JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 116 (2000) 243–262

www.elsevier.nl/locate/cam

Generalized Hahn's theorem

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Received 2 March 1999

Abstract

Let $\{P_n(x)\}_{n=0}^{\infty}$ be an orthogonal polynomial system and

$$L[\,\cdot\,] = \sum_{i=0}^{k} a_i(x)\mathrm{D}^i \quad \left(\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x}\right)$$

a linear differential operator of order k ($\geqslant 0$) with polynomial coefficients. We find necessary and sufficient conditions for a polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ defined by $Q_n(x) := L[P_{n+r}^{(r)}(x)], n \geqslant 0$, to be also an orthogonal polynomial system. We also give a few applications of this result together with the complete analysis of the cases: (i) $k = 0, 1, 2$ and $r = 0$, and (ii) $k = r = 1$. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 33C45; 34L05

Keywords: Differential equations; Orthogonal polynomials; Hahn's theorem

1. Introduction

In 1935, Hahn [4] proved: if $\{P_n(x)\}_{n=0}^{\infty}$ and $\{P_n'(x)\}_{n=0}^{\infty}$ are positive-definite orthogonal polynomial systems (OPS'), then $\{P_n(x)\}_{n=0}^{\infty}$ must be one of classical OPS' (Jacobi, Laguerre, or Hermite). Krall [8] and Webster [23] extended Hahn's theorem to quasi-definite OPS' (including Bessel polynomials [12]). Later, Hahn [5] and Krall [9] also showed: If for any fixed integer $r \geqslant 1$, $\{P_n(x)\}_{n=0}^{\infty}$ and $\{P_{n+r}^{(r)}(x)\}_{n=0}^{\infty}$ are OPS', then $\{P_n(x)\}_{n=0}^{\infty}$ must be a classical OPS. Recently, it is extended further as: If $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS and $\{P_n^{(r)}(x)\}_{n=0}^{\infty}$ is a WOPS, then $\{P_n(x)\}_{n=0}^{\infty}$ must be a classical OPS (cf. [16,19]).

Generalizing Hahn's theorem, we now ask: Given an OPS $\{P_n(x)\}_{n=0}^{\infty}$ and a linear differential operator $L[\cdot]=\sum_{i=0}^{k} a_i(x)\mathrm{D}^i$ with polynomial coefficients, when is the polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$

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PII: S0377-0427(99)00319-2

defined by

$$Q_n(x) := L[P_{n+r}^{(r)}(x)] = \sum_{i=0}^{k} a_i(x) P_{n+r}^{(i+r)}$$

also an OPS? Here, r is any nonnegative integer.

Krall and Sheffer [14] raised and solved the above problem for $r = 0, 1$ using the moments and the characterization of OPS' via formal generating series $G(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$ of a PS $\{P_n(x)\}_{n=0}^{\infty}$ (cf. [13]). Their method is quite complicated so that it seems to be impossible to be extended to the case $r \geqslant 2$. We solve the problem completely for any $r \geqslant 0$ by using the formal calculus of moment functionals (see Theorems 3.1 and 3.2), by which we can refine the characterizations of classical orthogonal polynomials in [19] (see Theorem 3.4). Finally, we analyse completely the cases for $k = 0, 1, 2$ and $r = 0$ or $k = r = 1$ and as by products, we obtain some new relations between classical orthogonal polynomials and classical-type orthogonal polynomials.

2. Preliminaries

All polynomials in this work are assumed to be real polynomials in one variable and we let $\mathscr{P}$ be the space of all real polynomials. We denote the degree of a polynomial $\pi(x)$ by $\deg(\pi)$ with the convention that $\deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ with $\deg(\phi_n) = n$, $n \geqslant 0$. Note that a PS forms a basis of $\mathscr{P}$.

We call any linear functional σ on $\mathscr{P}$ a moment functional and denote its action on a polynomial $\pi(x)$ by $\langle \sigma, \pi \rangle$. For a moment functional σ, we call

$$\sigma_n := \langle \sigma, x^n \rangle, \quad n = 0, 1, \ldots$$

the moments of σ. We say that a moment functional σ is quasi-definite (respectively, positive-definite) [2] if its moments $\{\sigma_n\}_{n=0}^{\infty}$ satisfy the Hamburger condition

$$\Delta_n(\sigma) := \det[\sigma_{i+j}]_{i,j=0}^{n} \neq 0 \quad (\text{respectively, } \Delta_n(\sigma) > 0), \quad n \geqslant 0.$$

Any PS $\{\phi_n(x)\}_{n=0}^{\infty}$ determines a moment functional σ (uniquely up to a nonzero constant multiple), called a canonical moment functional of $\{\phi_n(x)\}_{n=0}^{\infty}$, by the conditions

$$\langle \sigma, \phi_0 \rangle \neq 0 \quad \text{and} \quad \langle \sigma, \phi_n \rangle = 0, \quad n \geqslant 1.$$

Definition 2.1. A PS $\{P_n(x)\}_{n=0}^{\infty}$ is a weak orthogonal polynomial system (WOPS) if there is a nontrivial moment functional σ such that

$$\langle \sigma, P_m P_n \rangle = 0 \quad \text{if } 0 \leqslant m < n. \tag{2.1}$$

If we further have

$$\langle \sigma, P_n^2 \rangle = K_n, \quad n \geqslant 0,$$

where K_n are nonzero real constants, then we call $\{P_n(x)\}_{n=0}^{\infty}$ an orthogonal polynomial system (OPS). In either case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is a WOPS or an OPS relative to σ and call σ an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

It is immediate from (2.1) that for any WOPS $\{P_n(x)\}_{n=0}^{\infty}$, its orthogonalizing moment functional σ must be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. It is well known (see [Chapters 1 and 2]) that a moment functional σ is quasi-definite if and only if there is an OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ and then each $P_n(x)$ is uniquely determined up to a nonzero multiplicative constant. For a moment functional σ and a polynomial $\pi(x)$, we let σ' (the derivative of σ) and $\pi\sigma$ (the left multiplication of σ by $\pi(x)$) be the moment functionals defined by

$$\langle \sigma', \phi \rangle = -\langle \sigma, \phi' \rangle$$

and

$$\langle \pi\sigma, \phi \rangle = \langle \sigma, \pi\phi \rangle, \quad \phi \in \mathscr{P}.$$

Then it is easy to obtain the following (see [16,18]).

Lemma 2.1. *For a moment functional σ and a polynomial $\pi(x)$, we have*
(i) *Leibniz' rule*: $(\pi\sigma)' = \pi'\sigma + \pi\sigma'$;
(ii) $\sigma' = 0$ *if and only if* $\sigma = 0$.
Assume that σ is quasi-definite and $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ. Then
(iii) $\pi\sigma = 0$ *if and only if* $\pi(x) = 0$;
(iv) *for any other moment functional τ, $\langle \tau, P_n \rangle = 0$, $n \geqslant k+1$ for some integer $k \geqslant 0$ if and only if $\tau = \phi\sigma$ for some polynomial $\phi(x)$ of degree $\leqslant k$.*

It is well known [1,17] that there are essentially four distinct classical OPS' satisfying second-order differential equations with polynomial coefficients

$$\mathscr{L}[y](x) = \ell_2(x)y''(x) + \ell_1(x)y'(x) = \lambda_n y(x). \tag{2.2}$$

They are:
(i) Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ (orthogonal relative to $e^{-x^2}\,dx$) satisfying

$$y''(x) - 2xy'(x) = -2ny(x).$$

(ii) Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ (orthogonal relative to $x_+^{\alpha}e^{-x}dx$) satisfying

$$xy''(x) + (\alpha + 1 - x)y'(x) = -ny(x) \quad (\alpha \notin \{-1, -2, \ldots\}).$$

(iii) Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ (orthogonal relative to $(1-x)_+^{\alpha}(1+x)_+^{\beta}\,dx$) satisfying

$$(1-x^2)y''(x) + (\beta - \alpha - (\alpha+\beta+2)x)y'(x) = -n(n+\alpha+\beta+1)y(x)$$
$$(\alpha, \beta, \alpha+\beta+1 \notin \{-1, -2, \cdots\}).$$

(iv) Bessel polynomials $\{B_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ (see [12,15]) satisfying

$$x^2y''(x) + (\alpha x + 2)y'(x) = n(n+\alpha-1)y(x) \quad (\alpha \notin \{0, -1, -2, \ldots\}).$$

Here, x_+^{α} is the distribution with support in $[0, \infty)$, which is obtained by the regularization of the function

$$f_\alpha(x) = \begin{cases} x^\alpha & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

(see in [6, Chapter 3.3.2]).

More generally, Krall [10] (see also [18,21]) found necessary and sufficient conditions for an OPS to be eigenfunctions of differential equations with polynomial coefficients:

Proposition 2.2. *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be an OPS relative to* σ *and* $L_N[\cdot]=\sum_{i=1}^{N}\ell_i(x)\mathrm{D}^i$ $(\mathrm{D}=\mathrm{d}/\mathrm{d}x)$ *be a linear differential operator of order* N $(\geqslant 1)$ *with polynomial coefficients* $\ell_i(x)$ *of order* $\leqslant i$. *Then*

$$L_N[P_n](x)=\sum_{i=1}^{N}\ell_i(x)P_n^{(i)}(x)=\lambda_n P_n(x),\quad n\geqslant 0,$$

where

$$\lambda_n=\sum_{i=1}^{N}\frac{1}{i!}\ell_i^{(i)}(x)n(n-1)\cdots(n-i+1)$$

if and only if σ *satisfies* $r:=[(N+1)/2]$ *moment equations*

$$R_k(\sigma):=\sum_{i=2k+1}^{N}(-1)^i\binom{i-k-1}{k}(\ell_i\sigma)^{(i-2k-1)}=0,\quad k=0,1,\ldots,r-1.$$

Moreover, in this case, $N=2r$ *must be even.*

Using this characterization, Krall [11] classified all OPS' that are eigenfunctions of fourth-order differential equations. They are the four classical OPS' above and the three new OPS', now known as classical-type OPS' [7]:

(v) Legendre-type polynomials $\{P_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ (orthogonal relative to $(H(1-x^2)+(1/\alpha)(\delta(x-1)+\delta(x+1)))\,\mathrm{d}x)$ satisfying

$$(x^2-1)^2y^{(4)}+8x(x^2-1)y^{(3)}+4(\alpha+3)(x^2-1)y''+8\alpha xy'=\lambda_n y\quad\left(\alpha\neq\frac{-n(n-1)}{2},n\geqslant 0\right).$$

(vi) Laguerre-type polynomials $\{R_n(x)\}_{n=0}^{\infty}$ (orthogonal relative to $(\mathrm{e}^{-x}H(x)+(1/R)\delta(x))\,\mathrm{d}x)$ satisfying

$$x^2y^{(4)}-(2x^2-4x)y^{(3)}+[x^2-(2R+6)x]y''+[(2R+2)x-2R]y'$$
$$=\lambda_n y\quad(R\neq 0,-1,-2,\ldots).\tag{2.3}$$

(vii) Jacobi-type polynomials $\{S_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ (orthogonal relative to $((1-x)_+^{\alpha}H(x)+(1/M)\delta(x))\,\mathrm{d}x)$ satisfying

$$(x^2-x)^2y^{(4)}+2x(x-1)[(\alpha+4)x-2]y^{(3)}+x[(\alpha^2+9\alpha+14+2M)x-(6\alpha+12+2M)]y''$$
$$+[(\alpha+2)(2\alpha+2+2M)x-2M]y'=\lambda_n y\quad(\alpha\neq-1,-2,\ldots,\text{ and }n^2+\alpha n+M\neq 0,n\geqslant 0).$$

Here, $H(x)$ is the Heaviside step function.

In [20], we showed that if a fourth- (or higher) order differential equation has a classical OPS $\{P_n(x)\}_{n=0}^{\infty}$ as solutions, then the differential equation must be a linear combination of iterations of a second-order differential equation (2.2) having $\{P_n(x)\}_{n=0}^{\infty}$ as solutions.

3. Main results

In the following, we always let $\{P_n(x)\}_{n=0}^{\infty}$ be a monic OPS relative to σ and $L[\cdot]=\sum_{i=0}^{k} a_i(x)\mathrm{D}^i$ ($\mathrm{D}=\mathrm{d}/\mathrm{d}x$) a linear differential operator of order k with polynomial coefficients $a_i(x)=\sum_{j=0}^{i} a_{ij}x^j$, $0\leqslant i\leqslant k\,(a_k(x)\not\equiv 0)$. For an integer $r\geqslant 0$, we also let

$$Q_n(x)=L[P_{n+r}^{(r)}(x)]=\alpha_n x^n+\text{lower degree terms},\quad n\geqslant 0 \tag{3.1}$$

and assume that

$$\alpha_n:=\sum_{i=0}^{k} a_{ii}(n+r)_{(i+r)}\neq 0,\quad n\geqslant 0 \tag{3.2}$$

so that $\{Q_n(x)\}_{n=0}^{\infty}$ is also a PS, where

$$n_{(i)}=\begin{cases}1 & \text{if } i=0,\\ n(n-1)\cdots(n-i+1) & \text{if } i\geqslant 1.\end{cases}$$

We now ask: When is the PS $\{Q_n(x)\}_{n=0}^{\infty}$ also an OPS?

Then our main result is:

Theorem 3.1. *The* PS $\{Q_n(x)\}_{n=0}^{\infty}$ *defined by* (3.1) *is a WOPS if and only if there is a moment functional* $\tau\neq 0$ *and* $k+r+1$ *polynomials* $\{b_i(x)\}_{i=r}^{k+2r}$ *with* $\deg(b_i)\leqslant i$ *satisfying*

$$\sum_{i=j}^{k+r}(-1)^i\binom{i}{j}(a_{i-r}(x)\tau)^{(i-j)}=b_{j+r}(x)\sigma,\quad 0\leqslant j\leqslant k+r \tag{3.3}$$

or equivalently

$$\sum_{i=j}^{k+r}(-1)^i\binom{i}{j}(b_{i+r}(x)\sigma)^{(i-j)}=a_{j-r}(x)\tau,\quad 0\leqslant j\leqslant k+r, \tag{3.4}$$

where $a_i(x)=0$ *for* $i<0$. *In this case*, $\deg(b_r)=r$ *and*

$$\langle\tau,a_i\rangle=(-1)^{i+r}\langle\sigma,b_{i+2r}\rangle,\quad 0\leqslant i\leqslant k \tag{3.5}$$

so that $\langle\sigma,b_{2r}\rangle\neq 0$ *and* $b_{2r}(x)\not\equiv 0$. *Furthermore*, $\{Q_n(x)\}_{n=0}^{\infty}$ *is an OPS if and only if the polynomials* $\{b_i(x)\}_{i=r}^{k+2r}$ *satisfy, in addition to* (3.3),

$$\sum_{i=0}^{k+r} b_{i+r,i+r}n_{(i)}\neq 0,\quad n\geqslant 0, \tag{3.6}$$

where $b_i(x)=\sum_{j=0}^{i} b_{ij}x^j$. *In this case*, $\deg(b_r)=r$ *and* $b_{k+2r}(x)\neq 0$.

Proof. Assume that $\{Q_n(x)\}_{n=0}^{\infty}$ is a WOPS and let τ be a canonical moment functional of $\{Q_n(x)\}_{n=0}^{\infty}$. Then $\langle\tau,Q_mQ_n\rangle=0$, $0\leqslant m<n$. We shall prove that there are polynomials $\{b_i(x)\}_{i=r}^{k+2r}$ with $\deg(b_i)\leqslant i$ satisfying (3.3) by induction on $i=0,1,\ldots,k+r$. For $n\geqslant 1$,

$$0=\langle\tau,Q_n(x)\rangle=\left\langle\tau,\sum_{i=0}^{k} a_i(x)P_{n+r}^{(i+r)}(x)\right\rangle=\left\langle\sum_{i=0}^{k+r}(-1)^i(a_{i-r}\tau)^{(i)},P_{n+r}\right\rangle.$$

By Lemma 2.1(iv), there is a polynomial $b_r(x)$ of degree $\leqslant r$ such that

$$b_r(x)\sigma = \sum_0^{k+r} (-1)^i (a_{i-r}\tau)^{(i)}$$

so that (3.3) holds for $j=0$. Assume that for some ℓ with $0 \leqslant \ell < k+r$, there exist polynomials $\{b_i(x)\}_{i=r}^{\ell+r}$ of $\deg(b_i) \leqslant i$ such that (3.3) holds for $j=0,1,\dots,\ell$. Then for $n \geqslant \ell+2$,

$$0 = \langle \tau, Q_{\ell+1}Q_n\rangle = \left\langle \tau, Q_{\ell+1}\sum_{i=0}^{k} a_{i+r}P_{n+r}^{(i+r)}\right\rangle = \left\langle \sum_{i=0}^{k}(-1)^{i+r}(Q_{\ell+1}a_i\tau)^{(i+r)}, P_{n+r}\right\rangle$$

$$= \left\langle \sum_{i=0}^{k+r}(-1)^i \sum_{j=0}^{i}\binom{i}{j} Q_{\ell+1}^{(j)}(a_{i-r}\tau)^{(i-j)}, P_{n+r}\right\rangle$$

$$= \left\langle \sum_{j=0}^{k+r} Q_{\ell+1}^{(j)} \sum_{i=j}^{k+r}(-1)^i \binom{i}{j} (a_{i-r}\tau)^{(i-j)}, P_{n+r}\right\rangle$$

$$= \left\langle \sum_{j=0}^{\ell+1} Q_{\ell+1}^{(j)} \sum_{i=j}^{k+r}(-1)^i \binom{i}{j} (a_{i-r}\tau)^{(i-j)}, P_{n+r}\right\rangle$$

$$= Q_{\ell+1}^{(\ell+1)} \left\langle \sum_{i=\ell+1}^{k+r}(-1)^i \binom{i}{\ell+1} (a_{i-r}\tau)^{(i-\ell-1)}, P_{n+r}\right\rangle + \left\langle \sum_{j=0}^{\ell} Q_{\ell+1}^{(j)} b_{j+r}\sigma, P_{n+r}\right\rangle$$

$$= \alpha_{\ell+1}(\ell+1)! \left\langle \sum_{i=\ell+1}^{k+r}(-1)^i \binom{i}{\ell+1} (a_{i-r}\tau)^{(i-\ell-1)}, P_{n+r}\right\rangle + \left\langle \sigma, \sum_{j=0}^{\ell} Q_{\ell+1}^{(j)} b_{j+r}P_{n+r}\right\rangle.$$

Since $\deg(\sum_{j=0}^{\ell} Q_{\ell+1}^{j} b_{j+r}) \leqslant r+\ell+1 < n+r, \langle \sigma, \sum_{j=0}^{\ell} Q_{\ell+1}^{(j)} b_{j+r}P_{n+r}\rangle = 0$, so that

$$\left\langle \sum_{i=\ell+1}^{k+r}(-1)^i \binom{i}{\ell+1} (a_{i-r}\tau)^{(i-\ell-1)}, P_{n+r}\right\rangle = 0, \quad n \geqslant \ell+2.$$

Therefore, by Lemma 2.1(iv), there is a polynomial $b_{r+\ell+1}(x)$ with $\deg(b_{r+\ell+1}) \leqslant r+\ell+1$ such that $\sum_{i=\ell+1}^{k+r}(-1)^i \binom{i}{\ell+1} (a_{i-r}\tau)^{(i-\ell-1)} = b_{r+\ell+1}\sigma$, that is, (3.3) also holds for $j=\ell+1$.

Conversely, assume that there are moment functionals $\tau \neq 0$ and polynomials $\{b_i(x)\}_{i=r}^{k+2r}$ with $\deg(b_i) \leqslant i$ satisfying (3.3). Then

$$\langle \tau, Q_mQ_n\rangle = \left\langle \tau, Q_m \sum_0^{k} a_iP_{n+r}^{(i+r)}\right\rangle = \left\langle \sum_{i=0}^{k}(-1)^{i+r}(Q_ma_i\tau)^{(i+r)}, P_{n+r}\right\rangle$$

$$= \left\langle \sum_{i=0}^{k}(-1)^{i+r} \sum_{j=0}^{i+r}\binom{i+r}{j} Q_m^{(j)}(a_i\tau)^{(i+r-j)}, P_{n+r}\right\rangle$$

$$= \left\langle \sum_{j=0}^{k+r} Q_m^{(j)} \sum_{i=j}^{k+r} (-1)^i \binom{i}{j} (a_{i-r}\tau)^{(i-j)}, P_{n+r} \right\rangle$$

$$= \left\langle \sum_{j=0}^{k+r} Q_m^{(j)} b_{j+r}\sigma, P_{n+r} \right\rangle = \left\langle \sigma, \left(\sum_{j=0}^{k+r} Q_m^{(j)} b_{j+r} \right) P_{n+r} \right\rangle.$$

Hence,

$$\langle \tau, Q_m Q_n \rangle = 0, \quad 0 \leqslant m < n$$

since $\deg(\sum_0^{k+r} b_{j+r} Q_m^{(j)}) \leqslant r + m < n + r$. Thus $\{Q_n(x)\}_{n=0}^{\infty}$ is a WOPS relative to τ.

(3.3) $\Rightarrow$ (3.4): For $j = 0, 1, \ldots, k + r$

$$\sum_{i=j}^{k+r} (-1)^i \binom{i}{j} (b_{i+r}(x)\sigma)^{(i-j)} = \sum_{i=j}^{k+r} (-1)^i \binom{i}{j} \left[\sum_{\ell=j}^{k+r} (-1)^\ell \binom{\ell}{j} (a_{\ell-r}\tau)^{(\ell-i)} \right]^{(i-j)}$$

$$= \sum_{\ell=j}^{k+r} (-1)^{\ell+j} \binom{\ell}{j} \sum_{i=0}^{\ell-j} (-1)^i \binom{\ell-j}{i} (a_{\ell-r}\tau)^{(\ell-j)}$$

$$= \sum_{\ell=j}^{k+r} (-1)^{\ell+j} \binom{\ell}{j} \delta_{\ell j} (a_{\ell-r}(x)\tau)^{(\ell-j)}$$

$$= a_{j-r}\tau \quad (a_j(x) \equiv 0 \text{ if } j < 0)$$

since $\sum_{i=0}^{\ell-j} (-1)^i \binom{\ell-j}{i} = \delta_{\ell j}$.

(3.4) $\Rightarrow$ (3.3): The proof is similar as above.

Now we shall show (3.5). Since $\deg(b_{j+r}) \leqslant j + r$, there are constants $\{c_k^j\}_{k=0}^{j+r}$ such that $b_{j+r}(x) = \sum_{k=0}^{j+r} c_k^j P_k(x)$ so that $b_{j+r,j+r} = c_{j+r}^j$. Then by applying (3.3) to $P_{j+r}(x)$, we have

$$b_{j+r,j+r} = \frac{\langle \sum_{i=j}^{k+r} (-1)^i \binom{i}{j} (a_{i-r}\tau)^{(i-j)}, P_{j+r} \rangle}{\langle \sigma, P_{j+r}^2 \rangle}, \quad 0 \leqslant j \leqslant k + r.$$

In particular,

$$b_{rr} = \frac{\langle \tau, a_0 P_r^{(r)} \rangle}{\langle \sigma, P_r^2 \rangle} = \frac{r! a_0 \langle \tau, 1 \rangle}{\langle \sigma, P_r^2 \rangle} \neq 0$$

so that $\deg(b_r) = r$. Applying (3.4) to $P_0(x) = 1$, we can obtain (3.5).

Now assume that $\{Q_n(x)\}_{n=0}^{\infty}$ is a WOPS relative to τ. Then by (3.7), $\{Q_n(x)\}_{n=0}^{\infty}$ is an OPS relative to τ if and only if $\langle \tau, Q_n^2 \rangle = \langle \sigma, (\sum_{j=0}^{k+r} Q_n^{(j)} b_{j+r}) P_{n+r} \rangle \neq 0$, $n \geqslant 0$, which is equivalent to the condition (3.6).

In this case, (3.3) for $j = k + r$ implies that $b_{k+2r}(x)\sigma = (-1)^{k+r} a_k(x)\tau \neq 0$. Thus $b_{k+2r}(x) \neq 0$ since $a_k(x) \neq 0$ and τ is quasi-definite. $\square$

Set $j = r$ in (3.4). Then

$$a_0\tau = \sum_{i=r}^{k+r} (-1)^i \binom{i}{r} (b_{i+r}\sigma)^{(i-r)}. \tag{3.8}$$

Hence, we may restate Theorem 3.1 as:

Theorem 3.2. *The* PS $\{Q_n(x)\}_{n=0}^{\infty}$ *defined by* (3.1) *is a WOPS if and only if there are* $k+r+1$ *polynomials* $\{b_i(x)\}_{i=r}^{k+2r}$ *with* $\deg(b_i) \leqslant i$, *which are not all zero*, *satisfying*

$$a_0(x) \sum_{i=j}^{k+r} (-1)^i \binom{i}{j} (b_{i+r}\sigma)^{(i-j)} = \begin{cases} 0 & \text{if } 0 \leqslant j \leqslant r-1, \\ a_{j-r}(x) \sum_{i=r}^{k+r} (-1)^i \binom{i}{r} (b_{i+r}\sigma)^{(i-r)} & \text{if } r+1 \leqslant j \leqslant k+r. \end{cases} \tag{3.9}$$

In this case, $\deg(b_r) = r$, $b_{2r}(x) \not\equiv 0$, *and* $\{Q_n(x)\}_{n=0}^{\infty}$ *is a WOPS relative to*

$$\tau := \frac{1}{a_0} \sum_{i=r}^{k+r} (-1)^i \binom{i}{r} (b_{i+r}\sigma)^{(i-r)}.$$

Moreover, $\{Q_n(x)\}_{n=0}^{\infty}$ *is an OPS if and only if* $\{b_i(x)\}_{i=r}^{k+2r}$ *also satisfy the condition* (3.6). *In this case*, *we also have* $b_{k+2r}(x) \not\equiv 0$.

Proof. Assume that there are $k+r+1$ polynomials $\{b_i(x)\}_{i=r}^{k+2r}$ with $\deg(b_i) \leqslant i$, which are not all zero, and (3.9) holds. Define τ by (3.8). Then (3.4) holds so that we only need to show $\tau \neq 0$. If $\tau = 0$, then $\sum_{i=j}^{k+r} (-1)^i \binom{i}{j} (b_{i+r}\sigma)^{(i-j)} = 0$, $0 \leqslant j \leqslant k+r$. Then for $j = k+r$, $(-1)^{k+r}(b_{k+2r}\sigma) = 0$ so that $b_{k+2r}(x) = 0$. By induction on $j = k+r, k+r-1, \ldots, 0$, we can see $b_i(x) = 0$, for $r \leqslant i \leqslant k+2r$, which is a contradiction. The converse is trivial by Theorem 3.1. □

Theorem 3.3. *If the* PS $\{Q_n(x)\}_{n=0}^{\infty}$ *defined by* (3.1) *is also an OPS*, *then there are nonzero constants* λ_n, $n \geqslant r$, *such that*

$$M[Q_{n-r}(x)] = \lambda_n P_n(x), \quad n \geqslant r, \tag{3.10}$$

where $M[\cdot] = \sum_{i=0}^{k+r} b_{i+r}(x) D^i$ *and both* $\{P_n(x)\}_{n=0}^{\infty}$ *and* $\{Q_n(x)\}_{n=0}^{\infty}$ *must be eigenfunctions of linear differential operators of order* $2(k+r)$:

$$\mathrm{MLD}^r[P_n(x)] = \lambda_n P_n(x), \quad n \geqslant 0, \tag{3.11}$$

where $\lambda_n = 0$, $0 \leqslant n \leqslant r-1$ *and*

$$LD^r M[Q_n(x)] = \lambda_{n+r} Q_n(x), \quad n \geqslant 0. \tag{3.12}$$

Proof. Define a sequence of polynomials $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ by

$$\tilde{P}_n(x) = \begin{cases} P_n(x), & 0 \leqslant n \leqslant r-1, \\ M[Q_{n-r}(x)] = \sum_{i=0}^{k+r} b_{i+r}(x) Q_{n-r}^{(i)}(x), & n \geqslant r. \end{cases}$$

Then $\deg(\tilde{P}_n) = n$, $n \geqslant 0$, by (3.6) so that $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ is a PS.

Now we shall show that $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ. For $0 \leqslant m \leqslant n \leqslant r-1$, $\langle \sigma, \tilde{P}_m \tilde{P}_n \rangle = \langle \sigma, P_m P_n \rangle = \langle \sigma, P_n^2 \rangle \delta_{mn}$. For $0 \leqslant m \leqslant n$ and $n \geqslant r$,

$$\begin{aligned} \langle \sigma, \tilde{P}_m \tilde{P}_n \rangle &= \left\langle \sigma, \tilde{P}_m \sum_{i=0}^{k+r} b_{i+r} Q_{n-r}^{(i)} \right\rangle = \left\langle \sum_{i=0}^{k+r} (-1)^i (\tilde{P}_m b_{i+r} \sigma)^{(i)}, Q_{n-r} \right\rangle \\ &= \left\langle \sum_{i=0}^{k+r} (-1)^i \sum_{j=0}^{i} \binom{i}{j} \tilde{P}_m^{(j)} (b_{i+r}\sigma)^{(i-j)}, Q_{n-r} \right\rangle \\ &= \left\langle \sum_{j=0}^{k+r} \tilde{P}_m^{(j)} \sum_{i=j}^{k+r} (-1)^i \binom{i}{j} (b_{i+r}\sigma)^{(i-j)}, Q_{n-r} \right\rangle \\ &= \left\langle \tau, \left(\sum_{j=0}^{k+r} \tilde{P}_m^{(j)} a_{j-r} \right) Q_{n-r} \right\rangle \\ &= \left\langle \tau, \left(\sum_{j=r}^{k+r} \tilde{P}_m^{(j)} a_{j-r} \right) Q_{n-r} \right\rangle = \left\langle \tau, \left(\sum_{j=0}^{k} \tilde{P}_m^{(j+r)} a_j \right) Q_{n-r} \right\rangle = \begin{cases} 0 & \text{if } m < n, \\ \text{nonzero} & \text{if } m = n \end{cases} \end{aligned}$$

since $\deg(\sum_{j=0}^{k} \tilde{P}_m^{(j+r)} a_j) = m - r$ by (3.2) and $\{Q_n(x)\}_{n=0}^{\infty}$ is an OPS relative to τ.

Hence $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ so that $\tilde{P}_n(x) = M[Q_{n-r}(x)] = \lambda_n P_n(x)$, for some $\lambda_n \neq 0$ for $n \geqslant r$. Now

$$\mathrm{MLD}^r[P_n] = ML[P_n^{(r)}] = M[Q_{n-r}] = \tilde{P}_n = \lambda_n P_n, \quad n \geqslant r.$$

For $0 \leqslant n \leqslant r-1$, $D^r[P_n] = 0$ so that $\mathrm{MLD}^r[P_n] = 0$. We also have

$$LD^r M[Q_n] = LD^r[\tilde{P}_{n+r}] = LD^r(\lambda_{n+r} P_{n+r}) = \lambda_{n+r} L[P_{n+r}^{(r}\,)] = \lambda_{n+r} Q_n, \quad n \geqslant 0.$$

Finally since $b_{k+2r}(x) \not\equiv 0$, $M[\,\cdot\,]$ is of order $k+r$ and so $\mathrm{MLD}^r[\,\cdot\,]$ and $LD^r M[\,\cdot\,]$ are of order $2(k+r)$. □

Krall and Sheffer proved Theorem 3.1 only for $r = 0$ (see [14, Theorem 2.1]) and $r = 1$ (see [14, Theorem 3.1]) and Theorem 3.4 only for $r = 0$ (see [14, Theorem 2.3]), using the moments $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\tau_n\}_{n=0}^{\infty}$ of σ and τ, respectively. They used the characterization of OPS' via their formal (cf. [13]) generating series

$$G(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} \phi_n(t) x^n,$$

where $\phi_n(t)$ is a power series in t starting from t^n. Their method seems to be too much complicated to be extended to the case $r \geqslant 2$.

It is well-known (cf. [4,5,9,23]) that an OPS $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS if and only if $\{P_{n+r}^{(r)}(x)\}_{n=0}^{\infty}$ is also an OPS for some integer $r \geqslant 1$.

As a special case of Theorems 3.1, 3.2 and 3.4, we obtain:

Theorem 3.4. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to σ and $r \geqslant 1$ an integer. Then, the following are all equivalent.*

(i) *$\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS.*

(ii) *$\{P_{n+r}^{(r)}(x)\}_{n=0}^{\infty}$ is a WOPS.*

(iii) *There are nonzero moment functional τ and $r+1$ polynomials $\{b_k(x)\}_{k=r}^{2r}$ with $\deg(b_k) \leqslant k$ such that*

$$(-1)^r \binom{r}{j} \tau^{(r-j)} = b_{j+r}\sigma, \quad 0 \leqslant j \leqslant r. \tag{3.13}$$

(iv) *There are $r+1$ polynomials $\{b_k(x)\}_{k=r}^{2r}$ with $\deg(b_k) \leqslant k$ such that $\{b_k(x)\}_{k=r}^{2r}$ are not all zero and*

$$\sum_{i=j}^{r} (-1)^i \binom{i}{j} (b_{i+r}\sigma)^{(i-j)} = 0, \quad 0 \leqslant j \leqslant r-1.$$

Moreover, in this case, $\deg(b_r) = r$, $b_{2r}(x) \neq 0$, and

$$\sum_{k=r}^{2r} b_k(x) P_n^{(k)}(x) = \lambda_n P_n(x), \quad n \geqslant 0 \tag{3.14}$$

for some constants λ_n with $\lambda_0 = \lambda_1 = \cdots = \lambda_{r-1} = 0$ and

$$\sum_{i=0}^{r} \frac{(-1)^{i+r} \binom{r}{i} \langle \sigma, b_{2r} P_{i+r}^{(r-i)} \rangle}{\langle \sigma, P_{i+r}^2 \rangle} n_{(i)} \neq 0, \quad n \geqslant 0. \tag{3.15}$$

Proof. (i) $\Rightarrow$ (ii): It is well known that for a classical OPS $\{P_n(x)\}_{n=0}^{\infty}, \{P'_{n+1}(x)\}_{n=0}^{\infty}$ is also a classical OPS.

(ii) $\Rightarrow$ (i): See Theorems 3.2 and 3.3 in [19].

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv): It is a special case of Theorems 3.1 and 3.2 when $k=0$ so that $L[\,\cdot\,] = a_0 \mathrm{Id}$ (Id = the identity operator) and $Q_n(x) = a_0 P_{n+r}^{(r)}(x)$, $n \geqslant 0$. In (iii), $\deg(b_r) = r$ and $b_{2r}(x) \not\equiv 0$ by Theorem 3.1. Eq. (3.14) comes from Theorem 3.4 and (3.15) comes from (3.6), (3.7), and (3.13). □

Equivalences of (i)–(iii) in Theorem 3.4 are first proved in [19]. Moreover, the condition (3.14) also implies that $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS (see [19]).

4. Examples

As in Section 3, we always let $\{P_n(x)\}_{n=0}^{\infty}$ be the monic OPS relative to σ and write $a_0(x)=a_{00}=a_0$. If $k=r=0$, then $\{Q_n(x)\}_{n=0}^{\infty}$, where $Q_n(x)=a_0P_n(x)$, $n\geqslant 0$, and $a_0\neq 0$, is also an OPS if and only if $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS.

4.1. $k=1$ and $r=0$

Let $L[\cdot]=a_1(x)D+a_0$, where $a_1(x)=a_{11}x+a_{10}\neq 0$, $a_0\neq 0$, and $a_{11}n+a_0\neq 0$, $n\geqslant 0$. Define a monic PS $\{Q_n(x)\}_{n=0}^{\infty}$ by

$$(a_{11}n+a_0)Q_n(x)=L[P_n(x)]=a_1(x)P_n'(x)+a_0(x)P_n(x),\quad n\geqslant 0.$$

Then, $\{Q_n(x)\}_{n=0}^{\infty}$ is also a monic OPS (relative to $\tau:=a_0^{-1}((b_1(x)\sigma)'-b_0(x)\sigma)$) if and only if there are polynomials $b_1(x)=b_{11}x+b_{10}$ and $b_0(x)=b_0$ satisfying

$$a_0b_1\sigma=a_1\{(b_1\sigma)'-b_0\sigma\}\quad\text{and}\quad b_{11}n+b_0\neq 0,\quad n\geqslant 0.$$

Hence, if $\{Q_n(x)\}_{n=0}^{\infty}$ is also a monic OPS, then (3.11) and (3.14) become

$$\begin{aligned} ML[P_n]&=(a_1b_1)P_n''+(a_1'b_1+a_0b_1+a_1b_0)P_n'+a_0b_0P_n=\lambda_nP_n,\\ LM[Q_n]&=(a_1b_1)Q_n''+(a_1b_1'+a_1b_0+a_0b_1)Q_n'+a_0b_0Q_n=\lambda_nQ_n,\end{aligned}\tag{4.1}$$

where $\lambda_n=(a_{11}n+a_0)(b_{11}n+b_0), n\geqslant 0$. Hence both $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ are classical OPS' of the same type. Note

$$a_1(x)b_1(x)=a_{11}b_{11}x^2+(a_{11}b_{10}+a_{10}b_{11})x+a_{10}b_{10}.\tag{4.2}$$

Case 1: $\deg(a_1b_1)=0$. Then $a_{11}b_{11}=a_{11}b_{10}+a_{10}b_{11}=0$ so that $a_{11}=b_{11}=0$. Hence $ML[\cdot]=LM[\cdot]$ and so $P_n(x)=Q_n(x)$, $n\geqslant 0$, and

$$a_1(x)P_n'(x)=a_{11}nP_n(x)=0,\quad n\geqslant 0.$$

Therefore, $a_1(x)\equiv 0$, which is a contradiction.

Case 2: $\deg(a_1b_1)=1$. Then we may assume $a_1(x)=1$ and $b_1(x)=x$ or $a_1(x)=x$ and $b_1(x)=1$.

Case 2.1: $a_1(x)=1$ *and* $b_1(x)=x$. Then for $n\geqslant 0$

$$\begin{aligned} ML[P_n(x)]&=xP_n''(x)+(a_0x+b_0)P_n'(x)+a_0b_0P_n(x)=\lambda_nP_n(x),\\ LM[Q_n(x)]&=xQ_n''(x)+(a_0x+b_0+1)Q_n'(x)+a_0b_0Q_n(x)=\lambda_nQ_n(x).\end{aligned}\tag{4.3}$$

We may also assume $a_0=-1$ and $b_0=\alpha+1$ ($\alpha\neq -1,-2,\ldots$) by a real linear change of variable. Then (4.3) becomes

$$ML[P_n(x)]=xP_n''(x)+(\alpha+1-x)P_n'(x)-(\alpha+1)P_n(x)=\lambda_nP_n(x),$$

$$LM[Q_n(x)]=xQ_n''(x)+(\alpha+2-x)Q_n'(x)-(\alpha+1)Q_n(x)=\lambda_nQ_n(x).$$

Thus, $\{P_n(x)\}_{n=0}^{\infty}=\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}=\{L_n^{(\alpha+1)}(x)\}_{n=0}^{\infty}$, where $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is the monic Laguerre polynomials. Hence, we have (see [22, (5.1.13)]):

$$L_n^{(\alpha+1)}(x)=L_n^{(\alpha)}(x)-nL_{n-1}^{(\alpha+1)}(x),\quad n\geqslant 0\tag{4.4}$$

since $(L_n^{(\alpha)}(x))'=nL_{n-1}^{(\alpha+1)}(x)$, $n\geqslant 0$.

Case 2.2: $a_1(x)=x$ *and* $b_1(x)=1$. Then we may assume $a_0=\alpha$ ($\alpha\neq 0,-1,-2,\ldots$) and $b_0=-1$ so that (4.1) becomes

$$ML[P_n(x)] = xP_n''(x)+(\alpha+1-x)P_n'(x)-\alpha P_n(x)=\lambda_n P_n(x),$$
$$LM[Q_n(x)] = xQ_n''(x)+(\alpha-x)Q_n'(x)-\alpha Q_n(x)=\lambda_n Q_n(x).$$

Thus, $\{P_n(x)\}_{n=0}^{\infty}=\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}=\{L_n^{(\alpha-1)}(x)\}_{n=0}^{\infty}$ so that we have (see [22, (5.1.14)]):

$$(n+\alpha)L_n^{(\alpha-1)}(x)=x(L_n^{(\alpha)}(x))'+\alpha L_n^{(\alpha)}(x),\quad n\geqslant 0. \tag{4.5}$$

Case 3: $\deg(a_1b_1)=2$ *and* $(a_1b_1)(x)$ has a double root. Then from (4.2), $a_{11}b_{10}=a_{10}b_{11}$ so that $ML[\cdot]=LM[\cdot]$. Thus $\{P_n(x)\}_{n=0}^{\infty}=\{Q_n(x)\}_{n=0}^{\infty}$ and

$$(a_{11}x+a_{10})P_n'(x)=a_{11}nP_n(x),\quad n\geqslant 0,$$

which is impossible (cf. Proposition 2.2).

Case 4: $\deg(a_1b_1)=2$ *and* $(a_1b_1)(x)$ has 2 distinct real roots. Then we may assume that $(a_1b_1)(x)=1-x^2$ and $a_1(x)=1-x$ or $1+x$.

Case 4.1: $a_1(x)=1-x$ *and* $b_1(x)=1+x$. Then (4.1) becomes

$$ML[P_n(x)]=(1-x^2)P_n''(x)+((a_0+b_0-1)-(b_0-a_0+1)x)P_n'(x)+a_0b_0P_n(x)=\lambda_nP_n(x),$$

$$LM[Q_n(x)]=(1-x^2)Q_n''(x)+((a_0+b_0+1)-(b_0-a_0+1)x)Q_n'(x)+a_0b_0Q_n(x)=\lambda_nQ_n(x).$$

We may also assume $a_0+b_0-1=\beta-\alpha$ and $b_0-a_0+1=\alpha+\beta+2$ ($\alpha,\beta,\alpha+\beta+1\neq -1,-2,\ldots,$ and $\alpha\neq 0$). Then

$$ML[P_n(x)]=(1-x^2)P_n''(x)+(\beta-\alpha-(\alpha+\beta+2)x)P_n'(x)-\alpha(\beta+1)P_n(x)=\lambda_nP_n(x),$$

$$LM[Q_n(x)]=(1-x^2)Q_n''(x)+(\beta-\alpha+2-(\alpha+\beta+2)x)Q_n'(x)-\alpha(\beta+1)Q_n(x)=\lambda_nQ_n(x).$$

Therefore, we have

$$\{P_n(x)\}_{n=0}^{\infty}=\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}\quad\text{and}\quad \{Q_n(x)\}_{n=0}^{\infty}=\{P_n^{(\alpha-1,\beta+1)}(x)\}_{n=0}^{\infty},$$

where $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ is the monic Jacobi polynomials. Hence, we have $a_1(x)=1-x$, $a_0(x)=-\alpha$, $b_1(x)=1+x$, $b_0(x)=\beta+1$ so that

$$\begin{aligned}(n+\alpha)P_n^{(\alpha-1,\beta+1)}(x)&=(x-1)(P_n^{(\alpha,\beta)}(x))'+\alpha P_n^{(\alpha,\beta)}(x)\\ &=n(x-1)P_{n-1}^{(\alpha+1,\beta+1)}(x)+\alpha P_n^{(\alpha,\beta)}(x)\end{aligned} \tag{4.6}$$

since $(P_n^{(\alpha,\beta)}(x))'=nP_{n-1}^{(\alpha+1,\beta+1)}(x)$, $n\geqslant 0$.

Case 4.2: $a_1(x)=1+x$, $b_1(x)=1-x$. Then (4.1) becomes

$$ML[P_n]=(1-x^2)P_n''(x)+[(a_0+b_0+1)-(a_0-b_0+1)x]P_n'(x)+a_0b_0P_n(x)=\lambda_nP_n(x),$$

$$LM[Q_n]=(1-x^2)Q_n''(x)+[(a_0+b_0-1)-(a_0-b_0+1)x]Q_n'(x)+a_0b_0Q_n(x)=\lambda_nQ_n(x).$$

We may also assume $a_0 + b_0 + 1 = \beta - \alpha$ and $a_0 - b_0 + 1 = \alpha + \beta + 2$ ($\alpha, \beta, \alpha + \beta + 1 \neq -1, -2, \ldots,$ and $\beta \neq 0$). Then $a_0 = \beta$ and $b_0 = -\alpha - 1$ so that

$$ML[P_n] = (1 - x^2)P_n''(x) + [(\beta - \alpha) - (\alpha + \beta + 2)x]P_n'(x) - \beta(\alpha + 1)P_n(x) = \lambda_n P_n(x),$$

$$LM[Q_n] = (1 - x^2)Q_n''(x) + [(\beta - \alpha - 2) - (\alpha + \beta + 2)x]Q_n'(x) - \beta(\alpha + 1)Q_n(x) = \lambda_n Q_n(x).$$

Therefore, $\{P_n(x)\}_{n=0}^{\infty} = \{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty} = \{P_n^{(\alpha+1,\beta-1)}\}_{n=0}^{\infty}$ so that we have

$$\begin{aligned}(n + \beta)P_n^{(\alpha+1,\beta-1)}(x) &= (1 + x)[P_n^{(\alpha,\beta)}(x)]' + \beta P_n^{(\alpha,\beta)}(x)\\ &= n(1 + x)P_{n-1}^{(\alpha+1,\beta+1)}(x) + \beta P_n^{(\alpha,\beta)}(x).\end{aligned} \tag{4.7}$$

We have shown that if $\{P_n(x)\}_{n=0}^{\infty}$ is either Hermite and Bessel polynomials, then $\{a_1(x)P_n'(x) + a_0P_n(x)\}_{n=0}^{\infty}$ cannot be an OPS for any polynomials $a_1(x)$ and $a_0(x)$. This fact is closely related to the absence of Hermite or Bessel polynomials in Darboux transformations [3].

4.2. $k = 1$ and $r = 1$

Let $L[\cdot] = a_1(x)D + a_0$, where $a_1(x) \not\equiv 0$ and $a_{11}n + a_0 \neq 0$, $n \geqslant 0$. Define a monic PS $\{Q_n(x)\}_{n=0}^{\infty}$ by

$$(n + 1)(a_{11}n + a_0)Q_n(x) = L[P_{n+1}'] = a_1P_{n+1}''(x) + a_0P_{n+1}'(x), \quad n \geqslant 0. \tag{4.8}$$

We assume that $\{Q_n(x)\}_{n=0}^{\infty}$ is a monic OPS. Then there are $b_1(x) = b_{11}x + b_{10}$, $b_2(x) = b_{22}x^2 + b_{21}x + b_{20}$, $b_3(x) = b_{33}x^3 + b_{32}x^2 + b_{31}x + b_{30}$, not all zero, satisfying

$$\begin{aligned}&(b_3(x)\sigma)'' - (b_2(x)\sigma)' + b_1(x)\sigma = 0,\\ &a_0b_3(x)\sigma = a_1(x)\{2(b_3(x)\sigma)' - b_2(x)\sigma\}\end{aligned} \tag{4.9}$$

and $b_{33}n(n - 1) + b_{22}n + b_{11} \neq 0, n \geqslant 0$. Now (3.11) and (3.12) become

$$\begin{aligned}\mathrm{MLD}[P_n] &= (b_3D^2 + b_2D + b_1)(a_1D + a_0)[P_n']\\ &= a_1b_3P_n^{(4)} + (2a_1'b_3 + a_0b_3 + a_1b_2)P_n^{(3)} + (a_1'b_2 + a_0b_2 + a_1b_1)P_n'' + a_0b_1P_n'\\ &= \lambda_nP_n,\end{aligned} \tag{4.10}$$

$$LDM[Q_n] = (a_1D + a_0)D(b_3D^2 + b_2D + b_1)[Q_n] = \lambda_{n+1}Q_n.$$

Therefore, $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ must be either classical or classical-type OPS. Krall and Sheffer [14] considered this case only for $\{P_n(x)\}_{n=0}^{\infty} = \{P_n^{(\alpha,\alpha)}(x)\}_{n=0}^{\infty}$ the Gegenbauer polynomials. In case $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS, $\{P_{n+1}'(x)\}_{n=0}^{\infty}$ is also a classical OPS so that Case 4.2 is reduced to Case 4.1. Hence, $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ must be either Laguerre polynomials or Jacobi polynomials. We now claim that $\{P_n(x)\}_{n=0}^{\infty}$ cannot be a classical-type OPS. For example, assume that $\{P_n(x)\}_{n=0}^{\infty} = \{R_n(x)\}_{n=0}^{\infty}$ is the Laguerre-type OPS which is orthogonal relative to $\sigma = (\mathrm{e}^{-x}H(x) + (1/R)\delta(x))\,\mathrm{d}x$. Then, we may assume that $a_1(x)b_3(x) = x^2$ and $a_1(x) = 1$ or x. If $a_1(x) = 1$ and $b_3(x) = x^2$, then we obtain from (2.3) and (4.10)

$$2a_1'(x)b_3(x) + a_0b_3(x) + a_1(x)b_2(x) = 4x - 2x^2,$$

$$a_1'(x)b_2(x) + a_0b_2(x) + a_1(x)b_1(x) = x^2 - (2R + 6)x,$$

$$a_0b_1(x) = (2R + 2)x - 2R$$

from which we have

$$b_3(x)=x^2, \quad b_2(x)=-x^2+4x, \quad b_1(x)=-2x.$$

Then, by (3.10), $P_n(0)=0$, $n\geqslant 1$, which is a contradiction. If $a_1(x)=x$ and $b_3(x)=x$, then we have similarly as above either (i) $a_0=2$, $R=0$, $b_2(x)=-2x$, $b_1(x)=x$ or (ii) $a_0=-1$, $R=-\frac{3}{2}$, $b_2(x)=-2x+3$, $b_1(x)=x-3$. In case (i), $P_n(0)=0$, $n\geqslant 1$ by (3.10), which is a contradiction. In case (ii), we can see that $(b_3\sigma)''-(b_2\sigma)'+b_1\sigma=2\delta'(x)\neq 0$, which contradicts to (4.9). By similar arguments, we can see that $\{P_n(x)\}_{n=0}^{\infty}$ can be neither a Legendre-type OPS nor a Jacobi-type OPS.

4.3. $k=2$ and $r=0$

Let $L[\cdot]=a_2(x)D^2+a_1(x)D+a_0$, where $a_2(x)\not\equiv 0$ and

$$\alpha_n := a_{22}n(n-1)+a_{11}n+a_0\neq 0, \quad n\geqslant 0. \tag{4.11}$$

Then, the monic PS $\{Q_n(x)\}_{n=0}^{\infty}$ defined by

$$\alpha_n Q_n(x)=L[P_n](x)=a_2(x)P_n''(x)+a_1(x)P_n'(x)+a_0P_n(x), \quad n\geqslant 0$$

is an OPS relative to $\tau(=a_0^{-1}\{(b_2\sigma)''-(b_1\sigma)'+b_0\sigma\}$) if and only if there exist b_0, $b_1(x)$, $b_2(x)$ (not all zero) satisfying

$$\begin{aligned} &a_2(x)\tau=b_2(x)\sigma,\\ &2(a_2(x)\tau)'-a_1(x)\tau=b_1(x)\sigma,\\ &(a_2(x)\tau)''-(a_1(x)\tau)'+a_0\tau=b_0\sigma. \end{aligned} \tag{4.12}$$

and $b_{22}n(n-1)+b_{11}n+b_0\neq 0$, $n\geqslant 0$. In this case, $b_0\neq 0$ and $b_2(x)\neq 0$ and

$$\begin{aligned} ML[P_n] &= (b_2D^2+b_1D+b_0)(a_2D^2+a_1D+a_0)[P_n]\\ &= a_2b_2P_n^{(4)}+(2a_2'b_2+a_1b_2+a_2b_1)P_n^{(3)}\\ &\quad +(a_2''b_2+2a_1'b_2+a_0b_2+a_2'b_1+a_1b_1+a_2b_0)P_n''\\ &\quad +(a_1'b_1+a_0b_1+a_1b_0)P_n'+a_0b_0P_n=\lambda_nP_n,\\ LM[Q_n] &= (a_2D^2+a_1D+a_0)(b_2D^2+b_1D+b_0)[Q_n]=\lambda_nQ_n. \end{aligned} \tag{4.13}$$

Hence, $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ must be either classical or classical-type OPS. We first consider the case when $\{P_n(x)\}_{n=0}^{\infty}$ is a classical-type OPS.

Case 1: $\{P_n(x)\}_{n=0}^{\infty}=\{R_n(x)\}_{n=0}^{\infty}$ *the Laguerre-type OPS*. Then, $a_2(x)b_2(x)=x^2$ so that $a_2(x)=1,x,x^2$.

Case 1.1: $a_2(x)=1$ *or* $a_2(x)=x^2$. If $a_2(x)=1$, then $b_2(x)=x^2$ and from (2.3) and (4.13), we obtain

$$\begin{aligned} &a_1(x)x^2+b_1(x)=4x-2x^2,\\ &2a_1'(x)x^2+a_0x^2+a_1(x)b_1(x)+b_0=x^2-(2R+6)x,\\ &a_1'(x)b_1(x)+a_0b_1(x)+a_1(x)b_0=(2R+2)x-2R, \end{aligned}$$

from which we have

$$a_1(x)=-2, \quad a_0=1, \quad b_1(x)=4x, \quad b_0=0.$$

It is a contradiction since $b_0 \neq 0$. If $a_2(x) = x^2$, then $b_2(x) = 1$ and $R = -1$, which is also a contradiction.

Case 1.2: $a_2(x) = x$. Then $b_2(x) = x$ and (4.12) becomes

$$\begin{aligned} &x\tau = x\sigma, \\ &2(x\tau)' - a_1(x)\tau = b_1(x)\sigma, \\ &(x\tau)'' - (a_1(x)\tau)' - a_0(x)\tau = b_0(x)\sigma. \end{aligned} \tag{4.14}$$

Since $\sigma = (\mathrm{e}^{-x}H(x) + (1/R)\delta(x))\,\mathrm{d}x$,

$$(x\tau)' = (x\sigma)' = (1-x)\sigma \quad \text{and} \quad \sigma' = -\sigma + \delta(x). \tag{4.15}$$

Applying (4.15) to (4.14), we obtain $\tau = \mathrm{e}^{-x}H(x)\,\mathrm{d}x$ and

$$a_1(x) = -x + 2,\ a_0(x) = -R - 1 \quad \text{and} \quad b_1(x) = -x,\ b_0(x) = -R.$$

Hence, $\{Q_n(x)\}_{n=0}^{\infty} = \{L_n^{(0)}(x)\}_{n=0}^{\infty}$ and

$$(-n - R - 1)L_n^{(0)}(x) = xR_n''(x) + (2 - x)R_n'(x) - (R+1)R_n(x), \tag{4.16}$$

$$(-n - R)R_n(x) = xL_n^{(0)}(x)'' - xL_n^{(0)}(x)' - RR_n(x). \tag{4.17}$$

Case 2: $\{P_n(x)\}_{n=0}^{\infty} = \{P_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ *the Legendre-type OPS*. Then $a_2(x)b_2(x) = (x^2-1)^2$ so that $a_2(x) = x^2 - 1, (x+1)^2, (x-1)^2$. If $a_2(x) = (x+1)^2$ or $a_2(x) = (x-1)^2$, then by the same arguments as in Case 1.1, we can derive a contradiction.

Case 2.1: $a_2(x) = x^2 - 1$. Then $b_2(x) = x^2 - 1$ and (4.12) becomes

$$\begin{aligned} &(x^2-1)\tau = (x^2-1)\sigma, \\ &2((x^2-1)\tau)' - a_1(x)\tau = b_1(x)\sigma, \\ &((x^2-1)\tau)'' - (a_1(x)\tau)' - a_0(x)\tau = b_0(x)\sigma. \end{aligned} \tag{4.18}$$

Since $\sigma = \sigma_L + (1/\alpha)(\delta(x-1) + \delta(x+1))$, where $\sigma_L = H(1-x^2)\,\mathrm{d}x$ is the Legendre moment functional, we have

$$((x^2-1)\tau)' = ((x^2-1)\sigma)' = 2x\sigma_L \quad \text{and} \quad \sigma_L' = \delta(x+1) - \delta(x-1).$$

Applying these to (4.18) gives $\tau = \sigma_L$ and

$$a_1 = 4x,\ a_0 = 2\alpha + 2 \quad \text{and} \quad b_1 = 0,\ b_0 = 2\alpha.$$

Hence $\{Q_n(x)\}_{n=0}^{\infty} = \{P_n^{(0,0)}(x)\}_{n=0}^{\infty}$ and

$$(n(n-1) + 4n + 2\alpha + 2)P_n^{(0,0)}(x) = (x^2-1)P_n^{(\alpha)}(x)'' + 4xP_n^{(\alpha)}(x)' + (2\alpha+2)P_n^{(\alpha)}, \tag{4.19}$$

$$(n(n-1) + 2\alpha)P_n^{(\alpha)}(x) = (x^2-1)P_n^{(0,0)}(x)'' + 2\alpha P_n^{(0,0)}(x). \tag{4.20}$$

Case 3: $\{P_n(x)\}_{n=0}^{\infty} = \{S_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ *the Jacobi-type OPS*. Then $(a_2b_2)(x) = (x^2-x)^2$ and so that $a_2(x) = x^2 - x, x^2, (x-1)^2$. Recall that $\{S_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is orthogonal relative to

$$\sigma = \sigma_\alpha + \frac{1}{M}\delta(x), \tag{4.21}$$

where $\sigma_\alpha = (1-x)_+^\alpha H(x)\,\mathrm{d}x$ is a classical moment functional satisfying the moment equation

$$(x^2 - x)\sigma_\alpha' = \alpha x\sigma_\alpha, \quad \alpha \neq -1, -2, \ldots\ . \tag{4.22}$$

Since $\langle \sigma_\alpha, 1 \rangle = 1/(\alpha+1)$, we obtain from (4.22),

$$((x-1)\sigma_\alpha)' = (\alpha+1)\sigma_\alpha - \delta(x). \tag{4.23}$$

Case 3.1: $a_2(x) = x^2 - x$ *and* $b_2(x) = x^2 - x$. Then (4.12) becomes

$$(x^2 - x)\tau = (x^2 - x)\sigma, \tag{4.24}$$

$$2((x^2 - x)\tau)' - a_1(x)\tau = b_1(x)\sigma, \tag{4.25}$$

$$((x^2 - x)\tau)'' - (a_1(x)\tau)' + a_0\tau = b_0\sigma. \tag{4.26}$$

From (4.21) and (4.24), we have

$$\tau = \sigma_\alpha + \lambda\delta(x) + \mu\delta(x-1) \tag{4.27}$$

for some constants λ and μ. By (4.22) and (4.27), (4.25) becomes

$$((2\alpha+4)x - 2 - a_1(x) - b_1(x))\sigma_\alpha = \left(\lambda a_1(0) + \frac{1}{M} b_1(0)\right)\delta(x) + \mu a_1(1)\delta(x-1).$$

Hence $\mu a_1(1) = 0, a_1(x) + b_1(x) = (2\alpha+4)x - 2$, and

$$\lambda a_1(0) = -\frac{1}{M} b_1(0). \tag{4.28}$$

Multiply (4.26) by $(x^2 - x)$ and apply (4.22). Then we have

$$(\alpha + 2 + a_0 - a_1'(x) - b_0)(x-1) = \alpha((\alpha+2)x - a_1(x) - 1)$$

and $\lambda a_1(0) = 0$. Thus from (4.28), $b_1(0) = 0$ and so $a_1(x) = Ax - 2,\ b_1(x) = (2\alpha + 4 - A)x$ for some constant A so that $\lambda = 0$ since $a_1(0) = -2$. There are two cases: $\alpha = 0$ or $A = \alpha + 3$.

Case 3.11: $A = \alpha + 3$. Then $a_1(x) = (\alpha+3)x - 2, b_1(x) = (\alpha+1)x$ and $\mu = 0$. Thus $\tau = \sigma_\alpha$ and (4.26) becomes

$$((1-x)\sigma_\alpha)' = (b_0 - a_0)\sigma_\alpha + \frac{1}{M} b_0\delta(x). \tag{4.29}$$

From (4.23) and (4.29), we obtain $a_0(x) = \alpha + M + 1$ and $b_0(x) = M$. Hence, we have

$$\begin{aligned}(n^2 + 2n + \alpha n + \alpha + M + 1)Q_n(x) \\ = (x^2 - x)S_n^{(\alpha)}(x)'' + ((\alpha+3)x - 2)S_n^{(\alpha)}(x)' + (\alpha + M + 1)S_n^{(\alpha)}(x),\end{aligned} \tag{4.30}$$

$$(n^2 + \alpha n + M)S_n^{(\alpha)}(x) = (x^2 - x)Q_n(x)'' + (\alpha+1)Q_n(x)' + MQ_n(x). \tag{4.31}$$

Note that $Q_n(x) = (-2)^{-n}P_n^{(0,\alpha)}(1-2x),\ n \geqslant 0$.

Case 3.12: $\alpha = 0$. Then $\tau = \sigma_\alpha + \mu\delta(x-1), a_1(x) = Ax - 2$, and $b_1(x) = (4 - A)x$. Since $\sigma_\alpha = H(x)H(1-x)\,\mathrm{d}x, \sigma_\alpha' = \delta(x) - \delta(x-1)$ so that we obtain from (4.26)

$$b_0\sigma_\alpha + \frac{1}{M} b_0\delta(x) = (a_0 - A + 2)\sigma_\alpha + \delta(x) + (a_0\mu - 3 + A)\delta(x-1).$$

Thus, $b_0 = M,\ a_0 = b_0 + A - 2$, and $A = -a_0\mu + 3$. If $\mu = 0$, then $A = 3,\ a_1(x) = 3x - 2,\ a_0 = M + 1,\ b_1(x) = x$, and $b_0 = M$ so that it becomes the Case 3.11 with $\alpha = 0$. If $\mu \neq 0$, then we have $A = 2$ so that $\mu = 1/M,\ a_1(x) = 2(x-1), b_1(x) = 2x, a_0 = b_0 = M$, and

$$\tau = \left(H(x)H(1-x) + \frac{1}{M}\delta(x)\right)\,\mathrm{d}x.$$

Hence

$$(n^2+n+M)Q_n(x)=(x^2-x)P_n''(x)+2(x-1)P_n'(x)+MP_n(x), \tag{4.32}$$

$$(n^2+n+M)P_n(x)=(x^2-x)Q_n''(x)+2xQ_n'(x)+MQ_n(x). \tag{4.33}$$

Note that $Q_n(x)=(-1)^nS_n^{(0)}(1-x), n\geqslant 0$, are also Jacobi-type polynomials.

Case 3.2: $a_2(x)=x^2$ *or* $a_2(x)=(x-1)^2$. Then by the same argument as in Case 1.1, we have if $a_2(x)=x^2$, then $a_1(x)=0, a_0(x)=-2$ and if $a_2(x)=(x-1)^2$, then $a_0(x)=0$ and $M=0$. Hence, these contradict our assumptions that $\alpha_2\neq 0$ in (4.11) and $M\neq 0$.

We now consider the case when $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS. If $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the differential equation (2.2), then the differential operator $ML[\cdot]$ in (4.13) must be a linear combination of $I, \mathscr{L}, \mathscr{L}^2$ (see [20, Proposition 1]), where I is the identity operator.

Krall and Sheffer [14] considered this case only for $\{P_n(x)\}_{n=0}^{\infty}=\{L_n^{(0)}(x)\}_{n=0}^{\infty}$ or $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ through the factorization of fourth order differential equations satisfied by $\{P_n(x)\}_{n=0}^{\infty}$ into the product of two second order differential equations. Instead, we use moment functional relations (4.12), which is much easier to handle.

Case 4: $\{P_n(x)\}_{n=0}^{\infty}=\{H_n(x)\}_{n=0}^{\infty}$ *the Hermite polynomials*. Then we may assume $a_2(x)=b_2(x)=1$. Hence $\tau=\sigma$ by (4.12) so that $\{Q_n(x)\}_{n=0}^{\infty}=\{H_n(x)\}_{n=0}^{\infty}$.

Case 5: $\{P_n(x)\}_{n=0}^{\infty}=\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ *the Laguerre polynomials*. Then we may assume $a_2(x)b_2(x)=x^2$ so that $a_2(x)=x^2, x, 1$.

Case 5.1: $a_2(x)=x^2$. Then $b_2(x)=1$ and (4.12) becomes

$$\begin{aligned}&x^2\tau=\sigma,\\&2(x^2\tau)'-a_1(x)\tau=b_1(x)\sigma,\\&(x^2\tau)''-(a_1(x)\tau)'+a_0\tau=b_0\sigma.\end{aligned} \tag{4.34}$$

Multiplying the second equation in (4.34) by x^2 and using $(x\sigma)'=(\alpha+1-x)\sigma$, we have $a_1(x)=2\alpha x$ and $b_1(x)=-2$. Similarly from the third equation in (4.34), we have $a_0=\alpha^2-\alpha$ and $b_0=1$ so that $\alpha\neq 0,1$. Since $\sigma=x_+^{\alpha}\mathrm{e}^{-x}\,\mathrm{d}x$,

$$\tau=a_0^{-1}\{(b_2\sigma)''-(b_1\sigma)'+b_0\sigma\}=x_+^{\alpha-2}\mathrm{e}^{-x}\,\mathrm{d}x \quad (\alpha\neq 1,0,-1,\ldots)$$

so that $\{Q_n(x)\}_{n=0}^{\infty}=\{L_n^{(\alpha-2)}(x)\}_{n=0}^{\infty}$ and

$$(n(n-1)+2\alpha n+\alpha^2-\alpha)L_n^{(\alpha-2)}(x)=x^2L_n^{(\alpha)}(x)''+2\alpha xL_n^{(\alpha)}(x)'+(\alpha^2-\alpha)L_n^{(\alpha)}(x), \tag{4.35}$$

$$L_n^{(\alpha)}(x)=L_n^{(\alpha-2)}(x)''-2L_n^{(\alpha-2)}(x)'+L_n^{(\alpha-2)}(x). \tag{4.36}$$

Case 5.2: $a_2(x)=x$. Then $b_2(x)=x$ and (4.12) becomes

$$\begin{aligned}&x\tau=x\sigma,\\&2(x\tau)'-a_1(x)\tau=b_1(x)\sigma,\\&(x\tau)''-(a_1(x)\tau)'+a_0\tau=b_0\sigma\end{aligned}$$

so that $\tau=\sigma+\lambda\delta(x)$ for some constant λ. If $\lambda=0$, then $\tau=\sigma$ so that $\{Q_n(x)\}_{n=0}^{\infty}=\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$. If $\lambda\neq 0$, then we have $a_1(x)=-x, b_1(x)=-x+2, b_0=a_0-1$, and $\alpha=0$. Then $\tau=\sigma-(1/a_0)\delta(x)$ so that $\{Q_n(x)\}_{n=0}^{\infty}$ is the Laguerre-type OPS $\{R_n(x)\}_{n=0}^{\infty}$ with $R=-a_0\neq 0,-1,-2,\ldots$.

Case 5.3: $a_2(x)=1$. Then $b_2(x)=x^2$ and $\tau=x^2\sigma=x_+^{\alpha+2}\mathrm{e}^{-x}\,\mathrm{d}x$ so that $\{Q_n(x)\}_{n=0}^{\infty}=\{L_n^{(\alpha+2)}(x)\}_{n=0}^{\infty}$.

Case 6: $\{P_n(x)\}_{n=0}^{\infty}=\{B_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ *the Bessel polynomials*. Then we may assume that $a_2(x)b_2(x)=x^4$ so that $a_2(x)=x^2$ and $b_2(x)=x^2$ and (4.12) becomes

$$x^2\tau=x^2\sigma,$$
$$2(x^2\tau)'-a_1(x)\tau=b_1(x)\sigma,$$
$$(x^2\tau)''-(a_1(x)\tau)'+a_0\tau=b_0\sigma.$$

Hence $\tau=\sigma+\lambda\delta(x)+\mu\delta'(x)$ for some constants λ and μ. In this case, by the same arguments as in Case 5 using $(x^2\sigma)'=(\alpha x+2)\sigma$, we can obtain

$$a_1(x)=b_1(x)=\alpha x+2,\quad a_0=b_0\quad\text{and}\quad\lambda=\mu=0$$

so that $\{Q_n(x)\}_{n=0}^{\infty}=\{B_n^{(\alpha)}(x)\}_{n=0}^{\infty}$.

Case 7: $\{P_n(x)\}_{n=0}^{\infty}=\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ *the Jacobi polynomials*. Then we may assume $a_2(x)b_2(x)=(1-x^2)^2$ so that $a_2(x)=(1-x)^2,(1+x)^2,1-x^2$.

Case 7.1: $a_2(x)=(1-x)^2$. Then $b_2(x)=(1+x)^2$ and (4.12) becomes

$$\begin{aligned}&(1-x)^2\tau=(1+x)^2\sigma,\\&2((1-x)^2\tau)'-a_1(x)\tau=b_1(x)\sigma,\\&((1-x)^2\tau)''-(a_1(x)\tau)'+a_0\tau=b_0\sigma.\end{aligned}\tag{4.37}$$

Then using $((1-x^2)\sigma)'=(\beta-\alpha-(\alpha+\beta+2)x)\sigma$, we can easily obtain from (4.37) $a_1(x)=2\alpha(x-1), b_1(x)=(2\beta+4)(x+1)$, $a_0=\alpha(\alpha-1)$, $b_0=(\beta+2)(\beta+1)$ so that $\alpha\neq 0,1$. Since $\sigma=(1-x)_+^{\alpha}(1+x)_+^{\beta}\,dx$,

$$\tau=a_0^{-1}\{(b_2\sigma)''-(b_1\sigma)'+b_0\sigma\}=(1-x)_+^{\alpha-2}(1+x)_+^{\beta+2}\,dx$$

so that $\{Q_n(x)\}_{n=0}^{\infty}=\{P_n^{(\alpha-2,\beta+2)}(x)\}_{n=0}^{\infty}$ and

$$\begin{aligned}&(n(n-1)+2\alpha n+\alpha(\alpha-1))P_n^{(\alpha-2,\beta+2)}(x)\\&\quad=(x-1)^2P_n^{(\alpha,\beta)}(x)''1+2\alpha(x-1)P_n^{(\alpha,\beta)}(x)'+\alpha(\alpha-1)P_n^{(\alpha,\beta)}(x),\end{aligned}\tag{4.38}$$

$$\begin{aligned}&(n(n-1)+(2\beta+4)n+(\beta+2)(\beta+1))P_n^{(\alpha,\beta)}(x)\\&\quad=(x+1)^2P_n^{(\alpha-2,\beta+2)}(x)''+(2\beta+4)(x+1)P_n^{(\alpha-2,\beta+2)}(x)'\\&\qquad+(\beta+2)(\beta+1)P_n^{(\alpha-2,\beta+2)}(x).\end{aligned}\tag{4.39}$$

Case 7.2: $a_2(x)=(1+x)^2$. This case is reduced to Case 7.1 by replacing x by $-x$.

Case 7.3 $a_2(x)=1-x^2$. Then $b_2(x)=1-x^2$ and (4.12) becomes

$$\begin{aligned}&(1-x^2)\tau=(1-x^2)\sigma,\\&2((1-x^2)\tau)'-a_1(x)\tau=b_1(x)\sigma,\\&((1-x^2)\tau)''-(a_1(x)\tau)'+a_0\tau=b_0\sigma.\end{aligned}\tag{4.40}$$

Then we have for some constants λ and μ

$$\tau=\sigma+\lambda\delta(x-1)+\mu\delta(x+1),$$

$$b_1(x)=2(\beta-\alpha-(\alpha+\beta+2)x)-a_1(x),$$

$$\lambda(a_{11}+a_{10})=\mu(a_{11}-a_{10})=0.$$

Case 7.3.1: $\lambda=\mu=0$. Then $\tau=\sigma$ so that $\{Q_n(x)\}_{n=0}^{\infty}=\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$.

Case 7.3.2: $a_{11} + a_{10} = a_{11} - a_{10} = 0$, *that is* $a_1(x) = 0$. Then

$$\alpha = \beta = 0, \quad b_1(x) = -4x, \quad b_0 = a_0 - 2$$

so that $\sigma = H(1-x)H(1+x)\,\mathrm{d}x$ and

$$\tau = a_0^{-1}\{(b_2\sigma)'' - (b_1\sigma)' + b_0\sigma\} = \sigma - \frac{2}{a_0}(\delta(x+1) + \delta(x-1)).$$

Hence, $\{Q_n(x)\}_{n=0}^{\infty} = \{P_n^{(-a_0/2)}(x)\}_{n=0}^{\infty}$ is the Legendre-type OPS.

Case 7.3.3: $\lambda = 0$ *and* $a_{11} - a_{10} = 0$. Then $\tau = \sigma + \mu\delta(x+1)$ and (4.40) gives

$$a_1(x) = -(\alpha+1)(x+1), \quad b_1(x) = -(\alpha+3)x - \alpha + 1, \quad b_0 = a_0 - \alpha - 1$$

and $\beta = 0, \mu = -2^{\alpha+1}a_0^{-1}$ since $\sigma = H(x+1)(1-x)_+^{\alpha}\,\mathrm{d}x$. Hence $\tau = \sigma - 2^{\alpha+1}a_0^{-1}\delta(x+1)$ ($a_0 \neq n(n+\alpha)$, $n \geqslant 0$) and so $\{Q_n(x)\}_{n=0}^{\infty}$ is the Jacobi-type OPS satisfying

$$\begin{aligned} LM[y] = {} & (x^2-1)^2 y^{(iv)} + 2(x^2-1)((\alpha+4)x+\alpha)y'' + (x+1)\{(\alpha^2+9\alpha - \\ & - 2a_0 + 14)x + \alpha^2 - 3\alpha + 2a_0 - 10\}y'' - 2\{(\alpha+2)(a_0-\alpha-1)x \\ & - \alpha^2 - 3\alpha + a_0\alpha - 2\}y' + a_0(a_0-\alpha-1)y = \lambda_n y. \end{aligned}$$

In fact, $\{Q_n(x)\}_{n=0}^{\infty} = \{2^n S_n^{(\alpha)}((x+1)/2)\}_{n=0}^{\infty}(M = -\alpha_0)$ and

$$(n^2 + \alpha n - a_0)Q_n(x) = (x^2-1)P_n^{(\alpha,0)}(x)'' + (\alpha+1)(x+1)P_n^{(\alpha,0)}(x)' - a_0P_n^{(\alpha,0)}(x), \tag{4.41}$$

$$\begin{aligned} (n^2 + \alpha n + 2n + \alpha - a_0 + 1)P_n^{(\alpha,0)}(x) = {} & (x^2-1)Q_n''(x) + ((\alpha+3)x + \alpha - 1)Q_n'(x) \\ & + (\alpha - a_0 + 1)Q_n(x). \end{aligned} \tag{4.42}$$

Case 7.3.4: $\mu = 0$ *and* $a_{11} + a_{10} = 0$. This case is reduced to Case 7.3.3 by replacing x by $-x$.

Acknowledgements

This work is partially supported by Korea Ministry of Education (BSRI-1998-015-D00028) and KOSEF(98-0701-03-01-5).

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