qquad (n \geq 1),$$

where each $\beta_n \neq 0$ (respectively, $\beta_n > 0$).

In the case of WOPS's, Favard's theorem can be improved as follows.

PROPOSITION 2.7 (Krall and Sheffer [21]). *A monic* WOPS $\{P_n(x)\}_{n=0}^{\infty}$ *is an OPS* (*respectively, a positive-definite OPS*) *if and only if*

(2.19)
$$\beta_n \neq 0 \qquad (\textit{respectively, } \beta_n > 0)$$

for $n \geq 1$, *where* β_n *is the constant given in* (2.15).

Proof. See Lemma 1.1 in [21].

Once we know a PS $\{P_n(x)\}_{n=0}^{\infty}$ is a WOPS (it is so if $\{P_n(x)\}_{n=0}^{\infty}$ satisfies an admissible equation (1.3) : see Theorem 2.6), the advantage of applying Proposition 2.7 over Favard's theorem is evident. In order to check condition (2.19), we only need to know the coefficients of x^{n-1} and x^{n-2} of each $P_n(x)$ from a monic WOPS $\{P_n(x)\}_{n=0}^{\infty}$. More precisely, we have the following result from Proposition 2.7 and equation (2.17).

COROLLARY 2.8. *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be a WOPS and* $P_n(x) = \sum_{k=0}^{\infty} C_k^n x^k$ $(C_n^n = 1)$ *for* $n \geq 0$. *Then* $\{P_n(x)\}_{n=0}^{\infty}$ *is an OPS* (*respectively, a positive-definite OPS*) *if and only if*

(2.20)
$$\beta_n = C_{n-2}^n - (C_{n-1}^n - C_n^{n+1})C_{n-1}^n - C_{n-1}^{n+1} \neq 0 \quad (\textit{respectively, } \beta_n > 0)$$

for $n \geq 1$, *where* $C_{-1}^1 = 0$.

Now combining Theorem 2.5, Theorem 2.6, and Corollary 2.8, we can obtain necessary and sufficient conditions for the differential equation (1.3) to have an OPS of solutions in terms of only the coefficients of the differential expression $L[\cdot]$.

THEOREM 2.9. *The differential equation* (1.3) *has an OPS* (*respect- ively, a positive-definite OPS*) *of solutions if and only if*

(i) $\ell_{11} \notin \{-n\ell_{22} \mid n = 0, 1, 2, \dots\}$

and

(ii) *the condition* (2.20) *holds; i.e.* $\beta_n \neq 0$ $(respectively,\ \beta_n > 0)$, *where*

$$C_{n-1}^n = \frac{n[\ell_{10} + \ell_{21}(n-1)]}{\ell_{11} + 2\ell_{22}(n-1)} \tag{2.21}$$

and

$$C_{n-2}^n = \frac{n(n-1)[\ell_{20}(\ell_{11} + 2\ell_{22}(n-1)) + (\ell_{10} + \ell_{21}(n-2))(\ell_{10} + \ell_{21}(n-1)]}{2[\ell_{11} + 2\ell_{22}(n-1)][\ell_{11} + \ell_{22}(2n-3)]} \tag{2.22}$$

$(n \geq 1\ ;\ C_{-1}^1 = 0)$.

Proof. By Lemma 2.4, the above condition (i) is just the admissibility of $L[\cdot]$ which is also equivalent to the fact that the differential equation (1.3) has a unique monic PS $\{P_n(x)\}_{n=0}^{\infty}$ of solutions. If we set $P_n(x) = \sum_{k=0}^{n} C_k^n x^k$ $(C_n^n = 1\ ;\ n \geq 0)$, then C_{n-1}^n and C_{n-2}^n are given by (2.21) and (2.22), respectively, by solving the equation (2.14) for $k = n-1$ and $k = n-2$. Hence, Theorem 2.9 follows from Theorem 2.5, Theorem 2.6, and Corollary 2.8.

We end this section by the following remark.

REMARK 2.5. If we assume that the differential equation (1.3) has a monic PS $\{P_n(x)\}_{n=0}^{\infty}$ of solutions, then $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS if and only if the condition (2.20) holds. For a proof of this statement, see [24, Proposition 3.7]. Note here that apriori we do not assume that $\{P_n(x)\}_{n=0}^{\infty}$ is a WOPS (as in Proposition 2.7) or $L[\cdot]$ is admissible (as in Theorem 2.9). Furthermore, only condition (2.2) must be checked but not with the conditions given in (2.21) and (2.22). In general, these latter two equations are not well defined unless the expression $L[\cdot]$ is admissible.

3. Classification

We say that any two OPS's are equivalent if either one differs from the other by non-zero constant factors or one is obtained from the other by a real linear change of variable.

In this section, we will classify all classical OPS's up to equivalence classes using Theorem 2.9.

In the following, we let $\mathbb{N}$ be the set of all positive integers and use the notation

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

for any complex number α and any integer k in $\mathbb{N}$. As with Bochner, we divide the cases according to the roots of the leading coefficient $\ell_2(x)$ of the differential expression $L[\cdot]$ in (1.3).

Cases 1: Jacobi polynomials

We assume $\ell_{22} \neq 0$ and $\ell_{21}^2 - 4\ell_{22}\ell_{20} > 0$. Then, by a real linear change of variable, the equation (1.3) can be transformed into

$$\begin{aligned} L[y](x) &= (1-x^2)y''(x) + [(\beta-\alpha) - (\alpha+\beta+2)x]y'(x) \\ &= -n(n+\alpha+\beta+1)y(x). \end{aligned} \tag{3.1}$$

We assume $-(\alpha+\beta+1) \notin \mathbb{N}$ so that $L[\cdot]$ in (3.1) is admissible. Then the equation (3.1) has a unique monic PS $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, called the Jacobi PS, of solutions :

(3.2)

$$P_n^{(\alpha,\beta)}(x) = \binom{2n+\alpha+\beta}{n}^{-1} \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k$$

$(n \geq 0)$.

PROPOSITION 3.1. *The Jacobi PS* $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ *is*

(i) *a WOPS if* $-(\alpha+\beta+1) \notin \mathbb{N}$ *;*

(ii) *an OPS if and only if* $-\alpha$, $-\beta$, *and* $-(\alpha+\beta+1) \notin \mathbb{N}$ *:*

(iii) *a positive-definite OPS if and only if* α *and* $\beta > -1$.

Proof. The proof of (i) follows from Theorem 2.6. Now we assume $-(\alpha+\beta+1) \notin \mathbb{N}$. We then have, from (2.20), (2.21), (2.22), and (3.1), (3.3)

$$\beta_n = \frac{4n(\alpha+\beta+n)(\alpha+n)(\beta+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)} \qquad (n \geq 1).$$

Hence, $\beta_n \neq 0$ for $n \geq 1$ if and only if $\alpha+n \neq 0$ and $\beta+n \neq 0$ for $n \geq 1$ so that (ii) follows from Theorem 2.9. To prove (iii), it suffices to show $\beta_n > 0$ for $n \geq 1$ if and only if α and $\beta > -1$. If α and $\beta > -1$, then every factor in (3.3) is positive so that $\beta_n > 0$ for $n \geq 1$. Conversely, assume $\beta_n > 0$ for $n \geq 1$ but $\alpha < -1$ (when $\beta < -1$, the proof is essentially the same). Then, from $\beta_1 > 0$, we have $(\beta+1)(\alpha+\beta+3) < 0$. If $\beta+1 < 0$ and $\alpha+\beta+3 > 0$, then $\alpha+\beta+2 < 0$ and $0 < \alpha+2,\ \beta+2 < 1$ so that $\beta_2 < 0$, which is a contradiction. If $\beta+1 > 0$ and $\alpha+\beta+3 < 0$, then $\alpha < -2$. Then, from $\beta_2 > 0$, we have $\alpha+\beta+5 < 0$ and so $\alpha < -4$. Continuing the same process, we have that $\alpha < -2k$ for any integer $k \geq 1$, which is impossible.

The explicit orthogonality of the Jacobi PS $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ for α or $\beta < -1$ (but $-\alpha$ and $-\beta \notin \mathbb{N}$) has been treated by Morton and Krall [32].

Case 2: Bessel polynomials

We assume $\ell_{22} \neq 0$ and $\ell_{21}^2 - 4\ell_{22}\ell_{20} = 0$. Then, by a real linear change of variable, the equation (1.3) can be transformed into

$$L[y](x) = x^2 y''(x) + (\alpha x + \beta) y'(x) = n(n+\alpha-1)y(x). \tag{3.4}$$

We assume $-(\alpha-1) \notin \mathbb{N}$ so that $L[\cdot]$ in (3.4) is admissible. Then the equation (3.4) has a unique monic PS $\{B_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ of solutions :

$$B_n^{(\alpha,\beta)}(x) = \begin{cases} x^n & \text{if } \beta = 0 \\ \dfrac{1}{\beta^n \Gamma(\alpha+2n-1)} \displaystyle\sum_{k=0}^{n} \frac{n!\,\Gamma(\alpha+n+k-1)}{(n-k)!\,k!} \left(\frac{x}{\beta}\right)^k & \text{if } \beta \neq 0 \end{cases} \tag{3.5}$$

$(n \geq 0)$. When $\beta \neq 0$, we call $\{B_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ the Bessel PS. The PS $\{x^n\}_{n=0}^{\infty}$ is a WOPS by Theorem 2.6 but it cannot be an OPS.

PROPOSITION 3.2. *The Bessel PS* $\{B_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ *is an OPS* (*but not a positive-definite OPS*) *if and only if* $-(\alpha-1)\notin\mathbb{N}$ *and* $\beta\neq 0$.

Proof. We assume $-(\alpha-1)\notin N$. We then have, from (2.20), (2.21), (2.22), and (3.4),

$$\beta_n = \frac{-n\beta^2(\alpha+n-2)}{(\alpha+2n-3)(\alpha+2n-2)^2(\alpha+2n-1)} \qquad (n\geq 1). \tag{3.6}$$

Hence $\beta_n\neq 0$ for $n\geq 0$ if and only if $\beta\neq 0$ and $\beta_n<0$ for n large enough. Therefore, we have the proposition by Theorem 2.9.

The Bessel PS, as an OPS, was first observed by H.L. Krall [18]. Earlier these polynomials were discussed by Romanovski [33] and Bochner [3]. In [19], Krall and Frink studied the Bessel polynomials in detail and found, explicitly, their complex orthogonality.

Case 3: Laguerre polynomials

We assume $\ell_{22}=0$ and $\ell_{21}\neq 0$. Then by a real linear change of variable, the equation (1.3) can be transformed into

$$L[y](x) = xy''(x) + (\alpha+1-x)y'(x) = -ny(x). \tag{3.7}$$

The differential expression $L[\cdot]$ in (3.7) is admissible and so the equation (3.7) has a unique monic PS $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$, called the Laguerre polynomials, of solutions :

$$L_n^{(\alpha)}(x) = (-1)^n n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \qquad (n\geq 0). \tag{3.8}$$

PROPOSITION 3.3. *The Laguerre PS* $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ *is*

(i) *a WOPS for every* α ;
(ii) *an OPS if and only if* $-\alpha\notin\mathbb{N}$;
(iii) *a positive-definite OPS if and only if* $\alpha>-1$.

Proof. (i) follows from Theorem 2.6. We have from (2.20), (2.21), (2.22), and (3.7)

$$\beta_n = n(\alpha + n) \qquad (n \geq 1). \tag{3.9}$$

Hence $\beta_n \neq 0$ (respectively, $\beta_n > 0$) for $n \geq 1$ if and only if $\alpha + n \neq 0$ for $n \geq 1$ (respectively, $\alpha > -1$) so that (ii) and (iii) follow from Theorem 2.9.

The case $\alpha = 0$ is the one originally studied by Laguerre [26]. The case $\alpha > -1$ is due to Sonine [34] and the generalized Laguerre PS for $\alpha < -1$ and $-\alpha \notin \mathbb{N}$ has been recently studied by Morton and Krall [32].

Case 4: Hermite polynomials

We assume $\ell_{22} = \ell_{21} = 0$, $\ell_{20} \neq 0$, and $\ell_{11} < 0$. Then, by a real linear change of variable, the equation (1.3) can be transformed into

$$L[y](x) = y''(x) - 2xy'(x) = -2ny(x). \tag{3.10}$$

The differential expression $L[\cdot]$ in (3.10) is admissible and so the equation (3.10) has a unique monic PS $\{H_n(x)\}_{n=0}^{\infty}$ of solutions called the Hermite polynomials :

$$H_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!\,(n-2k)!} \frac{x^{n-2k}}{4^k} \qquad (n \geq 0), \tag{3.11}$$

where $[x]$ is the integer part of x.

PROPOSITION 3.4. *The Hermite PS $\{H_n(x)\}_{n=0}^{\infty}$ is a positive-definite OPS.*

Proof. We have from (2.20), (2.21), (2.22), and (3.10)

$$\beta_n = \frac{n}{2} \qquad (n \geq 0). \tag{3.12}$$

Hence, the proposition follows from Theorem 2.9.

Case 5: Twisted Hermite polynomials

Assume $\ell_{22} = \ell_{21} = 0$, $\ell_{20} \neq 0$, and $\ell_{11} > 0$. Then, by a real linear change of variable, the equation (1.3) can be transformed into

$$L[y](x) = y''(x) + 2xy'(x) = 2ny(x). \tag{3.13}$$

The differential expression $L[\cdot]$ in (3.13) is admissible and so the equation (3.13) has a unique monic PS $\{\check{H}_n(x)\}_{n=0}^{\infty}$ of solutions. We call $\{\check{H}_n(x)\}_{n=0}^{\infty}$ the *twisted Hermite* PS. In order to find $\check{H}_n(x)$ explicitly, we set $x = it$ and $\check{H}_n(x) = \check{H}_n(it) = i^n Z_n(t)$ with $i = \sqrt{-1}$. Then $Z_n(t)$ is a monic polynomial of degree n and satisfies the Hermite differential equation (3.10) so that $Z_n(t) = H_n(t)$. Hence, we have

$$\check{H}_n(x) = i^n H_n(-ix) = n! \sum_{k=0}^{[n/2]} \frac{1}{k!\,(n-2k)!} \frac{x^{n-2k}}{4^k} \qquad (n \geq 0). \tag{3.14}$$

PROPOSITION 3.5. *The twisted Hermite PS* $\{\check{H}_n(x)\}_{n=0}^{\infty}$ *is an OPS but not a positive-definite OPS.*

Proof. We have from (2.20), (2.21), (2.22), and (3.13)

$$\beta_n = \frac{-n}{2} \qquad (n \geq 1). \tag{3.15}$$

Hence, the proposition follows from Theorem 2.9.

Case 6: Twisted Jacobi polynomials

We assume $\ell_{22} \neq 0$ and $\ell_{21}^2 - 4\ell_{22}\ell_{20} < 0$. Then, by a real linear change of variable, the equation (1.3) can be transformed into

$$L[y](x) = (1 + x^2)y''(x) + (dx + e)y'(x) = n(n + d - 1)y(x). \tag{3.16}$$

We assume $-(d - 1) \notin \mathbb{N}$ so that $L[\cdot]$ in (3.16) is admissible. Then the equation (3.16) has a unique monic PS $\{\check{P}_n(x; d, e)\}_{n=0}^{\infty}$ of solutions. We call $\{\check{P}_n(x; d, e)\}_{n=0}^{\infty}$ the *twisted Jacobi* PS. In order to find $\check{P}_n(x; d, e)$ explicitly, we set $x = it$ and $\check{P}_n(x; d, e) = \check{P}_n(it; d, e) = i^n Z_n(t)$. Then $Z_n(t)$ is a monic polynomial of degree n and satisfies

$$(1 - t^2)y''(t) + (ie - dt)y'(t) = -n(n + d - 1)y(t),$$

which is the Jacobi differential equation (3.1) when

$$ie = \beta - \alpha \quad \text{and} \quad d = \alpha + \beta + 2.$$

Hence we have $Z_n(t) = P_n^{(\alpha,\beta)}(t)$ and $\check{P}_n^{(\alpha,\beta)}(x) = i^n P_n^{(\alpha,\beta)}(-ix)$ so that

(3.17)

$$\check{P}_n^{(\alpha,\beta)}(x) = \binom{2n+\alpha+\beta}{n}^{-1} \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-i)^{n-k}(x+i)^k$$

$(n \geq 0)$, where $\check{P}_n(x; d, e) = \check{P}_n(x; \alpha+\beta+2, i(\alpha-\beta)) = \check{P}_n^{(\alpha,\beta)}(x)$. Note that even though the expression for $\check{P}_n^{(\alpha,\beta)}(x)$ in (3.17) involves i, $\check{P}_n^{(\alpha,\beta)}(x)$ is a real polynomial of degree n since $\beta = \bar{\alpha}$.

PROPOSITION 3.6. *The twisted Jacobi PS* $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ *is an OPS* (*but not a positive-definite OPS*) *if and only if* $-(\alpha+\beta+1) \notin \mathbb{N}$.

Proof. We have, from (2.20), (2.21), (2.22), and (3.16),

(3.18)

$$\beta_n = \frac{-4n(\alpha+\beta+n)(\alpha+n)(\beta+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)} \qquad (n \geq 1).$$

Since $\beta = \bar{\alpha}$, $\beta_n \neq 0$ for $n \geq 1$ if and only if $\alpha+\beta+n \neq 0$ for $n \geq 2$ and $\beta_n < 0$ for n large enough. Hence, the proposition follows from Theorem 2.9.

The twisted Jacobi PS first appeared in the paper [33] of Romanovski as a PS satisfying the differential equation

$$(x^2+a^2)y''(x) + [2(1-m)x - va]y'(x) - n(n+1-2m)y(x) = 0,$$

where $a, m, v > 0$. He provided identities for the twisted Jacobi PS including the three term recurrence relation, the differentiation formula, and the orthogonality (with an incorrect weight function ; see section four).

As discussed in the introduction, Bochner [3] classified the so-called Sturm-Liouville polynomial systems that can arise as eigenfunctions of the differential equation (1.3). His analysis allowed a complex linear change of variable in his classification. Consequently, he identified the Hermite PS with the twisted Hermite PS, and the Jacobi PS with the twisted Jacobi PS.

Later, Cryer [5] found the twisted Jacobi PS as the Jacobi PS with complex parameters in his characterization of the classical OPS's through the Rodrigues' type formula.

Lastly in this section, we discuss briefly the problem of finding moments of the classical OPS's.

For each classical OPS, we can compute the moments $\{\sigma_n\}_{n=0}^{\infty}$ of its canonical moment functional σ by solving the corresponding moment equation (2.7) successively starting from any non-zero value for σ_0. However, the moment equation is, in general, a three term recurrence relation, which is not easy to solve. Morton and Krall [32] introduced an idea by which we can always reduce a three term recurrence relation to a two term recurrence relation. Let $t = x - x_0$, where x_0 is a constant, possibly complex, that will be chosen later. Then, in terms of the new variable t, the differential equation (1.3) and the corresponding moment equation (2.7) become

$$\begin{aligned}[\ell_{22}t^2 + (2\ell_{22}x_0 + \ell_{21})t + \ell_{22}x_0^2 + \ell_{21}x_0 + \ell_{20}]y''(t) \\ + [\ell_{11}t + (\ell_{11}x_0 + \ell_{10})]y'(t) = \lambda_n y(t),\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}(\ell_{11} + n\ell_{22})\sigma_{n+1}(x_0) + [\ell_{11}x_0 + \ell_{10} + n(2\ell_{22}x_0 + \ell_{21})]\sigma_n(x_0) \\ + n(\ell_{22}x_0^2 + \ell_{21}x_0 + \ell_{20})\sigma_{n-1}(x_0) = 0 \quad (n \geq 0),\end{aligned} \tag{3.20}$$

where $\sigma_n(x_0) = \langle \sigma, (x - x_0)^n \rangle$ is the nth moment of σ about x_0. If we choose x_0 so that $\ell_{22}x_0^2 + \ell_{21}x_0 + \ell_{20} = 0$, then the equation (3.20) becomes a two term recurrence relation and we have

$$\sigma_n = \langle \sigma, x^n \rangle = \langle \sigma, [(x - x_0) + x_0]^n \rangle = \sum_{k=0}^{n} \binom{n}{k} x_0^{n-k} \sigma_k(x_0) \qquad (n \geq 0). \tag{3.21}$$

We illustrate the above procedure for the twisted Jacobi polynomials; see Morton and Krall [32] for a similar discussion of the moments for the other classical OPS's, except the twisted Hermite PS. The moment equation for the twisted Hermite PS is a two-term recurrence relation, which can be solved easily.

Let $\check{\sigma} = \check{\sigma}^{(\alpha,\beta)}$ be the canonical moment functional of the twisted Jacobi PS $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ with $\check{\sigma}_0 = \langle \check{\sigma}, 1 \rangle = 1$. The corresponding moment equation is

$$(\alpha + \beta + n + 2)\check{\sigma}_{n+1} + i(\alpha - \beta)\check{\sigma}_n + n\check{\sigma}_{n-1} = 0 \qquad (n \geq 0), \tag{3.22}$$

which is a three-term recurrence relation unless $\alpha = \beta$. If we choose x_0 to be i and let $\{\check{\sigma}_n(i)\}_{n=0}^{\infty}$ be the moments of $\check{\sigma}$ about i, then $\{\check{\sigma}_n(i)\}_{n=0}^{\infty}$ satisfies a two-term recurrence relation

$$(\alpha + \beta + n + 2)\check{\sigma}_{n+1}(i) + 2i(\alpha + n + 1)\check{\sigma}_n(i) = 0 \qquad (n \geq 0),$$

from which it follows that

$$\check{\sigma}_n(i) = \frac{(-1)^n(2i)^n(\alpha+1)_n}{(\alpha+\beta+2)_n} \qquad (n \geq 0),$$

where $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$ for any complex number α and integer $k \geq 1$. We now obtain, from (3.21),

$$\check{\sigma}_n = i^n \sum_{j=0}^{n} \frac{\binom{n}{j}(-2)^j(\alpha+1)_j}{(\alpha+\beta+2)_j} \qquad (n \geq 0). \tag{3.23}$$

Similarly, if we use $\check{\sigma}_n(-i)$ instead of $\check{\sigma}_n(i)$, we then obtain

$$\check{\sigma}_n = (-i)^n \sum_{j=0}^{n} \frac{\binom{n}{j}(-2)^j(\beta+1)_j}{(\alpha+\beta+2)_j} \qquad (n \geq 0). \tag{3.24}$$

Note that all $\check{\sigma}_n$ are real since the complex conjugate of $\check{\sigma}_n$ (recall $\beta = \bar{\alpha}$) in (3.23) is exactly $\check{\sigma}_n$ in (3.24).

4. Integral representation of orthogonality

Although using moment functionals to introduce orthogonality has many advantages as we have seen in previous sections, it is still desirable to express the orthogonality as an integral with respect to a suitable measure. Such an integral representation of orthogonality is always possible due to the following classical results on the moment problem .

Given any sequence $\{\sigma_n\}_{n=0}^{\infty}$ of real numbers,

(i) (Boas [2]) there is a function μ: $\mathbb{R} \to \mathbb{R}$ of bounded variation on $\mathbb{R}$ such that

$$\sigma_n = \int_{\mathbb{R}} x^n \, d\mu(x) \qquad (n \geq 0); \tag{4.1}$$

(ii) (Duran [7]) there is a C^∞-function ϕ: $\mathbb{R} \to \mathbb{R}$ in the Schwartz space S such that

$$\sigma_n = \int_{\mathbb{R}} x^n \phi(x) \, dx \qquad (n \geq 0). \tag{4.2}$$

Hence for any moment functional σ, there is a distribution $w_\sigma(x)$ on $\mathbb{R}$ (for example, we may take $w_\sigma(x)$ to be $\phi(x)$ in (4.2)) such that

$$\langle \sigma, \pi \rangle = \langle w_\sigma, \pi \rangle \qquad (\pi \in \mathcal{P}), \tag{4.3}$$

where $\langle w_\sigma, \pi \rangle$ is the action of the distribution $w_\sigma(x)$ on the test function $\pi(x)$. In particular, if σ is an orthogonalizing moment functional of an OPS $\{P_n(x)\}_{n=0}^\infty$, we call $w_\sigma(x)$ in (4.3) an orthogonalizing weight for $\{P_n(x)\}_{n=0}^\infty$.

Recently, there have been several attempts of effectively finding orthogonalizing weights for various classes of OPS's. Morton and Krall [32] introduced a formal δ-series expansion of a moment functional σ :

$$\sigma \approx \sum_{n=0}^{\infty} (-1)^n \sigma_n \delta^{(n)}(x)/n!$$

and found, via the Fourier transform, orthogonalizing weights for the Jacobi, Laguerre, and Hermite PS's. This formal δ-series expansion was also used in Kim and Kwon [14] to produce an orthogonalizing hyperfunctional weight for the Bessel PS $\{B_n^{(2,2)}(x)\}_{n=0}^\infty$.

In case of a classical OPS $\{P_n(x)\}_{n=0}^\infty$ satisfying the differential equation (1.3), we may use the corresponding weight equation (2.11) to find an orthogonalizing weight for $\{P_n(x)\}_{n=0}^\infty$. To do this, however, we must interpret (2.11) as a classical differential equation with the right-hand side of (2.11) replaced by a function (not necessarily identically zero) having zero moments.

To be precise we have the following Theorem, which is a special case of Theorem 2.3 in [22] for second-order differential equations (see also [28, Theorem 5.6]).

THEOREM 4.1. *Let $\{P_n(x)\}_{n=0}^\infty$ be a classical OPS satisfying the differential equation* (1.3). *If $w(x)$ is an orthogonalizing weight distribution for $\{P_n(x)\}_{n=0}^\infty$, then $w(x)$ satisfies the distributional differential equation*

$$(\ell_2(x)w(x))' - \ell_1(x)w(x) = g(x), \tag{4.4}$$

where $g(x)$ is a distribution having zero moments; that is,

$$\langle g(x), x^n \rangle = 0 \qquad (n \geq 0). \tag{4.5}$$

Conversely, if $w(x)$ is a distribution such that

(i) *$w(x)$ decays rapidly at infinity so that $< w, x^n >$ exists and is finite for all $n \geq 0$;*

(ii) *$w(x)$ is a solution to equation* (4.4) *on $\mathbb{R}$ distributionally ;*

and

(iii) $w(x)$ *is non-trivial as a moment functional,*

then $w(x)$ *is an orthogonalizing weight distribution for* $\{P_n(x)\}_{n=0}^{\infty}$.

Condition (iii) in the above Theorem 4.1 means that $\langle w, x^n \rangle \neq 0$ for some $n \geq 0$. For any classical OPS, there always exists a distributional orthogonalizing weight $w(x)$ satisfying the conditions (i), (ii), (iii) in Theorem 4.1. In fact, it is enough to take $w(x)$ to be $\phi(x)$ in (4.2) where $\{\sigma_n\}_{n=0}^{\infty}$ are the moments of any canonical moment functional σ of the given classical OPS.

We call the equation (4.4) the non-homogeneous weight equation for the differential equation (1.3). When $g(x) \equiv 0$, the homogeneous weight equation

$$(\ell_2(x)w(x))' - \ell_1(x)w(x) = 0 \tag{4.6}$$

is exactly the symmetry equation (2.12) of (1.3) (see Remark 2.1).

Although it turns out that it is enough to solve classically the *homogeneous* weight equation (4.6) for an orthogonalizing weight for any positive-definite classical OPS (as shown by Lesky [27]), we must, in general, consider the *non-homogeneous* weight equation (4.4) in the space of distributions; see, for example, Kwon, Kim, and Han [22] for the case of the Bessel PS and Littlejohn [28] and Krall and Littlejohn [16] for other classical OPS's as well as non-classical OPS's satisfying higher order differential equations.

There are several examples of non-trivial continuous functions having zero moments available. For example, the function $g(x)$ given by

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \exp(-x^{\frac{1}{4}})\sin(x^{\frac{1}{4}}) & \text{if } x > 0 \end{cases} \tag{4.7}$$

is continuous on $\mathbb{R}$ and has zero moments. This function was found by Stieltjes [35]. For more such examples, we refer to Hardy [12] and Maroni [30].

Once an orthogonalizing weight $w(x)$ (or any orthogonalizing moment functional σ) of an OPS $\{P_n(x)\}_{n=0}^{\infty}$ is chosen, the squared norms $\langle w, P_n^2 \rangle$ can be computed most easily from the three-term recurrence relation (2.18). In fact, we have (see [4, Theorem 4.2 in Chap. 1])

$$\langle w(x), P_n^2 \rangle = \prod_{j=0}^{n} \beta_j, \tag{4.8}$$

where $\beta_0 = \langle w, P_0^2 \rangle = \langle w, 1 \rangle$ and β_n $(n \geq 1)$ are the constants in (2.18).

We shall now construct an orthogonalizing weight for each classical OPS found in section three. We always assume that the parameters involved in each PS are restricted so that the PS is an OPS.

Case 1: Jacobi polynomials

In this case, the homogeneous weight equation corresponding to the Jacobi differential equation (3.1) is

$$(1 - x^2)w'(x) + [(\alpha + \beta)x - (\beta - \alpha)]w(x) = 0, \tag{4.9}$$

which is equivalent to

$$(1 - x^2)[(1 - x)^{-\alpha}(1 + x)^{-\beta} w(x)]' = 0 \tag{4.10}$$

for $x \neq \pm 1$. Then the general distributional solution of (4.10) for $x \neq \pm 1$ is

$$w(x) = [c_1 H(1 - x) + c_2 H(1 + x) + c_3](1 - x)^{\alpha}(1 + x)^{\beta},$$

where c_i $(i = 1, 2, 3)$ is an arbitrary constant and $H(x)$ is the Heaviside function. If we choose $c_1 = -1$, $c_2 = +1$, and $c_3 = 0$, then this $w(x)$ extends to a distribution on $\mathbb{R}$ (see Remark 4.1 below) :

$$w^{(\alpha,\beta)}(x) = (1 - x)_+^{\alpha}(1 + x)_+^{\beta}, \tag{4.11}$$

which is a non-trivial distributional solution to (4.9) on $\mathbb{R}$ with compact support $[-1, 1]$. Since $w^{(\alpha,\beta)}(x)$ satisfies the conditions (i), (ii), (iii) in Theorem 4.1, $w^{(\alpha,\beta)}(x)$ is an orthogonalizing weight for $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$. We then have, from (3.3) and (4.8),

$$\begin{aligned} &\langle w^{(\alpha,\beta)}(x), [P_n^{(\alpha,\beta)}(x)]^2 \rangle \\ &\qquad = \frac{2^{2n+\alpha+\beta+1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)\Gamma(n + \alpha + \beta + 1)n!}{\Gamma(2n + \alpha + \beta + 1)\Gamma(2n + \alpha + \beta + 2)} \end{aligned} \tag{4.12}$$

for $n \geq 0$, since $\langle w^{(\alpha,\beta)}(x), 1 \rangle = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$.

REMARK 4.1. For any complex number a, consider the function $f_a\colon \mathbb{R} \to \mathbb{C}$ defined by

$$f_a(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^a & \text{if } x > 0 \end{cases},$$

where we take $\log x$ to be real for $x > 0$ so that x^a is defined uniquely for $x > 0$. The function $f_a(x)$ always extends to a distribution x_+^a on $\mathbb{R}$ with support in $[0, \infty)$. For $\operatorname{Re} a > -1$, $f_a(x)$ is locally integrable on $\mathbb{R}$ so that $x_+^a = f_a(x)$ and for $\operatorname{Re} a \le -1$, x_+^a is obtained from $f_a(x)$ by analytic continuation and regularization. For details on the distribution x_+^a, we refer to Hörmander [13, Chap. 3.3.2]; see also Morton and Krall [32] for an explicit integral representation of the distribution $w^{(\alpha,\beta)}(x)$ in (4.11).

Case 2: Bessel polynomials

In this case, it is more convenient to replace x by $\frac{\beta x}{2}$ and α by $\alpha + 2$ so that the equation (3.4) becomes

$$L[y](x) = x^2 y''(x) + [(\alpha + 2)x + 2]y'(x) = n(n + \alpha + 1)y(x), \tag{4.13}$$

where $-(\alpha + 1) \notin \mathbb{N}$. We then denote $B_n^{(\alpha+2,2)}(x)$ by $B_n^{(\alpha)}(x)$. Now, the homogeneous weight equation corresponding to (4.13) is

$$x^2 w'(x) - (\alpha x + 2)w(x) = 0, \tag{4.14}$$

of which the only one linearly independent distributional solution with support in $[0, \infty)$ is

$$w_0(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^\alpha \exp(-2/x) & \text{if } x > 0. \end{cases} \tag{4.15}$$

Romanovski [33] used $w_0(x)$ as an orthogonalizing weight for Bessel PS, but $w_0(x)$ cannot be an orthogonalizing weight since it does not decay rapidly at infinity. In fact, we have

$$\lim_{x\to\infty} x^n w_0(x) = \infty$$

for $n + \alpha > 0$. We now consider the non-homogeneous weight equation

$$x^2 w'(x) - (\alpha x + 2)w(x) = g(x), \tag{4.16}$$

where $g(x)$ is a function with zero moments. For $x \neq 0$, the general solution of (4.16) is

$$w(x) = \begin{cases} c_1(-x)^\alpha e^{-2/x} & \text{if } x < 0 \\ x^\alpha e^{-2/x} \int_0^x e^{2/t} t^{-2-\alpha} g(t)\, dt + c_2 x^\alpha e^{-2/x} & \text{if } x > 0, \end{cases}$$

where c_1 and c_2 are arbitrary constants. With concern for the boundary condition (i) in Theorem 4.1, we choose $c_1 = 0$ and $c_2 = -\int_0^\infty e^{2/t} t^{-2-\alpha} g(t)\, dt$ to obtain

$$w^{(\alpha)}(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ -x^\alpha e^{-2/x} \int_x^\infty e^{2/t} t^{-2-\alpha} g(t)\, dt & \text{if } x > 0. \end{cases} \tag{4.17}$$

If we further take $g(x)$ to be the function given in (4.7), then $w^{(\alpha)}$ in (4.17) is a continuous function on $\mathbb{R}$ satisfying the conditions (i) and (ii) in Theorem 4.1 (see [9], [22], and [30]). Hence, $w^{(\alpha)}(x)$ in (4.17) (with $g(x)$ in (4.7)) is an orthogonalizing weight for Bessel PS $\{B_n^{(\alpha)}(x)\}_{n=0}^\infty$ if and only if

$$\langle w^{(\alpha)}(x), 1\rangle = -\int_0^\infty x^\alpha e^{-2/x} \left[\int_x^\infty e^{2/t} t^{-2-\alpha} g(t)\, dt\right] dx \neq 0. \tag{4.18}$$

Condition (4.18) was first proved in [22] for $\alpha = 0$ and, recently, Maroni [30] proved (4.18) for all $\alpha \geq 12(\frac{2}{\pi})^4 - 2$.

If we let $\sigma^{(\alpha)}$ be the canonical moment functional of $\{B_n^{(\alpha)}(x)\}_{n=0}^\infty$ with $\sigma_0^{(\alpha)} = 1$, then we have from (3.6) and (4.8)

$$\langle \sigma^{(\alpha)}, [B_n^{(\alpha)}(x)]^2\rangle = \frac{(-4)^n n!\Gamma(\alpha+2)\Gamma(\alpha+n+1)}{\Gamma(\alpha+2n+1)\Gamma(\alpha+2n+2)} \qquad (n \geq 0). \tag{4.19}$$

REMARK 4.2. Krall and Frink [19] found the complex orthogonality (now called the Bessel orthogonality) of the Bessel PS through the contour integral along the unit circle in the complex plane. Although the homogeneous weight equation (4.14) cannot yield a distributional orthogonalizing weight for the Bessel PS, it has a non-trivial hyperfunctional solution with support at $\{0\}$ with respect to which the Bessel PS is orthogonal (see [9], [14], and [15]).

Later in this section, we will discuss again real orthogonalizing weights for $\{B_n^{(\alpha)}(x)\}_{n=0}^\infty$ for any α with $-(\alpha+1) \notin \mathbb{N}$.

Case 3: Laguerre polynomials

In this case, the homogeneous weight equation corresponding to the Laguerre differential equation (3.7) is

$$xw'(x) + (x - \alpha)w(x) = 0. \tag{4.20}$$

If we set $v(x) = e^x w(x)$, then $v(x)$ satisfies the Euler equation

$$xv'(x) - \alpha v(x) = 0,$$

of which the general distributional solution is

$$v(x) = c_1 x_+^\alpha + c_2 x_-^\alpha,$$

where c_1 and c_2 are arbitrary constants and x_-^α is the distribution on $\mathbb{R}$ with support in $(-\infty, 0]$ (defined similarly as x_+^α; see Remark 4.1 and Hörmander [13, Chap. 3.3.2]). Hence, the general solution of (4.20) is

$$w(x) = c_1 x_+^\alpha e^{-x} + c_2 x_-^\alpha e^{-x}.$$

For this $w(x)$ to vanish at infinity, c_2 must be zero. Then by taking $c_1 = 1$, we obtain

$$w^{(\alpha)}(x) = x_+^\alpha e^{-x}. \tag{4.21}$$

Since $w^{(\alpha)}(x)$ in (4.21) satisfies the conditions (i), (ii), (iii) in Theorem 4.1, $w^{(\alpha)}(x)$ is an orthogonalizing weight for $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$. Since

$$\langle x_+^\alpha e^{-x}, 1\rangle = \Gamma(\alpha + 1)$$

we have, from (3.9) and (4.8),

$$\langle w^{(\alpha)}(x), [L_n^{(\alpha)}(x)]^2\rangle = n!\,\Gamma(n + \alpha + 1). \tag{4.22}$$

Case 4: Hermite polynomials

In this case, the homogeneous weight equation corresponding to the Hermite differential equation (3.10) is

$$w'(x) + 2xw(x) = 0, \tag{4.23}$$

of which the only one linearly independent distributional solution is

$$w(x) = \exp(-x^2). \tag{4.24}$$

Since $w(x)$ in (4.24) satisfies the conditions (i), (ii), (iii) in Theorem 4.1, $w(x)$ is an orthogonalizing weight for $\{H_n(x)\}_{n=0}^{\infty}$. We then have, from (3.12) and (4.8),

$$\langle w(x), H_n^2(x)\rangle = \int_{-\infty}^{\infty} H_n^2(x)\exp(-x^2)\,dx = \sqrt{\pi}n!\,2^{-n} \qquad (n \geq 0). \tag{4.25}$$

Case 5: Twisted Hermite polynomials

In this case, the homogeneous weight equation corresponding to the twisted Hermite differential equation (3.13) is

$$w'(x) - 2x w(x) = 0, \tag{4.26}$$

of which the only one linearly independent distributional solution is

$$w_0(x) = \exp(x^2),$$

which cannot be an orthogonalizing weight. However, from (3.14) and (4.25), we can obtain the complex orthogonality :

$$\int_{-i\infty}^{i\infty} \check{H}_m(x)\check{H}_n(x)\exp(x^2)\,dx = (-1)^n\sqrt{\pi}n!\,2^{-n}i\,\delta_{mn} \qquad (m \text{ and } n \geq 0). \tag{4.27}$$

Let us now consider the non-homogeneous weight equation

$$w'(x) - 2x w(x) = g(x), \tag{4.28}$$

where $g(x)$ is a non-trivial continuous function on $\mathbb{R}$ with zero moments and support in $[0, \infty)$. Then the general solution of (4.28) is

$$w(x) = ce^{x^2} + e^{x^2}\int_0^x e^{-t^2} g(t)\,dt,$$

where c is an arbitrary constant. For this $w(x)$ to vanish at infinity, c must be zero and

$$\int_0^\infty e^{-x^2} g(x)\,dx = 0. \tag{4.29}$$

Then we have

$$w(x) = \begin{cases} 0 & \text{if } x \le 0 \\ e^{x^2} \int_0^x e^{-t^2} g(t)\,dt & \text{if } x > 0. \end{cases} \tag{4.30}$$

Note that $w(x)$ in (4.30) is a classical solution to (4.26) on $\mathbb{R}$. If we further assume

$$\lim_{x\to\infty} x^n g(x) = 0 \qquad (n \ge 0), \tag{4.31}$$

then it is easy to see that

$$\lim_{x\to\infty} x^n w(x) = 0 \qquad (n \ge 0),$$

and so $w(x)$ satisfies the conditions (i), (ii) in Theorem 4.1. Consequently, $w(x)$ in (4.30) is a real orthogonalizing weight for $\{\check{H}_n(x)\}_{n=0}^\infty$ if and only if

$$\langle w(x), 1\rangle = \int_0^\infty e^{x^2} \left[\int_0^x e^{-t^2} g(t)\,dt\right] dx \ne 0. \tag{4.32}$$

The existence of a weight $w(x)$ for the twisted Hermite PS, of the form given in (4.30) and satisfying (4.29), is discussed below in Remark 4.3.

If we let σ be the orthogonalizing moment functional for $\{\check{H}_n(x)\}_{n=0}^\infty$ with $\sigma_0 = \sqrt{\pi}$, then we have, from (3.15) and (4.8),

$$\langle \sigma, [\check{H}_n(x)]^2\rangle = (-1)^n \sqrt{\pi}\, n! 2^{-n} \qquad (n \ge 0). \tag{4.33}$$

REMARK 4.3. We can easily see that there is a non-trivial function $g(x)$ with zero moments, which also satisfies the condition (4.29). Choose any two linearly independent continuous functions $g_1(x)$ and $g_2(x)$ with zero moments and support in $[0, \infty)$. Set

$$A_i = \int_0^\infty e^{-x^2} g_i(x)\,dx \qquad (i = 1, 2).$$

If $A_i \neq 0$ ($i = 1, 2$), then

$$g(x) = A_2 g_1(x) - A_1 g_2(x)$$

satisfies the condition (4.29) and has zero moments.

Case 6: Twisted Jacobi polynomials

In this case, the homogeneous weight equation corresponding to the twisted Jacobi differential equation (3.16) is

$$(1 + x^2)w'(x) + [(d - 2)x + e]w(x) = 0, \tag{4.34}$$

of which the only linearly independent distributional solution is

$$f(x) = (1 + x^2)^{\frac{2-d}{2}} \exp(-e \arctan x).$$

Romanovski [33] used $f(x)$ as an orthogonalizing weight for the twisted Jacobi PS, but $f(x)$ cannot be an orthogonalizing weight since it does not decay rapidly at infinity. In fact, we have

$$\lim_{x \to \infty} x^n f(x) = \infty$$

for $n + 2 - d > 0$. However, from (3.17) and (4.12), we can obtain the complex orthogonality

(4.35)

$$\begin{aligned} &\langle (1 - x)_+^\alpha (1 + x)_+^\beta, \check{P}_m^{(\alpha,\beta)}(ix)\check{P}_n^{(\alpha,\beta)}(ix)\rangle \\ &= \frac{(-1)^n 2^{2n+\alpha+\beta+1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)\Gamma(n + \alpha + \beta + 1)n!}{\Gamma(2n + \alpha + \beta + 1)\Gamma(2n + \alpha + \beta + 2)}\delta_{mn} \end{aligned}$$

(m and $n \geq 0$), where $ie = \beta - \alpha$ and $d = \alpha + \beta + 2$.

Let us now consider the non-homogeneous weight equation

$$(1 + x^2)w'(x) + [(d - 2)x + e]w(x) = g(x), \tag{4.36}$$

where $g(x)$ is a non-trivial continuous function on $\mathbb{R}$ with zero moments and support in $[0, \infty)$. Then the general solution of (4.36) is

$$w(x) = e^{f(x)}[c + \int_0^x e^{-f(t)}(1 + t^2)^{-1} g(t)\, dt],$$

where c is an arbitrary constant. For this $w(x)$ to vanish at infinity, c must be zero and

$$\int_0^\infty e^{-f(t)}(1+t^2)^{-1}g(t)\,dt = 0. \tag{4.37}$$

Then we have

$$\check{w}(x) = \begin{cases} 0 & \text{if } x \le 0 \\ e^{f(x)} \int_0^x e^{-f(t)}(1+t^2)^{-1}g(t)\,dt & \text{if } x > 0. \end{cases} \tag{4.38}$$

Note that $\check{w}(x)$ in (4.38) is a classical solution to (4.36). If $g(x)$ satisfies the condition (4.31), then $\check{w}(x)$ satisfies the conditions (i) and (ii) in Theorem 4.1. Consequently, $\check{w}(x)$ in (4.38) is a real orthogonalizing weight for $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$ if and only if

$$\langle \check{w}(x), 1\rangle = \int_0^\infty e^{f(x)}\left[\int_0^x e^{-f(t)}(1+t^2)^{-1}g(t)\,dt\right]dx \neq 0. \tag{4.39}$$

If we let $\check{\sigma}$ be the orthogonalizing moment functional of $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$ with

$$\check{\sigma}_0 = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},$$

then we have, from (3.18) and (4.8),

$$\begin{aligned} &\langle \check{\sigma}, [\check{P}_n^{(\alpha,\beta)}(x)]^2\rangle \\ &\quad = \frac{(-1)^n 2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)n!}{\Gamma(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+2)}. \\ &\qquad (n \ge 0) \end{aligned} \tag{4.40}$$

REMARK 4.4. In the formula (4.12), the parameters α and β are real numbers with $-(\alpha+\beta+1)$, $-\alpha$, and $-\beta \notin N$. However, by analytic continuation, the same formula holds for complex parameters α and β as long as $-\operatorname{Re}(\alpha+\beta+1)$, $-\operatorname{Re}\alpha$, and $-\operatorname{Re}\beta \notin -\mathbb{N}$. This fact is used in (4.35), where $\beta = \bar{\alpha}$, $\operatorname{Re}\alpha = \operatorname{Re}\beta = \frac{d-2}{2}$, and $-(d-1) = -(\alpha+\beta+1) \notin \mathbb{N}$.

Constructing explicit real orthogonalizing weights for classical OPS's by solving the non-homogeneous weight equation (4.4) has been successful except, at the moment, for the Bessel PS $\{B_n^{(\alpha)}(x)\}_{n=0}^\infty$ when $0 \neq \alpha < 12(\frac{2}{\pi})^4 - 2$, the twisted Hermite PS $\{\check{H}_n(x)\}_{n=0}^{\infty}$, and the twisted Jacobi PS $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$. For any OPS (classical or not), its real orthogonalizing weight can be explicitly constructed by the following remarkable result on the general moment problem.

THEOREM 4.2 (Duran [8]). *For any sequence of real or complex numbers $\{\sigma_n\}_{n=0}^{\infty}$, define a function $w(x)$ by*

$$w(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{2}\int_0^{\infty}\left(\sum_{n=0}^{\infty}\sigma_n c_n t^n h(\lambda_n t)\right) J_0(\sqrt{xt})\,dt & \text{if } x > 0, \end{cases} \tag{4.41}$$

where $c_n = \frac{(-1)^n}{2^{2n+1}(n!)^2}$, $\lambda_n = n + \sum_{k=0}^{n} c_n$, $J_0(x)$ is the Bessel function of the first kind, and $h(x)$ is a C^{∞}-function on $\mathbb{R}$ with compact support satisfying $h(0) = 1$ and $h^{(n)}(0) = 0$ $(n \ge 1)$. Then, $w(x)$ is a function in the Schwartz space S and satisfies

$$\int_{-\infty}^{\infty} x^n w(x)\,dx = \int_0^{\infty} x^n w(x)\,dx = \sigma_n \qquad (n \ge 0).$$

In particular, if we take $\{\sigma_n\}_{n=0}^{\infty}$ in Theorem 4.2 to be the moments of a canonical moment functional of any classical OPS, then the function $w(x)$ in (4.41) is a real orthogonalizing weight for the OPS. Moreover, by Theorem 4.1, the function

$$g(x) = (\ell_2(x)w(x))' - \ell_1(x)w(x)$$

is a function, in the Schwartz space S with zero moments and support in $[0, \infty)$, satisfying the condition (4.31). In the case of the twisted Hermite or the twisted Jacobi polynomials, this $g(x)$ also satisfies (4.29) and (4.32) or (4.37) and (4.39) respectively.

REMARK 4.5. In the case of the Bessel, the twisted Hermite, and the twisted Jacobi polynomials, their complex orthogonality seems more natural than their real orthogonality. In fact, through the hyperfunctional representations of orthogonalizing weights, we can see that any OPS has both real and complex orthogonality : see, for example, [14].

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