Elect. Comm. in Probab. **13** (2008), 67–84

ELECTRONIC COMMUNICATIONS in PROBABILITY

HERMITE AND LAGUERRE POLYNOMIALS AND MATRIX-VALUED STOCHASTIC PROCESSES

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Submitted March 13, 2007, *accepted in final form* January 29, 2008

AMS 2000 Subject classification: 60J60, 60J65, 15A52, 33C45.
Keywords: Brownian matrices, Wishart processes, Hermite polynomials, Laguerre polynomials, martingale polynomials, chaos representation property.

Abstract
We extend to matrix-valued stochastic processes, some well-known relations between real-valued diffusions and classical orthogonal polynomials, along with some recent results about Lévy processes and martingale polynomials. In particular, joint semigroup densities of the eigenvalue processes of the generalized matrix-valued Ornstein-Uhlenbeck and squared Ornstein-Uhlenbeck processes are respectively expressed by means of the Hermite and Laguerre polynomials of matrix arguments. These polynomials also define martingales for the Brownian matrix and the generalized Gamma process. As an application, we derive a chaotic representation property for the eigenvalue process of the Brownian matrix.

1 Introduction

To our knowledge, the first connection between orthogonal polynomials and stochastic processes appears in an attempt by Wong [26] to construct semigroup densities in closed form for a class of stationary Markov processes. The link between the two fields is established by noting that most of the orthogonal polynomials in the Askey scheme form a complete set of solutions for an eigenvalue equation for the infinitesimal generator of the process $\mathcal{L}$ (see [1], [14] and [25]), that is

$$\mathcal{L}Q_k(x) = \lambda_k Q_k(x), \quad k \in \mathbb{N}. \tag{1.1}$$

The polynomials $\{Q_k(x), k \geq 0\}$ are orthonormal with respect to the weight $w(y)$ and hence the Kolmogorov equation leads to the following expression for the semigroup densities:

$$p_t(x, y) = \sum_{k=0}^{\infty} e^{\lambda_k t} Q_k(x) Q_k(y) w(y). \tag{1.2}$$

[1]RESEARCH SUPPORTED IN PART BY THE LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES, CNRS-UMR 7599, PARIS, FRANCE

More recently Schoutens [23] revealed another probabilistic property of certain polynomials in the Askey scheme. Polynomials that satisfy a generating function relation of the form

$$\sum_{k=0}^{\infty} Q_k(x)\frac{z^k}{k!} = f(z)\exp(x\ u(z)), \tag{1.3}$$

for $u(z), f(z)$ analytic, $u(0) = 0$, $u'(0) \neq 0$ and $f(0) \neq 0$, are called Sheffer polynomials [24]. We however focus on Sheffer polynomials that satisfy an orthogonality relation, which were first characterized by Meixner [21] and eventually associated with his name. An extra parameter $t \geq 0$ can be introduced to the latter expression of the generating function to define the Lévy-Meixner systems.

Definition 1.1. A Lévy-Meixner system is a system of orthogonal polynomials $\{Q_k(x,t), k \geq 0\}$ defined by two analytic functions $f(z)$ and $u(z)$ with $u(0) = 0$, $u'(0) \neq 0$ and $f(0) = 1$ such that the generating function has the following form:

$$\sum_{k=0}^{\infty} Q_k(x,t)\frac{z^k}{k!} = (f(z))^t \exp(x\ u(z)), \tag{1.4}$$

for $(f(\tau(i\theta)))^{-1}$ an infinitely divisible characteristic function and $\tau(u(z)) = z$ the inverse function of $u(z)$.

As a consequence, Lévy-Meixner systems satisfy the following martingale relation:

$$\mathrm{E}\big[Q_k(X_t,t)\ \big|\ X_s\big] = Q_k(X_s,s), \tag{1.5}$$

for $0 \leq s \leq t$, $k \geq 0$ and $\{X_t, t \geq 0\}$ the Lévy process corresponding to the Lévy-Meixner polynomials $\{Q_k(x,t), k \geq 0\}$. For the interest of the present note, we mention the Brownian-Hermite system and the Gamma-Laguerre system, which are involved in the chaotic representation property of the Brownian motion and the Gamma process, respectively.
The matrix-valued counterparts of the classical Hermite, Laguerre and Jacobi polynomials have gained increasing interest as natural multidimensional extensions. These polynomials originated in conjunction with random matrix theory and multivariate statistics of the $2/\alpha$-ensembles. The parameter α refers to the field over which the random matrix entries are distributed and is usually confined to 2 for real, 1 for complex and 1/2 for quaternions. Herz [10], Constantine [7] and Muirhead [22] provided a generalization of the hypergeometric functions and exploited it to study the Hermite and Laguerre polynomials for $\alpha = 2$. Further results on their properties have been provided by James [13] and Chikuse [6] and Baker and Forrester [2] for all α. Lasalle computed the generating functions for the three types of polynomials for all $\alpha > 0$ in [16], [17] and [18].
Following a different line of research, matrix-valued stochastic processes originated in the work of Dyson [9], when eigenvalues of a random matrix were chosen to follow Brownian motions. Bru in [4], [3] and [5] defined the matrix equivalent to the squared Bessel process as the now celebrated Wishart process. König and O'Connell [15] showed that the eigenvalues of a complex Wishart process evolve like independent squared Bessel processes conditioned never to collide. Further extensions of the properties of squared Bessel processes to Wishart processes have been achieved by Donati-Martin et al. [8], such as local absolute continuity relationships between the laws of Wishart processes with different dimensions.
We propose to bring together these results and thus generalize (1.2) and (1.5) to their matrix counterparts for the Hermite and Laguerre polynomials. The paper is organized as follows.

Section 2 reviews the definitions and properties of the zonal, Hermite and Laguerre polynomials. We prove in Section 3 that (1.2) corresponds to the joint semigroup densities of the eigenvalue process for a matrix-valued Ornstein-Uhlenbeck process in case of the Hermite polynomials and a matrix-valued generalized squared Ornstein-Uhlenbeck process in case of the Laguerre polynomials. Section 4 gives the matrix equivalent of the martingale relation (1.5). As an application, we also give the chaotic representation property of the eigenvalue process of the Brownian matrix.

2 The Hermite and Laguerre polynomials

We restrict the framework to $\alpha = 2$, that is we consider random matrices with entries distributed over the real line. In particular, we only consider the set of real $m \times m$ symmetric matrices $\S_m$ and the set of real positive definite $m \times m$ symmetric matrices $\S_m^+$. The eigenvalues of $X \in \S_m$ or $X \in \S_m^+$ will be noted $(x_1, \dots, x_m)$.

2.1 Zonal polynomials and hypergeometric functions

Let κ be a partition of k, written symbolically $\kappa \vdash k$, such that $\kappa = (k_1, \dots, k_m)$, $k_1 \geq \dots \geq k_m \geq 0$ and $k = \sum_{i=1}^{m} k_i$. The first step towards the construction of orthogonal polynomials of matrix argument is to generalize of the monomial x^k.

Definition 2.1. For $X \in S_m$, the zonal polynomial $C_\kappa(X)$ is a symmetric polynomial in the eigenvalues $(x_1, \dots, x_m)$ of X. It is the only homogeneous polynomial eigenfunction of the Laplace-Beltrami operator

$$D^* = \sum_{i=1}^{m} x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i}, \tag{2.1}$$

with eigenvalue

$$\sum_{i=1}^{m} k_i(k_i - i) + k(m-1), \tag{2.2}$$

having highest order term corresponding to κ.

The zonal polynomial was defined by James [12] and corresponds to the special case $\alpha = 2$ of the Jack polynomial [11]. For some $Y \in S_m^+$, it moreover satisfies

$$C_\kappa(YX) = C_\kappa(\sqrt{Y} X \sqrt{Y}). \tag{2.3}$$

By analogy to single variable hypergeometric functions, we have the following (with the same notation as Muirhead [22]):

Definition 2.2. The hypergeometric functions of matrix argument are defined by

$${}_p\mathbf{F}_q(a_1, \dots, a_p; b_1, \dots, b_q; X) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(X)}{k!}, \tag{2.4}$$

where the second summation is over all partitions $\kappa = (k_1, \ldots, k_m)$, $k_1 \geq \cdots \geq k_m \geq 0$, of $k = \sum_{i=1}^{m} k_i$, $k! = k_1! \cdots k_m!$ and the generalized Pochhammer symbols are given by

$$(a)_\kappa = \prod_{i=1}^{m} \left(a - \frac{i-1}{2} \right)_{k_i}, \quad (a)_k = a(a+1)\cdots(a+k-1), \quad (a)_0 = 1. \tag{2.5}$$

The hypergeometric functions of two matrix arguments $X, Y \in S_m$ are defined by

$${}_p\mathbf{F}_q^{(m)}(a_1, \ldots, a_p; b_1, \ldots, b_q; X, Y) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(X) C_\kappa(Y)}{k! C_\kappa(I)}, \tag{2.6}$$

where $I = \mathrm{diag}_m(1)$.

Two important special cases of (2.4) are

$${}_0\mathbf{F}_0(X) = \mathrm{etr}(X) \quad \text{and} \quad {}_1\mathbf{F}_0(a; X) = \det(I - X)^{-a}, \tag{2.7}$$

where $\mathrm{etr}(X) = \exp\big(\mathrm{tr}(X)\big)$ is the exponential trace function. The relation between (2.4) and (2.6) is given by

$$\int_{O_m} {}_p\mathbf{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; XHYH')(dH) = {}_p\mathbf{F}_q^{(m)}(a_1, \ldots, a_p; b_1, \ldots, b_q; X, Y) \tag{2.8}$$

where (dH) denotes the invariant Haar measure on the orthogonal group O_m and H' the transpose of H.

2.2 The class of Meixner polynomials of matrix argument

The classical Meixner polynomials have the following generalization to polynomials with matrix arguments (see [6]):

Definition 2.3. A multivariate symmetric Meixner system is a system of orthogonal polynomials $\{P_\kappa(X)\}$ defined by two analytic $m \times m$ symmetric matrix-valued functions U with $U(0) = 0$ and F such that

$$\sum_{k=0}^{\infty} \sum_{\kappa} P_\kappa(X) \frac{C_\kappa(Z)}{k!\, C_\kappa(I)} = F(U(Z)) \int_{O_m} \mathrm{etr}\big(HXH'U(Z)\big)(dH) \tag{2.9}$$

for Z a $m \times m$ symmetric matrix.

We focus on two families of orthogonal polynomials, for which we give the definition along with the main properties.

2.2.1 The Hermite polynomials

The system of matrix-valued Hermite polynomials $\{H_\kappa(X)\}$ is defined by the generating function in Definition 2.3 for $U(Z) = Z$ and $F(Z) = \mathrm{etr}\left(-\frac{1}{2}Z^2\right)$, that is

$$\sum_{k=0}^{\infty} \sum_{\kappa} H_\kappa(X) \frac{C_\kappa(Z)}{k!\, C_\kappa(I)} = \mathrm{etr}\left(-\frac{Z^2}{2}\right) \int_{O_m} \mathrm{etr}\left(HXH'Z\right)(dH) \tag{2.10}$$

for X, Z two $m \times m$ symmetric matrices.
They form a complete orthogonal system in $\S_m$ with respect to the weight

$$W^H(X) = \mathrm{etr}\left(-\frac{X^2}{2}\right), \tag{2.11}$$

such that

$$\int_{\S_m} H_\kappa(X)\ H_\sigma(X)\ W^H(X)\ (dX) = \mathcal{N}_\kappa^{(H)}\ \delta_{\kappa\sigma}, \tag{2.12}$$

with normalization factor

$$\mathcal{N}_\kappa^{(H)} = k!\ C_\kappa(I)\ 2^{\frac{m}{2}} \pi^{\frac{m(m+1)}{4}}. \tag{2.13}$$

The Hermite polynomials $H_\kappa(X)$ are essentially m-dimensional functions of the eigenvalues $(x_1, \ldots, x_m)$ of the matrix X. They satisfy an eigenvalue equation,

$$\mathcal{L}^{(H)} H_\kappa(X) = \lambda_\kappa H_\kappa(X), \tag{2.14}$$

for the m-dimensional operator

$$\mathcal{L}^{(H)} = \sum_{i=1}^{m} \left(\frac{\partial^2}{\partial x_i^2} - x_i \frac{\partial}{\partial x_i} + \sum_{j \neq i} \frac{1}{x_i - x_j} \frac{\partial}{\partial x_i} \right) \tag{2.15}$$

and the eigenvalues $\lambda_\kappa = -k$.
Another useful representation of the Hermite polynomials is that of the so-called Rodrigues formula,

$$H_\kappa(X)\ \mathrm{etr}\left(-\frac{X^2}{2}\right) = \frac{1}{2^{\frac{m}{2}} \pi^{\frac{m(m+1)}{4}}} \int_{S_m} C_\kappa(iT)\ \mathrm{etr}\left(-\frac{T^2}{2} - iXT\right)(dT), \tag{2.16}$$

which is first proved by Chikuse in [6]. An immediate consequence is the inverse Fourier transform relation,

$$C_\kappa(iT)\ \mathrm{etr}\left(-\frac{T^2}{2}\right) = \frac{1}{2^{\frac{m}{2}} \pi^{\frac{m(m+1)}{4}}} \int_{S_m} H_\kappa(X)\ \mathrm{etr}\left(-\frac{X^2}{2} + iXT\right)(dX), \tag{2.17}$$

which can be proved rigorously using the Rodrigues formula with Theorem 3.4 in [6].

2.2.2 The Laguerre polynomials

The system of matrix-valued Laguerre polynomials $\{L_\kappa^\gamma(X)\}$ is defined by the generating function

$$\sum_{k=0}^{\infty} \sum_{\kappa} L_\kappa^\gamma(X)\ \frac{C_\kappa(Z)}{k!\ C_\kappa(I)} = (\det(I-Z))^{-\gamma-\frac{m-1}{2}} \int_{O_m} \mathrm{etr}\left(-HXH'Z(I-Z)^{-1}\right)(dH) \tag{2.18}$$

for $X, Z \in S_m$.
The matrix-valued Laguerre polynomials can be expressed in terms of the zonal polynomials as follows:

$$L_\kappa^\gamma(X) = \left(\gamma + \frac{m+1}{2}\right)_\kappa C_\kappa(I) \sum_{s=0}^{k} \sum_{\sigma} \frac{(-1)^s \binom{\kappa}{\sigma}}{(\gamma + \frac{m+1}{2})_\sigma}\ \frac{C_\sigma(X)}{C_\sigma(I)}. \tag{2.19}$$

where the generalized binomial coefficients $\binom{\kappa}{\sigma}$ are defined as follows:

$$\frac{C_\kappa(X+I)}{C_\kappa(I)} = \sum_{s=0}^{k} \sum_{\sigma \vdash s,\ \sigma \subseteq \kappa} \binom{\kappa}{\sigma} \frac{C_\sigma(X)}{C_\sigma(I)}, \tag{2.20}$$

with $\sigma \subseteq \kappa$ meaning $s_i \leq k_i,\ \forall i$.
They form a complete orthogonal system in $\S_m^+$ with respect to the weight

$$W^L(X) = \operatorname{etr}(-X)\ \ (\det X)^\gamma\,, \tag{2.21}$$

such that

$$\int_{S_m^+} L_\kappa^\gamma(X)\ L_\sigma^\gamma(X)\ W^L(X)\ (dX) = \mathcal{N}_\kappa^{(L)}\ \delta_{\kappa\sigma}, \tag{2.22}$$

with normalization factor

$$\mathcal{N}_\kappa^{(L)} = k!\ C_\kappa(I)\ \Gamma_m\left(\gamma + \frac{m+1}{2}\right)\left(\gamma + \frac{m+1}{2}\right)_\kappa. \tag{2.23}$$

The Laguerre polynomial $L_\kappa^\gamma(X)$ is moreover eigenfunctions of the m-dimensional operator

$$\mathcal{L}^{(L)} = \sum_{i=1}^{m}\left(x_i \frac{\partial^2}{\partial x_i^2} + (\gamma + 1 - x_i)\frac{\partial}{\partial x_i} + \sum_{j\neq i} \frac{x_i}{x_i - x_j}\frac{\partial}{\partial x_i}\right) \tag{2.24}$$

for the eigenvalue $-k$.

Remark 2.4. We have the following limit relation between the Laguerre and Hermite polynomials:

$$\lim_{\gamma\to\infty} \gamma^{-k/2} L_\kappa^\gamma(\gamma + \sqrt{\gamma}X) = (-1)^k H_\kappa(X). \tag{2.25}$$

3 Eigenvalue processes

Baker and Forrester investigate in [2] the properties of certain Schrödinger operators of Calogero-Sutherland-type quantum systems that have the Hermite or Laguerre polynomials as eigenfunctions. As an application, they derived semigroup densities for m-dimensional diffusions. In this section, we show that these processes relate to the eigenvalue processes of the generalized Ornstein-Uhlenbeck (OU) and the generalized squared Ornstein-Uhlenbeck (OUSQ) processes respectively.

3.1 The generalized OU process and Hermite polynomials

Definition 3.1. We call the generalized OU process the matrix-valued process, $(X_t, t \geq 0)$, solution to the stochastic differential equation

$$dX_t = \sqrt{2}\ d\beta_t - \lambda X_t dt, \quad X_0 = X, \tag{3.1}$$

for a Brownian matrix $(\beta_t, t \geq 0)$.

Similarly as in the one-dimensional case, the semigroup densities $\mathbf{Q}(t,X,Y)$ of the generalized OU process are related to the symmetric $m\times m$ Brownian matrix with semigroup densities $\mathbf{P}(\tau,X,Y)$ as follows:

$$\mathbf{Q}(t,X,Y)=e^{\lambda t\frac{m(m+1)}{2}}\mathbf{P}\left(\frac{e^{2\lambda t}-1}{\lambda},X,Ye^{\lambda t}\right) \tag{3.2}$$

The semigroup densities of the Brownian matrix have the form (see [20])

$$\mathbf{P}(\tau,X,Y)=\frac{\tau^{-\frac{m(m+1)}{4}}}{2^{\frac{m}{2}}\pi^{\frac{m(m+1)}{4}}}\ \mathrm{etr}\left(-\frac{(Y-X)^2}{2\tau}\right), \tag{3.3}$$

which yields for the generalized OU process,

$$\mathbf{Q}(t,X,Y)=\frac{\lambda^{\frac{m(m+1)}{4}}}{2^{\frac{m}{2}}\pi^{\frac{m(m+1)}{4}}}\left(\frac{e^{2\lambda t}}{e^{2\lambda t}-1}\right)^{\frac{m(m+1)}{4}}\mathrm{etr}\left(-\frac{\lambda\left(X^2+Y^2e^{2\lambda t}\right)}{2(e^{2\lambda t}-1)}\right){}_0\mathbf{F}_0\left(\frac{\lambda XYe^{\lambda t}}{e^{2\lambda t}-1}\right). \tag{3.4}$$

The latter densities are expressed with respect to the Haar measure (dY) on $\S_m$. Now Y has the representation

$$Y=HL^YH' \tag{3.5}$$

with some $H\in O_m$ and L^Y the diagonal matrix composed of the ordered eigenvalues ($y_1>y_2>\ldots>y_m$) of Y. Following Muirhead [22], the measure (dY) decomposes to

$$(dY)=(dH)(dL^Y), \tag{3.6}$$

where (dH) is the invariant Haar measure on O_m and (dL^Y) is the measure over the eigenvalues:

$$(dL^Y)=\frac{\pi^{\frac{m^2}{2}}}{\Gamma_m\left(\frac{m}{2}\right)}\prod_{i<j}^{m}(y_i-y_j)\bigwedge_{i=1}^{m}dy_i, \tag{3.7}$$

with $\bigwedge$ the exterior product. The joint semigroup densities of the eigenvalue processes with respect to the measure (dL^Y) are then obtained by integration over the orthogonal group O_m, which gives

$$\mathbf{q}(t,X,Y)=\frac{\lambda^{\frac{m(m+1)}{4}}}{2^{\frac{m}{2}}\pi^{\frac{m(m+1)}{4}}}\left(\frac{e^{2\lambda t}}{e^{2\lambda t}-1}\right)^{\frac{m(m+1)}{4}}\mathrm{etr}\left(-\frac{\lambda\left(X^2+Y^2e^{2\lambda t}\right)}{2(e^{2\lambda t}-1)}\right){}_0\mathbf{F}_0^{(m)}\left(\frac{\lambda e^{\lambda t}}{e^{2\lambda t}-1}X;Y\right). \tag{3.8}$$

These semigroup densities have however an equivalent formulation using the Hermite polynomials as follows:

Proposition 3.2. *Let $(X_t,t\geq 0)$ be the generalized OU process. The joint semigroup densities of the eigenvalue processes can be expressed as*

$$\mathbf{q}(t,X,Y)=\lambda^{\frac{m(m+1)}{4}}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{e^{-k\lambda t}}{\mathcal{N}_\kappa^{(H)}}\ H_\kappa(\sqrt{\lambda}X)\ H_\kappa(\sqrt{\lambda}Y)\ W^H(\sqrt{\lambda}Y) \tag{3.9}$$

with respect to the measure (dL^Y) on the eigenvalues $(y_1>y_2>\ldots>y_m)$ of Y.

Proof. We show that the characteristic function of the eigenvalue process coincide with the characteristic function expressed using the Hermite polynomials in (3.9). The characteristic function of the process is expressed as

$$G_1(t,X,W) = \int_{S_m} \mathrm{etr}\,(iWY)\,\mathbf{Q}(t,X,Y)(dY),$$

which leads to

$$G_1(t,X,W) = \mathrm{etr}\left(iWXe^{-\lambda t} - \frac{W^2}{2\lambda}\left(1-e^{-2\lambda t}\right)\right),$$

using the fact that $\int_{S_m} \mathrm{etr}\left(-\frac{Y^2}{2}\right)(dY) = 2^{\frac{m}{2}}\pi^{\frac{m(m+1)}{4}}$. By projection on the orthogonal group, the characteristic function of the eigenvalue process thus reads

$$g_1(t,X,W) = \mathrm{etr}\left(-\frac{W^2}{2\lambda}\left(1-e^{-2\lambda t}\right)\right)\ {}_0\mathbf{F}_0^{(m)}\left(iW, Xe^{-\lambda t}\right).$$

On the other hand, one could use equation (3.9) to write the characteristic function as

$$g_2(t,X,W) = \int_{S_m} \mathrm{etr}\,(iWY)\,\lambda^{\frac{m(m+1)}{4}} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{e^{-k\lambda t}}{\mathcal{N}_\kappa^{(H)}}\ H_\kappa(\sqrt{\lambda}X)\ H_\kappa(\sqrt{\lambda}Y)\ \mathrm{etr}\left(-\frac{\lambda Y^2}{2}\right)(dY).$$

The integral over the Hermite polynomial is easily evaluated using the inverse Fourier transform formula (2.17), which yields

$$g_2(t,X,W) = \sum_{k=0}^{\infty}\sum_{\kappa} \frac{e^{-k\lambda t}}{k!\ C_\kappa(I)}\ H_\kappa(\sqrt{\lambda}X)\ C_\kappa\left(\frac{iW}{\sqrt{\lambda}}\right)\ \mathrm{etr}\left(-\frac{W^2}{2\lambda}\right).$$

By definition of the generating function in (2.10), the expression reduces to

$$g_2(t,X,W) = \mathrm{etr}\left(\frac{W^2}{2\lambda}\ e^{-2\lambda t}\right)\ {}_0\mathbf{F}_0^{(m)}\left(iW; Xe^{-\lambda t}\right)\ \mathrm{etr}\left(-\frac{W^2}{2\lambda}\right).$$

which matches $g_1(t,X,W)$ and thus concludes the proof. □

The proof of the theorem contains a result, which extends to matrices a summation formula first derived by Baker and Forrester in [2] as Proposition 3.9. Indeed, by comparing the semigroup densities of the eigenvalue processes in (3.9) and (3.8), we have

Corollary 3.3.

$$\sum_{k=0}^{\infty}\sum_{\kappa} \frac{e^{-k\lambda t}}{k!C_\kappa(I)}\ H_\kappa(\sqrt{\lambda}X)H_\kappa(\sqrt{\lambda}Y)$$

$$= \left(\frac{e^{2\lambda t}}{e^{2\lambda t}-1}\right)^{\frac{m(m+1)}{4}}\ \mathrm{etr}\left(-\frac{\lambda(X^2+Y^2)}{2(e^{2\lambda t}-1)}\right){}_0\mathbf{F}_0^{(m)}\left(\frac{\lambda e^{\lambda t}\ X}{e^{2\lambda t}-1}; Y\right). \tag{3.10}$$

Note that the Hermite polynomials in [2] are defined with a different normalization as $2^{\frac{k}{2}}H_\kappa(\sqrt{2}X)$.

3.2 The generalized OUSQ process and Laguerre polynomials

Definition 3.4. Let $(X_t, t \geq 0)$ be the matrix-valued process solution to the stochastic differential equation

$$dX_t = \sqrt{X_t} d\beta_t + d\beta_t' \sqrt{X_t} + (\delta I - 2\lambda X_t)dt, \quad X_0 = X, \tag{3.11}$$

for a Brownian matrix $(\beta_t, t \geq 0)$. X_t is called the generalized OUSQ process.

The semigroup densities of a Wishart process, given by

$$\mathbf{P}(\tau, X, Y) = \frac{1}{(2\tau)^{\frac{\delta m}{2}} \Gamma_m(\frac{\delta}{2})} \operatorname{etr}\left(-\frac{1}{2\tau}(X+Y)\right) (\det Y)^{\frac{\delta-m-1}{2}} \; {}_0\mathbf{F}_1\left(\frac{\delta}{2}; \frac{XY}{4\tau^2}\right), \tag{3.12}$$

relate to the semigroup densities of the generalized OUSQ process as follows (see Bru [5]):

$$\mathbf{Q}(t, X, Y) = e^{\lambda m(m+1)t} \mathbf{P}\left(\frac{e^{2\lambda t}-1}{2\lambda}, X, Ye^{2\lambda t}\right), \tag{3.13}$$

which yields

$$\begin{aligned} \mathbf{Q}(t, X, Y) &= \frac{1}{\Gamma_m(\frac{\delta}{2})} \left(\frac{\lambda\, e^{2\lambda t}}{e^{2\lambda t}-1}\right)^{\frac{\delta m}{2}} \operatorname{etr}\left(-\frac{\lambda(X+Ye^{2\lambda t})}{e^{2\lambda t}-1}\right) (\det Y)^{\frac{\delta-m-1}{2}} \\ &\quad \cdot {}_0\mathbf{F}_1\left(\frac{\delta}{2}; \frac{XY\lambda^2 e^{2\lambda t}}{(e^{2\lambda t}-1)^2}\right). \end{aligned} \tag{3.14}$$

As for the OU process, the semigroup densities of the eigenvalue processes are derived by integration over the orthogonal group O_m, such that

$$\begin{aligned} \mathbf{q}(t, X, Y) &= \frac{1}{\Gamma_m(\frac{\delta}{2})} \left(\frac{\lambda\, e^{2\lambda t}}{e^{2\lambda t}-1}\right)^{\frac{\delta m}{2}} \operatorname{etr}\left(-\frac{\lambda(X+Ye^{2\lambda t})}{e^{2\lambda t}-1}\right) (\det Y)^{\frac{\delta-m-1}{2}} \\ &\quad \cdot {}_0\mathbf{F}_1^{(m)}\left(\frac{\delta}{2}; X, \frac{Y\lambda^2 e^{2\lambda t}}{(e^{2\lambda t}-1)^2}\right) \end{aligned} \tag{3.15}$$

with respect to the measure (dL^Y) on the eigenvalues $(y_1 > y_2 > \ldots > y_m)$ of Y. Alternatively, the latter densities can be expressed using the Laguerre polynomials as shown in the following proposition.

Proposition 3.5. *Let $(X_t, t \geq 0)$ be the generalized OUSQ process. The joint semigroup densities of the eigenvalue processes can be expressed as*

$$\mathbf{q}(t, X, Y) = \lambda^{-\frac{m(m+1)}{2}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{e^{-2\lambda kt}}{\mathcal{N}_\kappa^{(L)}} L_\kappa^\gamma(\lambda X)\; L_\kappa^\gamma(\lambda Y)\; W^L(\lambda Y) \tag{3.16}$$

for $\gamma = \frac{\delta-m-1}{2}$ and with respect to the measure (dL^Y) on the eigenvalues $(y_1 > y_2 > \ldots > y_m)$ of Y.

Proof. For the proof, we show that the Laplace transforms of (3.15) and (3.16) coincide and thus conclude by unicity. For convenience, we set $\tau = \frac{e^{2\lambda t}-1}{2\lambda}$. For all $W \in \S_m^+$, the Laplace transform of the semigroup densities of the process is defined as

$$G_1(t, X, W) = \int_{\S_m^+} \operatorname{etr}(-YW)\; \mathbf{Q}(t, X, Y)\; (dY)$$

which reads

$$\begin{aligned} G_1(t,X,W) &= \frac{\operatorname{etr}\left(-\frac{X}{2\tau}\right)e^{\lambda\delta m t}}{(2\tau)^{\frac{\delta m}{2}}\Gamma_m(\frac{\delta}{2})} \\ &\quad \cdot \int_{\S_m^+} \operatorname{etr}\left(-YW - \frac{Ye^{2\lambda t}}{2\tau}\right)\ (\det(Y))^{\frac{\delta-m-1}{2}}\ {}_0\mathbf{F}_1\left(\frac{\delta}{2};\frac{XYe^{2\lambda t}}{4\tau^2}\right)\ (dY) \end{aligned}$$

With $Z = 4\tau^2 e^{-2\lambda t}X^{-1/2}(W + \frac{e^{2\lambda t}}{2\tau}I)X^{-1/2}$ and the change of variables $\bar{Y} = \frac{e^{2\lambda t}}{4\tau^2}\ X^{1/2}YX^{1/2}$, we get

$$\begin{aligned} G_1(t,X,W) &= \frac{\operatorname{etr}\left(-\frac{X}{2\tau}\right)e^{\lambda\delta m t}}{(2\tau)^{\frac{\delta m}{2}}\Gamma_m(\frac{\delta}{2})}\left(\det\left(X\frac{e^{2\lambda t}}{4\tau^2}\right)\right)^{-\frac{\delta}{2}} \\ &\quad \cdot \int_{\S_m^+} \operatorname{etr}\left(-\bar{Y}Z\right)\left(\det(\bar{Y})\right)^{\frac{\delta-m-1}{2}}\ {}_0\mathbf{F}_1\left(\frac{\delta}{2};\bar{Y}\right)(d\bar{Y}). \end{aligned}$$

From Theorem 7.3.4 in Muirhead [22], we obtain

$$G_1(t,X,W) = \frac{\operatorname{etr}\left(-\frac{X}{2\tau}\right)e^{\lambda\delta m t}}{(2\tau)^{\frac{\delta m}{2}}}\ \left(\det\left(X\frac{e^{2\lambda t}}{4\tau^2}\right)\right)^{-\frac{\delta}{2}}(\det(Z))^{-\frac{\delta}{2}}\,{}_0\mathbf{F}_0\left(Z^{-1}\right),$$

which is equivalent to

$$G_1(t,X,W) = \left(\det\left(I + \frac{1-e^{-2\lambda t}}{\lambda}W\right)\right)^{-\frac{\delta}{2}}\ \operatorname{etr}\left(-XWe^{-2\lambda t}\left(I + \frac{1-e^{-2\lambda t}}{\lambda}W\right)^{-1}\right).$$

On the other hand, the Laplace transform of (3.16), defined by

$$g_2(t,X,W) = \int_{\S_m^+} \operatorname{etr}(-YW)\ \mathbf{q}(t,X,Y)\ (dY),$$

gives

$$g_2(t,X,W) = \lambda^{-\frac{m(m+1)}{2}}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{e^{-2\lambda k t}}{\mathcal{N}_\kappa^{(L)}}\ L_\kappa^\gamma(\lambda X)\ \int_{\S_m^+}\operatorname{etr}(-YW)\ L_\kappa^\gamma(\lambda Y)\ W^L(\lambda Y)(dY).$$

With the change of variables $\bar{Y} = \lambda Y$, we get

$$g_2(t,X,W) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{e^{-2\lambda k t}}{\mathcal{N}_\kappa^{(L)}}L_\kappa^\gamma(\lambda X)\int_{\S_m^+}\operatorname{etr}\left(-\bar{Y}\left(\frac{W}{\lambda}+I\right)\right)L_\kappa^\gamma(\bar{Y})\left(\det(\bar{Y})\right)^\gamma(d\bar{Y}).$$

which yields by Theorems 7.6.2 in Muirhead [22],

$$g_2(t,X,W) = \left(\det\left(I+\frac{W}{\lambda}\right)\right)^{-\gamma-\frac{m+1}{2}}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{e^{-2\lambda k t}}{k!C_\kappa(I)}\ L_\kappa^\gamma(\lambda X)\ C_\kappa\left(W(\lambda I+W)^{-1}\right).$$

From the generating function of the Laguerre polynomials given by (2.18), we obtain

$$g_2(t,X,W) = \left(\det\left(I + \frac{1-e^{-2\lambda t}}{\lambda}W\right)\right)^{-\frac{\delta}{2}}\ {}_0\mathbf{F}_0^{(m)}\left(-X, We^{-2\lambda t}\left(I + \frac{1-e^{-2\lambda t}}{\lambda}W\right)^{-1}\right).$$

Now, by the property of the hypergeometric functions given in (2.8), we have the following relation between the two Laplace transforms:

$$\int_{O_m} G_1(t, HXH', W)(dH) = g_2(t, X, W),$$

which concludes the proof by unicity of the Laplace transform. □

As a corollary, we state the following result, which is equivalent to Proposition 4.12 in [2].

Corollary 3.6.

$$\sum_{k=0}^{\infty}\sum_{\kappa} \frac{e^{-2\lambda k t}}{\mathcal{N}_\kappa^{(L)}}\ L_\kappa^\gamma(\lambda X) L_\kappa^\gamma(\lambda Y)$$

$$= \frac{\lambda^{m(m+1)}}{\Gamma_m(\frac{\delta}{2})}\left(\frac{e^{2\lambda t}}{e^{2\lambda t}-1}\right)^{\frac{\delta m}{2}} \operatorname{etr}\left(-\frac{\lambda\,(X+Y)}{e^{2\lambda t}-1}\right) {}_0\mathbf{F}_1^{(m)}\left(\frac{\delta}{2}; \frac{\lambda X e^{\lambda t}}{e^{2\lambda t}-1}, \frac{\lambda Y e^{\lambda t}}{e^{2\lambda t}-1}\right). \tag{3.17}$$

Proof. The proof is immediate by comparing (3.15) and (3.16). □

4 Martingale polynomials

By analogy to the work by Schoutens [23], we define the extension of the Lévy-Meixner systems to systems of orthogonal polynomials over symmetric random matrices.

Definition 4.1. A multivariate symmetric Lévy-Meixner system is a system of orthogonal polynomials $\{P_\kappa(X,t)\}$ defined by two analytic $m \times m$ symmetric matrix-valued functions U with $U(0) = 0$ and F such that

$$\sum_{k=0}^{\infty}\sum_{\kappa} P_\kappa(X,t)\ \frac{C_\kappa(Z)}{k!\ C_\kappa(I)} = F(U(Z))^t \int_{O_m} \operatorname{etr}\big(HXH'U(Z)\big) dH \tag{4.1}$$

for X, Z $m \times m$ symmetric matrices and $t > 0$.

4.1 The Hermite-Gaussian system

The extension of (2.10) to a Lévy-Meixner system yields:

$$\sum_{k=0}^{\infty}\sum_{\kappa} H_\kappa(X,t)\ \frac{C_\kappa(Z)}{k!\ C_\kappa(I)} = \operatorname{etr}\left(-\frac{Z^2}{2}t\right) \int_{O_m} \operatorname{etr}\,(HXH'Z)\, dH. \tag{4.2}$$

Proposition 4.2. *For $X \in S_m$, let $\{H_\kappa(X,t)\}$ be the Hermite space-time polynomials defined by*

$$H_\kappa(X,t) = t^{k/2}\ H_\kappa\left(\frac{X}{\sqrt{t}}\right). \tag{4.3}$$

Then for a symmetric $m \times m$ Brownian matrix $(X_t, t \geq 0)$, the following martingale equality holds $(s < t)$:

$$\mathrm{E}\Big[H_\kappa(X_t,t)\ \Big|\ X_s\Big] = H_\kappa(X_s, s). \tag{4.4}$$

Proof. From the characteristic function of a GOE ensemble (see [22]), we deduce

$$\mathrm{E}\Big[\mathrm{etr}\big(H(X_t - X_s)H'Z\big) \;\Big|\; X_s\Big] = \mathrm{etr}\left(\frac{Z^2}{2}(t-s)\right),$$

for $H \in O_m,\ Z \in S_m$ and $s < t$. Taking the conditional expectation of (4.2), we get

$$\sum_{k=0}^{\infty}\sum_{\kappa} \mathrm{E}\Big[H_\kappa(X_t,t) \;\Big|\; X_s\Big]\ \frac{C_\kappa(Z)}{k!\ C_\kappa(I)} = \mathrm{etr}\left(-\frac{Z^2}{2}t\right)\ \int_{O_m} \mathrm{E}\Big[\mathrm{etr}\left(HX_tH'Z\right) \;\Big|\; X_s\Big](dH).$$

Since the process X_t has independent increments, we obtain

$$\sum_{k=0}^{\infty}\sum_{\kappa} \mathrm{E}\Big[H_\kappa(X_t,t) \;\Big|\; X_s\Big]\ \frac{C_\kappa(Z)}{k!\ C_\kappa(I)} = \mathrm{etr}\left(-\frac{Z^2}{2}s\right)\ \int_{O_m} \mathrm{etr}\left(HX_sH'Z\right)(dH).$$

The RHS is thus the generating function of the Hermite polynomials as described by (4.2), so that

$$\sum_{k=0}^{\infty}\sum_{\kappa} \mathrm{E}\Big[H_\kappa(X_t,t) \;\Big|\; X_s\Big]\ \frac{C_\kappa(Z)}{k!\ C_\kappa(I)} = \sum_{k=0}^{\infty}\sum_{\kappa} H_\kappa\left(\frac{X_s}{\sqrt{s}}\right)\ \frac{C_\kappa(\sqrt{s}Z)}{k!\ C_\kappa(I)}.$$

Equating the terms of the sum yields

$$\mathrm{E}\Big[H_\kappa(X_t,t) \;\Big|\; X_s\Big] = s^{k/2}\ H_\kappa\left(\frac{X_s}{\sqrt{s}}\right)$$

and hence (4.3) follows. □

4.2 The Laguerre-Wishart system

Let $\mathbf{Q}_x^\delta$ be the law of a Wishart distribution $W_m(\delta, x)$ on $C(\mathbb{R}_+, S_m^+)$. If $W_m(\delta', y)$ is another Wishart distribution, then it is well-known (see [22]) that $W_m(\delta+\delta', x+y)$ is also a Wishart distribution. The laws $\mathbf{Q}_x^\delta$ are however not infinitely divisible. The variable δ must indeed belong to $T_m = \{1, 2, \ldots, m-1\} \cup (m-1, \infty)$, the so-called Gindikin's ensemble, and not $\mathbb{R}_+$, as was showed by Lévy in [19]. This remark restricts the matrix-valued extension of the Gamma process as follows:

Definition 4.3. The matrix-valued Gamma process is defined as $(X_t, t \in T_m)$ with $X_t \sim W_m(t, \frac{1}{2}I)$.

The one-dimensional Gamma process is recovered for $m = 1$, since $T_1 = \mathbb{R}_+$ and the Wishart distribution reduces to the Gamma distribution. It follows that (2.18) extends to

$$\sum_{k=0}^{\infty}\sum_{\kappa}\ L_\kappa(X,t)\ \frac{C_\kappa(Z)}{k!\ C_\kappa(I)} = (\det(I-Z))^{\frac{t}{2}}\ \int_{O_m} \mathrm{etr}\left(-HXH'Z(I-Z)^{-1}\right)(dH), \qquad (4.5)$$

which leads to the following proposition.

Proposition 4.4. *For $X \in S_m$, let $\{L_\kappa(X,t)\}$ be the Laguerre space-time polynomials defined by*

$$L_\kappa(X,t) = L_\kappa^{\left(\frac{t-m+1}{2}\right)}(X)\,. \qquad (4.6)$$

Then for a matrix-valued Gamma process $(X_t, t \in T_m)$, $X_t \sim W_m(t, \frac{1}{2}I)$, *the following martingale equality holds* $(s < t \in T_m)$:

$$\mathrm{E}\Big[L_\kappa(X_t, t) \;\Big|\; X_s\Big] = L_\kappa(X_s, s). \tag{4.7}$$

Proof. The additivity property of the Wishart distribution implies independence of the increments with $X_t - X_s \sim W_m(t-s, \frac{1}{2}I)$ (see [5]). From the characteristic function of a Wishart distribution (see [22]), we deduce

$$\mathrm{E}\Big[\mathrm{etr}\big(-H(X_t - X_s)H'Z(I-Z)^{-1}\big) \;\Big|\; X_s\Big] = (\det(I-Z))^{\frac{t-s}{2}},$$

for $H \in O_m,\ Z \in S_m$ and $s < t \in T_m$. Taking the conditional expectation of (2.18), we get

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{\kappa}\mathrm{E}\Big[L_\kappa(X_t,t)\;\Big|\;X_s\Big]\,\frac{C_\kappa(Z)}{k!\;C_\kappa(I)}\\ =\;\;&(\det(I-Z))^{-\frac{t}{2}}\int_{O_m}\mathrm{E}\Big[\mathrm{etr}\left(-HX_tH'Z(I-Z)^{-1}\right)\;\Big|\;X_s\Big](dH).\end{aligned}$$

Since the process X_t has independent increments, we obtain

$$\sum_{k=0}^{\infty}\sum_{\kappa}\mathrm{E}\Big[L_\kappa(X_t,t)\;\Big|\;X_s\Big]\,\frac{C_\kappa(Z)}{k!\;C_\kappa(I)} = (\det(I-Z))^{-\frac{s}{2}}\int_{O_m}\mathrm{etr}\left(-HX_sH'Z(I-Z)^{-1}\right)(dH).$$

The RHS is thus the generating function of the Laguerre polynomials as described by (2.18), so that

$$\sum_{k=0}^{\infty}\sum_{\kappa}\mathrm{E}\Big[L_\kappa(X_t,t)\;\Big|\;X_s\Big]\,\frac{C_\kappa(Z)}{k!\;C_\kappa(I)} = \sum_{k=0}^{\infty}\sum_{\kappa}L_\kappa^{(\frac{s-m+1}{2})}(X_s)\,\frac{C_\kappa(Z)}{k!\;C_\kappa(I)}.$$

Equating the terms of the sum yields

$$\mathrm{E}\Big[L_\kappa(X_t,t)\;\Big|\;X_s\Big] = L_\kappa^{(\frac{s-m+1}{2})}(X_s)$$

and hence (4.6) follows. □

5 A chaos representation property

Let $W = (W_t, t \geq 0)$ be a one-dimensional Brownian motion. The Wiener chaos $K_n(W)$ of order $n \geq 1$ is defined as the subspace of $L^2\left(\mathcal{F}_\infty^W\right)$ generated by the stochastic integrals

$$\int_0^\infty dW_{t_1}\int_0^{t_1}dW_{t_2}\ldots\int_0^{t_{n-1}}dW_{t_n}\;f_n(t_1,\ldots,t_n), \tag{5.1}$$

with $f_n : \Delta_n \to \mathbb{R}$ a measurable function such that

$$\int_{\Delta_n} dt_1\ldots dt_n f_n^2(t_1,\ldots,t_n) < \infty \tag{5.2}$$

and $\Delta_n = \{(t_1, \ldots, t_n); 0 < t_n < t_{n-1} < \ldots < t_1\}$. The L^2 space of the Brownian motion has then the direct sum decomposition,

$$L^2\left(\mathcal{F}_\infty^W\right) = \bigoplus_{n=0}^{\infty} K_n(W), \tag{5.3}$$

where $K_0(W)$ is the subspace of constants. It is also classicaly known that the one-dimensional Hermite space-time polynomials,

$$H_n(W_t, t) = t^{n/2} H_n\left(\frac{W_t}{\sqrt{t}}\right),\ n = 0, 1, 2, \ldots \tag{5.4}$$

form a basis of $L^2\left(\mathcal{F}_\infty^W\right)$. In order words, we have that

$$\begin{aligned} K_n(W) &= \operatorname{span}\left\{H_n\left(\int_{\Delta_n} f_n(t) dW_t,\ ||f_n||_{L^2(\Delta_N)}\right) \ \Big|\ f_n \in L^2(\Delta_n)\right\},\ n \geq 1 \\ K_0(W) &= \mathbb{R}. \end{aligned} \tag{5.5}$$

The result extends to the eigenvalue processes of the Brownian matrix. We first prove key properties of the Hermite space-time polynomials of matrix arguments:

Proposition 5.1. *The Hermite space-time polynomials are space-time harmonic, i.e.*

$$\left(\frac{\partial}{\partial t} + \mathcal{L}\right) H_\kappa(X, t) = 0, \tag{5.6}$$

for $\mathcal{L}$ the infinitesimal generator of the eigenvalue process of the Brownian matrix, given by

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \frac{1}{x_i - x_j}\frac{\partial}{\partial x_i}. \tag{5.7}$$

Proof. It suffices to show that the generating function of the polynomials, given by (4.2), or equivalently by

$$\sum_{k=0}^{\infty}\sum_{\kappa} H_\kappa(X, t)\ \frac{C_\kappa(Z)}{k!\ C_\kappa(I)} = \operatorname{etr}\left(-\frac{Z^2}{2}t\right)\ {}_0\mathbf{F}_0^{(m)}(X, Z), \tag{5.8}$$

is space-time harmonic. From Lemma 3.1 in [2], we immediately have

$$\mathcal{L}\ {}_0\mathbf{F}_0^{(m)}(X, Z)\ \operatorname{etr}\left(-\frac{Z^2}{2}t\right) = \frac{1}{2}\operatorname{Tr}(Z^2)\ {}_0\mathbf{F}_0^{(m)}(X, Z)\ \operatorname{etr}\left(-\frac{Z^2}{2}t\right), \tag{5.9}$$

which is clearly canceled by the time derivative,

$$\frac{\partial}{\partial t}\ \operatorname{etr}\left(-\frac{Z^2}{2}t\right)\ {}_0\mathbf{F}_0^{(m)}(X, Z) = \operatorname{Tr}\left(-\frac{Z^2}{2}\right)\operatorname{etr}\left(-\frac{Z^2}{2}t\right)\ {}_0\mathbf{F}_0^{(m)}(X, Z). \tag{5.10}$$

□

Proposition 5.2. *For $s \leq t$, we have*

$$\mathrm{E}\big[C_\kappa(X_t)\ |\ \mathcal{F}_s\big] = H_\kappa(X_s, t - s). \tag{5.11}$$

Proof. From the explicit knowledge of the semigroup density of the Brownian matrix, we have

$$\mathrm{E}\big[C_\kappa(X_t) \mid \mathcal{F}_s\big] = \int_{S_m} C_\kappa(X) \, \frac{(t-s)^{-\frac{m(m+1)}{4}}}{2^{\frac{m}{2}}\pi^{\frac{m(m+1)}{4}}} \, \mathrm{etr}\left(-\frac{(X-X_s)^2}{2(t-s)}\right) \, dX. \tag{5.12}$$

The change of variable $Y = \dfrac{X}{\sqrt{t-s}}$ yields

$$\mathrm{E}\big[C_\kappa(X_t) \mid \mathcal{F}_s\big] = \frac{(t-s)^{\frac{k}{2}}}{2^{\frac{m}{2}}\pi^{\frac{m(m+1)}{4}}} \int_{S_m} C_\kappa(Y) \, \mathrm{etr}\left(-\frac{1}{2}\left(Y - \frac{X_s}{\sqrt{t-s}}\right)^2\right) \, dY. \tag{5.13}$$

Using the Rodrigues formula for the Hermite polynomials (see (3.22) in [6]), which we recall here,

$$H_\kappa(Z) = \frac{1}{2^{\frac{m}{2}}\pi^{\frac{m(m+1)}{4}}} \int_{S_m} C_\kappa(Y) \, \mathrm{etr}\left(-\frac{1}{2}(Y-Z)^2\right) \, dY, \tag{5.14}$$

concludes the proof. □

Proposition 5.3. *For $u \leq s \leq t$, we have*

$$H_\kappa(X_s, t-s) = H_\kappa(X_u, t-u) + \sum_{i=1}^{m} \int_u^s H_\kappa^{(i)}(X_r, t-r) dW_r^{(i)}, \tag{5.15}$$

where $(W_t^{(i)}, t \geq 0)$ are m independent one-dimensional Brownian motions and

$$H_\kappa^{(i)}(X,t) = \frac{\partial}{\partial x_i} H_\kappa(X,t) \tag{5.16}$$

with x_i an eigenvalue of X.

Proof. The Itô formula gives

$$\begin{aligned} H_\kappa(X_t, t) \quad &= \quad H_\kappa(X_0, 0) + \int_0^t \frac{\partial}{\partial s} H_\kappa(X_s, s) ds + \sum_{i=1}^{m} \int_0^t H_\kappa^{(i)}(X_s, s) dX_s^{(i)} \\ &\quad + \frac{1}{2} \sum_{i,j} \int_0^t \frac{\partial^2}{\partial x_i x_j^2} H_\kappa(X_s, s) d\langle X^{(i)}, X^{(j)}\rangle_s. \end{aligned} \tag{5.17}$$

The single eigenvalue process satisfies

$$dX_t^{(i)} = dW_t^{(i)} + \sum_{j \neq i} \frac{1}{X_t^{(i)} - X_t^{(j)}} dt. \tag{5.18}$$

Using the infinitesimal generator $\mathcal{L}$ of the joint eigenvalue process, the Itô formula is equivalent to

$$H_\kappa(X_t, t) = H_\kappa(X_0, 0) + \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right) H_\kappa(X_s, s) ds + \sum_{i=1}^{m} \int_0^t H_\kappa^{(i)}(X_s, s) dW_s^{(i)} \tag{5.19}$$

Space-time harmonicity of the Hermite polynomials yields the result. □

By analogy to the Wiener chaos $K_n(W)$, we define for the eigenvalue process X the chaos $\mathbf{K}_n(X)$ as the subspace of $L^2(\mathcal{F}_\infty^X)$ generated by the stochastic integrals

$$\int_0^\infty dX_{t_1}^{(i_1)} \int_0^{t_1} dX_{t_2}^{(i_2)} \dots \int_0^{t_{n-1}} dX_{t_n}^{(i_n)}\ f_n(t_1,\dots,t_n), \tag{5.20}$$

for $(i_1,\dots,i_n) \in \{1,\dots,m\}^n$ with $f_n : \Delta_n \to \mathbb{R}$ a measurable function such that

$$\int_{\Delta_n} dt_1 \dots dt_n f_n^2(t_1,\dots,t_n) < \infty \tag{5.21}$$

and $\Delta_n = \{(t_1,\dots,t_n); 0 < t_n < t_{n-1} < \dots < t_1\}$.

Theorem 5.4. *The L^2 space of the eigenvalue process has the direct sum decomposition,*

$$L^2\left(\mathcal{F}_\infty^X\right) = \bigoplus_{n=0}^\infty \mathbf{K}_n(X), \tag{5.22}$$

where $\mathbf{K}_0(W)$ is the subspace of constants.

Proof. For $p(x_1,\dots,x_l)$ a polynomial on $(\mathbb{R}^m)^l$, $l \in \mathbb{N}^*$, the subspace generated by the random variables $p(X_{t_1},\dots,X_{t_l}),\ 0 \le t_1 < \dots < t_l$, is dense in $L^2(\mathcal{F}_\infty^X)$. Since any polynomial of that form has a decomposition in a linear combination of zonal polynomials, the random variables

$$Z = \prod_{i=1}^l C_{\kappa_i}(X_{t_i}), \tag{5.23}$$

with $\kappa_1,\dots,\kappa_l \in \mathbb{N}^m$, form a total subset of $L^2(\mathcal{F}_\infty^X)$.
Now by Proposition 5.2, we have $C_{\kappa_l}(X_{t_l}) = H_{\kappa_l}(X_{t_l},0)$. Then, using Proposition 5.3 with $s = t = t_l$ and $u = t_{l-1}$, we get

$$C_{\kappa_l}(X_{t_l}) = H_{\kappa_l}(X_{t_l-1}, t_l - t_{l-1}) + \sum_{i=1}^m \int_{t_{l-1}}^{t_l} H_{\kappa_l}^{(i)}(X_r, t_l - r) dW_r^{(i)}. \tag{5.24}$$

The random variable Z can thus be expressed as the sum of a random variable

$$Z_1 = H_{\kappa_l}(X_{t_l-1}, t_l - t_{l-1}) \prod_{j=1}^{l-1} C_{\kappa_j}(X_{t_j})$$

and a sum of stochastic integrals with respect to $dW_r^{(i)}$, $(i = 1,\dots,m$ and $t_{l-1} \le r \le t_l)$ with integrands

$$Z_2^{(i)}(r) = H_{\kappa_l}^{(i)}(X_r, t_l - r) \prod_{j=1}^{l-1} C_{\kappa_j}(X_{t_j}).$$

The chaos decomposition then follows by a decreasing induction on l. □

Remark 5.5. Equation (5.18) implies that the multidimensional Brownian motion is adapted to the filtration of the eigenvalue process. Theorem 5.4 gives the converse, which implies that the eigenvalue process has the same filtration as the m-dimensional Brownian motion.

6 Conclusion

Hypergeometric orthogonal polynomials have proven increasingly useful in describing important properties of stochastic processes. Polynomials classified in the Askey scheme for instance provide a way to construct semigroup densities of some Markov processes in closed form. Some of these polynomials moreover create martingale processes when combined with specific Lévy processes.
Far from generalizing these two properties to all matrix-valued polynomials, we specialize to the Hermite and Laguerre polynomials. We show their relation with joint semigroup densities of the eigenvalue processes of the generalized matrix-valued Ornstein-Uhlenbeck and squared Ornstein-Uhlenbeck processes. We also extend Lévy-Meixner systems of orthogonal polynomials to symmetric random matrices. The results are again restrictive to Hermite and Laguerre polynomials, but could lead the way to a general theory of hypergeometric polynomials and matrix-valued processes.
As an application of these results, we derive a chaos representation property for the eigenvalue process of the Brownian matrix. This merely shows that the eigenvalue process enjoys the chaos representation property, but further research could uncover a wider class of matrix-valued processes with the same representation property.

Acknowledgements

The author wishes to thank Marc Yor for many helpful discussions.

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