Ελληνικό Στατιστικό Ινστιτούτο
Πρακτικά 20ου Πανελληνίου Συνεδρίου Στατιστικής (2007), σελ 469-477

Asymptotic Behaviour of a q-Binomial Type Distribution Based on q-Krawtchouk Orthogonal Polynomials

A. Kyriakoussis and M.G. Vamvakari
Harokopio University of Athens

ABSTRACT

In this manuscript, we introduce for $0 < q < 1$ a new deformed q-binomial distribution by virtue of the normalized q-Krawtchouk orthogonal polynomials. We call this q-discrete probability distribution *q-binomial distribution of type I*, say $\mathcal{B}(n,p,q)$ with $p > 0$. Interpretation of this q-discrete probability distribution is provided by considering, for suitable values of q, $n-$dimensional vector spaces over the finite field of $1/q$ elements. Furthermore, we study its asymptotic behaviour for $n \to \infty$ by showing that is compatible with a standardized Stieltjes-Wigert distribution.

1. INTRODUCTION

Let v_n be a discrete probability measure in $S = \{x_i, i = 1, 2, \ldots, n;\ n = 0, 1, \ldots\}$ with finite moments of all orders. Then it is well known that their exist a sequence of normalized discrete orthogonal polynomials $\{P_m(x)\}$ with respect to the measure v_n satisfying the orthogonality relation

$$\sum_{x \in S} v_n(x) P_m(x) P_\nu(x) = \lambda_{n\nu} \delta_{m,\nu}, \tag{1}$$

where $\delta_{m,\nu}$ the Kronecker delta and $\lambda_{n\nu}$ a non-negative sequence of n and ν and the three term recurrence relation

$$xP_m(x) = P_{m+1}(x) + a_m P_m(x) + b_m P_{m-1}(x) \quad (m \geq 1), \tag{2}$$

where $a_m \in R$ and $b_m > 0$ and with initial conditions $P_0(x) = 1$ and $P_1(x) = x - a_0$. Conversely, Favard's theorem ensures the existence of a discrete probability measure v_n on S for which the sequence of polynomials determined by the recurrence relation (2) are orthogonal. The mean value and the variance of the discrete random variable X in the discrete spectrum S with probability function $v_n(x)$ are given respectively by $\mu = a_0$ and $\sigma^2 = b_1$. If $a_m = 0$ then all moments of odd order are zero.(see Saitoh and Yoshida (2000), Christiansen (2004)).

Saitoh and Yoshida(2000) introduced a q-deformed binomial distribution for $0 < q < 1$, by virtue of a q-deformed sequence of Krawtchouk orthogonal polynomials and studied its asymptotic behaviour by showing that is compatible with a q-deformed Gaussian distribution in a quantum probability space.

Various q-analogues for $0 < q < 1$, of the classical binomial distribution have also been studied by many authors. Among them we refer to Kemp(1992, 2002), Sicong(1994) and Charalambides(2005).

In this manuscript, we introduce for $0 < q < 1$ a new deformed q-binomial distribution by virtue of the normalized q-Krawtchouk orthogonal polynomials. Interpretation of this q-discrete probability distribution is provided by considering, for suitable values of q, $n-$dimensional vector spaces over the finite field of $1/q$ elements. Furthermore, we study its asymptotic behaviour for $n \to \infty$ by showing that is compatible with a standardized Stieltjes-Wigert distribution.

2.INTRODUCTORY DEFINITIONS AND NOTATIONS

For the needs of this manuscript we recall some usual definitions and notation used in q-analysis (see Koekoek and Swarttouw (1998)).

Let $0 < q < 1$, x a real number and k a positive integer. The *q-shifted factorial* is defined by

$$(a;q)_n := \prod_{j=1}^{n}(1 - aq^{j-1})$$

and the general q-shifted factorial is given by

$$(a;q)_\infty := \prod_{j=1}^{\infty}(1 - aq^{j-1}), \ \ (a;q)_0 = 1.$$

Also,

$$(a, b; q)_n := (a;q)_n(b;q)_n.$$

The *q-number* is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}$$

and the *q-binomial coefficient* by

$$\binom{n}{k}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

or

$$\binom{n}{k}_{1/q} = q^{-k(n-k)}\binom{n}{k}_q .$$

Also

$$(-t;q)_n = \sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}}t^k.$$

3. q-BINOMIAL DISTRIBUTION OF TYPE I–ASYMPTOTIC BEHAVIOUR

Let $0 < q < 1$ be a rational number such that $1/q$ be an integer larger than 1. Also, let V_n be an $n-$dimensional vector space over the finite field $GF(1/q)$ of $1/q$ elements. Then V_n contains a total of $1/q^n$ vectors and the number of x-dimensional subspaces of V_n, $x = 0, 1, \ldots, n$ is given by $\binom{n}{x}_{1/q}$ (see Exton(1983)).
Supposing that each x-dimensional subspace assigns a weight, say $g_q(x; V_n)$, with

$$g_q(x;V_n) = q^{\binom{x+1}{2}}p^{-x}, \ \ p > 0, \ \ x = 0, 1, \ldots, n$$

we have that the probability of appearance of a x-dimensional subspace of V_n is given by

$$f_X(x) = \frac{\binom{n}{x}_{1/q} q^{\binom{x+1}{2}}p^{-x}}{\sum_{x=0}^{n}\binom{n}{x}_{1/q} q^{\binom{x+1}{2}}p^{-x}} \tag{3}$$

or

$$f_X(x) = \binom{n}{x}_q q^{\binom{x}{2}}(p^{-1}q^{-n})^x \prod_{i=1}^{n}(1+p^{-1}q^{i-n-1})^{-1}, \ x = 0, 1, \ldots, n, \tag{4}$$

with $0 < q < 1, \ \ p > 0$.
Next, we define the following deformation of the q-analogue binomial distribution (4) considering the random variable $Y = q^{-X}$ with probability function

$$u_Y(y) = \frac{\binom{n}{-\frac{\ln y}{\ln q}}_q q^{\binom{-\frac{\ln y}{\ln q}}{2}} \alpha_n^{-\frac{\ln y}{\ln q}}}{\prod_{i=1}^{n}(1+\alpha_n q^{i-1})}, \ y = q^{-0}, q^{-1}, \ldots, q^{-n}, \tag{5}$$

where

$$\alpha_n = p^{-1}q^{-n}, \ p > 0 \tag{6}$$

such that

$$\alpha_n > 1, \ \alpha_n q^n \to 0 \tag{7}$$

and

$$\alpha_n \to \infty \text{ as } n \to \infty. \tag{8}$$

Proposition 3.1. *The probability function $u_Y(y)$ is induced by the normalized q-Krawtchouk orthogonal polynomials, say $p_m(y; p, n, q)$, $0 < q < 1$ given by*

$$p_m(y; p, n, q) = \frac{(q^{-n}; q)_m}{(-pq^m; q)_m} K_m(y; p, n, q), \tag{9}$$

where $y = q^{-x}$, $x = 0, 1, \ldots, n$ and $K_m(y; p, n, q)$ the q-Krawtchouk orthogonal polynomials with parameter $p = \alpha_n^{-1} q^{-n}$.
Moreover the random variable Y has the following rth order moments

$$\mu_r = \sum_{y \in S_1} y^r u_Y(y) = q^{-rn} \frac{\prod_{i=1}^{r}(1 + a_n^{-1} q^i)}{\prod_{i=1}^{r}(1 + a_n^{-1} q^{-n+i})}. \tag{10}$$

Proof. The normalized q-Krawtchouk orthogonal polynomials $p_m(q^{-x}; p, n, q)$, $0 < q < 1$ with parameter $p = \alpha_n^{-1} q^{-n}$ satisfies the orthogonality relation

$$\sum_{x=0}^{n} \frac{(q^{-n}; q)_x}{(q; q)_x} (-1)^x \alpha_n^x q^{nx} p_m(q^{-x}; p, n, q) p_\nu(q^{-x}; p, n, q)$$
$$= \gamma_{n\nu}(1 + \alpha_n^{-1} q^{-n})(-\alpha_n^{-1} q^{-n+1}; q)_n \alpha_n^n q^{n^2} q^{-\binom{n+1}{2}} \delta_{m\nu}$$

where $\delta_{m\nu}$ the Kronecker delta and $\gamma_{n\nu}$ a non-zero sequence of n and ν (see Koekoek and Swarttouw(1998)). Thus their weight function, say $u(x; n, q)$, can be written as

$$u(x; n, q) = \frac{\frac{(q^{-n}; q)_x}{(q; q)_x} (-1)^x \alpha_n^x q^{nx}}{(1 + \alpha_n^{-1} q^{-n})(-\alpha_n^{-1} q^{-n+1}; q)_n \alpha_n^n q^{n^2} q^{-\binom{n+1}{2}}}. \tag{11}$$

Using the identities

$$\frac{(q^{-n}; q)_x}{(q; q)_x} = (-1)^x q^{-nx} \alpha_n^{-x} \binom{n}{x}_q q^{\binom{x}{2}}$$

and

$$(1+\alpha_n^{-1}q^{-n})(-\alpha_n^{-1}q^{-n+1};q)_n = \alpha_n^{-n}q^{-n^2}q^{\binom{n+1}{2}}\prod_{i=1}^{n}(1+\alpha_n q^j)$$

we have from (11)

$$u(x;n,q) = u_Y(y),\ y = q^{-x}.$$

Moreover, by the probability function (5) the rth moment of the random variable Y is

$$\begin{aligned}
\mu_r &= \sum_{y\in S_1} y^r u_Y(y) \\
&= \frac{\sum_{y\in S_1} y^r \binom{n}{-\frac{\ln y}{\ln q}}_q q^{\binom{-\frac{\ln y}{\ln q}}{2}} \alpha_n^{-\frac{\ln y}{\ln q}}}{\prod_{i=1}^{n}(1+\alpha_n q^{i-1})} \\
&= \frac{\sum_{x=0}^{n} q^{-xr}\binom{n}{x}_q q^{\binom{x}{2}}\alpha_n^x}{\prod_{i=1}^{n}(1+\alpha_n q^{i-1})} \\
&= \frac{\prod_{i=1}^{n}(1+a_n q^{i-r-1})}{\prod_{i=1}^{n}(1+\alpha_n q^{i-1})} \\
&= \frac{(1+\alpha_n q^{-r})(1+\alpha_n q^{-r+1})\cdots(1+\alpha_n^{-1}q^{-1})\cdots(1+\alpha_n q^{n-r-1})}{(1+\alpha_n q^{-1})\cdots(1+\alpha_n q^{n-r+1})(1+\alpha_n q^{n-r})\cdots(1+\alpha_n q^{n-1})} \qquad (12)
\end{aligned}$$

From the last expression the equation (10) is obtained.

Definition 3.1. *For $0 < q < 1$, the q-discrete probability distribution with probability function $u_Y(y)$ defined in the spectrum $S_1 = \{q^{-k},\ k = 0,1,\ldots,n\}$ and based on normalized q-Krawtchouk orthogonal polynomials is called q-Binomial of type I distribution and is denoted by $\mathcal{B}(n,p,q)$.*

Remark 3.1. Using the moments of rth order (10) the random variable Y of the q-binomial of type I distribution has respectively the following mean value and variance

$$\mu_Y = q^{-n}\frac{1+a_n^{-1}q}{1+a_n^{-1}q^{-n+1}} \qquad (13)$$

and

$$\sigma_Y^2 = q^{-2n+1}\alpha_n^{-1}\frac{(1+\alpha_n^{-1}q)(q^{-n}-1)(1-q)}{(1+a_n^{-1}q^{-n+1})^2(1+a_n^{-1}q^{-n+2})} \qquad (14)$$

Equivalently, since the q-binomial of type I distribution is induced from the normalized q-Krawtchouk orthogonal polynomials $p_m(y;p,n,q)$, $y = q^{-x}$, $p = \alpha_n^{-1}q^{-n}$, the mean value and variance of the random variable Y can be derived from the recurrence relation of these polynomials. Analytically,the recurrence relation of the orthogonal polynomials $p_m(y;p,n,q)$ with $p = \alpha_n^{-1}q^{-n}$ is

$$\begin{aligned} yp_m(y;p,n,q) &= p_{m+1}(y;p,n,q) + a_m^{(p,n,q)} p_m(y;p,n,q) \\ &+ b_m^{(p,n,q)} p_{m-1}(y;p,n,q) (m \geq 1) \end{aligned} \tag{15}$$

with initial conditions

$$p_0(y;p,n,q) = 1, \quad p_1(y;p,n,q) = y - a_0,$$

where

$$a_m^{(p,n,q)} = 1 - (A_m + C_m), \; b_m^{(p,n,q)} = A_{m-1}C_m \tag{16}$$

and

$$\begin{aligned} A_m &= \frac{(1-q^{m-n})(1+\alpha_n^{-1}q^{m-n})}{(1+\alpha_n^{-1}q^{2m-n})(1+\alpha_n^{-1}q^{2m-n+1})} \\ C_m &= -\alpha_n^{-1}q^{2m-2n-1}\frac{(1+\alpha_n^{-1}q^{m-n})(1-q^m)}{(1+\alpha_n^{-1}q^{2m-n-1})(1+\alpha_n^{-1}q^{2m-n})} \end{aligned} \tag{17}$$

(see Koekoek and Swarttouw (1998)).
So,

$$\mu_Y = 1 - A_0 - C_0, \quad \sigma_Y^2 = A_0C_1$$

and on using (17) we recapture (13) and (14).

Theorem 3.1. *The limit distribution for $n \to \infty$ of the standardized q-binomial of type I distribution $\mathcal{B}(n,p,q)$ is the standardized Stieltjes-Wigert distribution with probability density function*

$$u_q^{SW}(y) = \frac{q^{-11/8}(q^{-3/2}(1-q)^{1/2}y + q^{-1})^{-1/2}}{\sqrt{2\pi \log q^{-1}}} e^{\frac{(\log(q^{-3/2}(1-q)^{1/2}y+q^{-1}))^2}{2\log q}},$$
$$y > -q^{1/2}(1-q)^{-1/2}. \tag{18}$$

Proof. Shifting so as to have zero mean in the q-Binomial of type I distribution with mean value μ_Y, its orthogonal polynomials are $p_m(y+\mu_Y;p,n,q)$,

$p = \alpha_n^{-1} q^{-n}$. The coefficients of the recurrence relation of $p_m(y + \mu_Y; p, n, q)$ are represented as $a_m^{(n,p,q)} - \mu_Y = A_0 + C_0 - (A_m + C_m)$ and $b_m^{(n,p,q)} = A_{m-1} C_m$. In addition, to standardized so as to be of variance 1, it can be realized by replacing the recurrence relation coefficients respectively by

$$\frac{a_m^{(n,p,q)} - \mu_Y}{\sigma_Y} = \frac{A_0 + C_0 - (A_m + C_m)}{\sqrt{A_0 C_1}} \tag{19}$$

and

$$\frac{b_m^{(n,p,q)}}{\sigma_Y^2} = \frac{A_{m-1} C_m}{A_0 C_1}, \tag{20}$$

where A_m and C_m are defined in (17).

Thus the orthogonal polynomials, say $Q_m(y; p, q, n)$, $p = \alpha_n^{-1} q^{-n}$, of the standardized q-binomial of type I distribution are determined by the recurrence relation

$$\begin{aligned} yQ_m(y;p,q,n) &= Q_{m+1}(y;p,q,n) + \frac{A_0 + C_0 - (A_m + C_m)}{\sqrt{A_0 C_1}} Q_m(y;p,q,n) \\ &+ \frac{A_{m-1} C_m}{A_0 C_1} Q_{m-1}(y;p,q,n), \ (m \geq 1) \end{aligned} \tag{21}$$

with initial conditions

$$Q_0(y;p,q,n)(y) = 1 \qquad Q_1(y;p,q,n)(y) = y.$$

Using (13), (14) and (17) the coefficients (19) and (20) respectively become

$$\begin{aligned} &\frac{a_m^{(n,p,q)} - \mu_Y}{\sigma_Y} = \\ &\frac{1 - \frac{(1-q^{m-n})(1+\alpha_n^{-1} q^{m-n})}{(1+\alpha_n^{-1} q^{2m-n})(1+\alpha_n^{-1} q^{2m-n+1})} + \frac{\alpha_n^{-1} q^{2m-2n-1}(1+\alpha_n^{-1} q^{m-n})(1-q^m)}{(1+\alpha_n^{-1} q^{2m-n-1})(1+\alpha_n^{-1} q^{2m-n})}}{q^{-n+1/2} \alpha_n^{-1/2} \frac{(1+\alpha_n^{-1} q)^{1/2} (q^{-n}-1)^{1/2} (1-q)^{1/2}}{(1+a_n^{-1} q^{-n+1})(1+a_n^{-1} q^{-n+2})^{1/2}}} \\ &- \frac{q^{-n} \frac{1+a_n^{-1} q}{1+a_n^{-1} q^{-n+1}}}{q^{-n+1/2} \alpha_n^{-1/2} \frac{(1+\alpha_n^{-1} q)^{1/2} (q^{-n}-1)^{1/2} (1-q)^{1/2}}{(1+a_n^{-1} q^{-n+1})(1+a_n^{-1} q^{-n+2})^{1/2}}} \end{aligned} \tag{22}$$

and

$$\frac{b_m^{(n,p,q)}}{\sigma_Y^2} = q^{2n-2} \frac{\frac{(1-q^{m-1-n})(1+\alpha_n^{-1} q^{m-1-n})(1+\alpha_n^{-1} q^m)(1-q^m)}{(1+\alpha_n^{-1} q^{2m-2-n})(1+\alpha_n^{-1} q^{2m-1-n})^2 (1+\alpha_n^{-1} q^{2m-n})}}{\frac{(1+\alpha_n^{-1} q)(q^{-n}-1)(1-q)}{(1+a_n^{-1} q^{-n+1})^2 (1+a_n^{-1} q^{-n+2})}}. \tag{23}$$

Taking the limit $n \to \infty$ of (22) and (23) and using (7) and (8) , we obtain the recurrence relation of the orthogonal polynomials, say $Q_m(y)$, for the limit distribution, as

$$\begin{aligned} yQ_m(y) &= Q_{m+1}(y) \\ &+ q^{3/2}(1-q)^{-1/2}(q^{-2m-1}(1+q-q^{m+1})-q^{-1})Q_m(y) \\ &+ q^{-4m+4}(1-q)(1-q^m)Q_{m-1}(y),\ (m \geq 1) \end{aligned} \tag{24}$$

with initial conditions

$$Q_0(y) = 1 \quad Q_1(y) = y.$$

The continuous Stieltjes-Wigert distribution with probability density function

$$v_q^{SW}(x) = \frac{q^{1/8}}{\sqrt{2\pi \log q^{-1}}} \frac{1}{x} e^{\frac{(\log x)^2}{2\log q}},\ x > 0 \tag{25}$$

with

$$\mu = q^{-1},\ \sigma^2 = q^{-3}(1-q), \tag{26}$$

is based on the normalized Stieltjes-Wigert orthogonal polynomials, say $p_m^{SW}(x)$, satisfying the recurrence relation

$$\begin{aligned} xp_m^{SW}(x) &= p_{m+1}^{SW}(x) + q^{-2m-1}(1+q-q^{m+1})p_m^{SW}(x) \\ &+ q^{-4m+1}(1-q^m)p_{m-1}^{SW}(x)\,(m \geq 1) \end{aligned} \tag{27}$$

with initial conditions

$$p_0^{SW}(x) = 1, \quad p_1^{SW}(x) = x - q^{-1}$$

(see Christiansen (2003)).
By standardizing (27) and (25) we obtain respectively (24) and (18) which complete our proof.

ΠΕΡΙΛΗΨΗ

Στην εργασία αυτή, εισάγεται για $0 < q < 1$ μία μετασχηματισμένη q-διωνυμική κατανομή μέσω των κανονικοποιημένων q-Krawtchouk ορθογωνίων πολυωνύμων. Η q-διακριτή αυτή κατανομή ονομάζεται *q-διωνυμική κατανομή τύπου I* και συμβολίζεται με $\mathcal{B}(n,p,q)$ με $p > 0$. Επίσης. δίδεται μία ερμηνεία αυτής της q-διακριτής κατανομής θεωρώντας, για κατάλληλες τιμές του q, $n-$διάστατους διανυσματικούς χώρους επί πεπερασμένων συνόλων με $1/q$ στοιχεία. Επιπλέον, μελετάται η ασυμπτωτική της συμπεριφορά για $n \to \infty$ αποδεικνύοντας ότι η αντίστοιχη τυποποιημένη q-διωνυμική κατανομή τύπου I συγκλίνει στην τυποποιημένη Stieltjes-Wigert κατανομή.

REFERENCES

Charalambides Ch.A. (2005): Moments of a class of discrete q-distributions, *J. Stat. Plan. Infer., 135,64–76.*

Christiansen,J.S.(2003): The moment problem associated with the Stiltjes-Wigert polynomials, *J. Math. Anal. Appl., 277, 218-245.*

Christiansen,J.S.(2004): Inderminate moment problems within the Askey-scheme, PH.D. thesis, Institute for Mathematical Sciences, University of Copenhagen.

Exton,H.(1983): *q-Hypergeometric Function and Applications* , Ellis Horwood Series, *Mathematics and Its Applications* .

Kemp,A.W.(1992): Steady state Markov chain models for the heine and Euler distributions, *J. Appl. Probab., 29, 869-876.*

A.W. Kemp,A.W.(2002): Certain q-analogue of the binomial distribution, *Sankhy a: The Indian Jounal of Statistics, 64, 293-305.*

Koekoek,R. and Swarttouw,R.F.(1998): The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Delft University of Technology, Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics, Report no., *98-17.*

Saitoh, A. and Yoshida,H.(2000): A q-deformed Poisson distribution based on orthogonal polynomials, *J. Phys. A: Math. Gen., 33, 1435-1444.*

Sicong,J (1994): The q-deformed binomial distribution and its asymptotic behaviour, *J. Phys. A: Math. Gen., 27, 493-499.*