

Research Article

Derivation of Identities Involving Bernoulli and Euler Numbers

Imju Lee and Dae San Kim

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea

Correspondence should be addressed to Dae San Kim, dskim@sogang.ac.kr

Received 14 June 2012; Accepted 3 August 2012

Academic Editor: Cheon Ryoo

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We derive some new and interesting identities involving Bernoulli and Euler numbers by using some polynomial identities and p -adic integrals on \mathbb{Z}_p .

1. Introduction and Preliminaries

Let p be a fixed odd prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will, respectively, denote the ring of p -adic integers, the field of p -adic rational numbers, and the completion of algebraic closure of \mathbb{Q}_p . The p -adic absolute value $|\cdot|_p$ on \mathbb{C}_p is normalized so that $|p|_p = 1/p$. Let $\mathbb{Z}_{>0}$ be the set of natural numbers and $\mathbb{Z}_{\geq 0} = \mathbb{Z}_{>0} \cup \{0\}$.

As is well known, the Bernoulli polynomials $B_n(x)$ are defined by the generating function as follows:

$$F(t, x) = \frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (1.1)$$

with the usual convention of replacing $B(x)^n$ by $B_n(x)$.

In the special case, $x = 0$, $B_n(0) = B_n$ is referred to as the n th Bernoulli number. That is, the generating function of Bernoulli numbers is given by

$$F(t) = F(t, 0) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = e^{Bt}, \quad (1.2)$$

with the usual convention of replacing B^n by B_n , (cf. [1–23]).

From (1.2), we see that the recurrence formula for the Bernoulli numbers is

$$(B + 1)^n - B_n = \delta_{1,n}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}, \quad (1.3)$$

where $\delta_{k,n}$ is the Kronecker symbol.

By (1.1) and (1.2), we easily get the following:

$$B_n(x) = (B + x)^n = \sum_{l=0}^n \binom{n}{l} B_l x^{n-l} = \sum_{l=0}^n \binom{n}{l} B_{n-l} x^l, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.4)$$

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable \mathbb{C}_p -valued functions on \mathbb{Z}_p . For $f \in UD(\mathbb{Z}_p)$, the bosonic p -adic integral on \mathbb{Z}_p is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (1.5)$$

(cf. [12]). Then it is easy to see that

$$I(f_1) = I(f) + f'(0), \quad (1.6)$$

where $f_1(x) = f(x + 1)$ and $f'(0) = df(x)/dx|_{x=0}$.

By (1.6), we have the following:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (1.7)$$

(cf. [12–14]). From (1.7), we can derive the Witt's formula for the n th Bernoulli polynomial as follows:

$$\int_{\mathbb{Z}_p} (x + y)^n d\mu(y) = B_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.8)$$

By (1.1), we have the following:

$$B_n(1 - x) = (-1)^n B_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.9)$$

Thus, from (1.3), (1.4), and (1.9), we have the following:

$$B_n(1) = B_n + \delta_{1,n} = (-1)^n B_n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.10)$$

By (1.4), we have the following:

$$B_n(x + y) = \sum_{k=0}^n \binom{n}{k} B_k(x) y^{n-k}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.11)$$

Especially, for $x = 1$ and $y = 1$,

$$B_n(2) = \sum_{k=0}^n \binom{n}{k} B_k(1) = \sum_{k=0}^n \binom{n}{k} (B_k + \delta_{1,k}), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.12)$$

Therefore, from (1.9), (1.10), and (1.12), we can derive the following relation. For $n \in \mathbb{Z}_{\geq 0}$,

$$(-1)^n B_n(-1) = B_n(2) = n + B_n(1) = n + B_n + \delta_{1,n} = n + (-1)^n B_n. \quad (1.13)$$

Let $f(y) = (x + y)^{n+1}$. By (1.6), we have the following:

$$\int_{\mathbb{Z}_p} (x + y + 1)^{n+1} d\mu(y) - \int_{\mathbb{Z}_p} (x + y)^{n+1} d\mu(y) = (n + 1)x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.14)$$

By (1.8) and (1.14), we have the following:

$$B_{n+1}(x + 1) - B_{n+1}(x) = (n + 1)x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.15)$$

Thus, by (1.11) and (1.15), we have the following identity.

$$x^n = \frac{1}{n + 1} \sum_{l=0}^n \binom{n + 1}{l} B_l(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.16)$$

As is well known, the Euler polynomials $E_n(x)$ are defined by the generating function as follows:

$$G(t, x) = \frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (1.17)$$

with the usual convention of replacing $E(x)^n$ by $E_n(x)$.

In the special case, $x = 0$, $E_n(0) = E_n$ is referred to as the n th Euler number. That is, the generating function of Euler numbers is given by

$$G(t) = G(t, 0) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = e^{Et}, \quad (1.18)$$

with the usual convention of replacing E^n by E_n , (cf. [1–23]).

From (1.18), we see that the recurrence formula for the Euler numbers is

$$(E + 1)^n + E_n = 2\delta_{0,n}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.19)$$

By (1.17) and (1.18), we easily get the following:

$$E_n(x) = (E + x)^n = \sum_{l=0}^n \binom{n}{l} E_l x^{n-l} = \sum_{l=0}^n \binom{n}{l} E_{n-l} x^l, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.20)$$

Let $C(\mathbb{Z}_p)$ be the space of continuous \mathbb{C}_p -valued functions on \mathbb{Z}_p . For $f \in C(\mathbb{Z}_p)$, the fermionic p -adic integral on \mathbb{Z}_p is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x, \quad (1.21)$$

(cf. [9]). Then it is easy to see that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad (1.22)$$

where $f_1(x) = f(x+1)$.

By (1.22), we have the following:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \quad (1.23)$$

From (1.23), we can derive the Witt's formula for the n -th Euler polynomial as follows:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.24)$$

By (1.17), we have the following:

$$E_n(1-x) = (-1)^n E_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.25)$$

Thus, from (1.19), (1.20), and (1.25), we have the following:

$$E_n(1) = -E_n + 2\delta_{0,n} = (-1)^n E_n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.26)$$

By (1.20), we have the following:

$$E_n(x+y) = \sum_{k=0}^n \binom{n}{k} E_k(x) y^{n-k}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.27)$$

Especially, for $x = 1$ and $y = 1$,

$$E_n(2) = \sum_{k=0}^n \binom{n}{k} E_k(1) = \sum_{k=0}^n \binom{n}{k} (-E_n + 2\delta_{0,k}), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.28)$$

Therefore, from (1.25), (1.26), and (1.28), we can derive the following relations. For $n \in \mathbb{Z}_{\geq 0}$,

$$(-1)^n E_n(-1) = E_n(2) = 2 - E_n(1) = 2 + E_n - 2\delta_{0,n} = 2 - (-1)^n E_n. \quad (1.29)$$

Let $f(y) = (x + y)^n$. By (1.22), we have the following:

$$\int_{\mathbb{Z}_p} (x + y + 1)^n d\mu_{-1}(y) + \int_{\mathbb{Z}_p} (x + y)^n d\mu_{-1}(y) = 2x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.30)$$

By (1.24) and (1.30), we have the following:

$$E_n(x + 1) + E_n(x) = 2x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.31)$$

Thus, by (1.27) and (1.31), we get the following identity.

$$x^n = \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l(x) + E_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (1.32)$$

The Bernstein polynomials are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } k, n \in \mathbb{Z}_{\geq 0}, \quad (1.33)$$

with $0 \leq k \leq n$ (cf. [14]).

By the definition of $B_{k,n}(x)$, we note that

$$B_{k,n}(x) = B_{n-k,n}(1-x). \quad (1.34)$$

In this paper, we derive some new and interesting identities involving Bernoulli and Euler numbers from well-known polynomial identities. Here, we note that our results are “complementary” to those in [6], in the sense that we take a fermionic p -adic integral where a bosonic p -adic integral is taken and vice versa, and we use the identity involving Euler polynomials in (1.32) where that involving Bernoulli polynomials in (1.16) is used and vice versa. Finally, we report that there have been a lot of research activities on this direction of research, namely, on derivation of identities involving Bernoulli and Euler numbers and polynomials by exploiting bosonic and fermionic p -adic integrals (cf. [6–8]).

2. Identities Involving Bernoulli Numbers

Taking the bosonic p -adic integral on both sides of (1.16), we have the following:

$$\begin{aligned}
 \int_{\mathbb{Z}_p} x^m d\mu(x) &= \int_{\mathbb{Z}_p} \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k(x) d\mu(x) \\
 &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} \int_{\mathbb{Z}_p} B_k(x) d\mu(x) \\
 &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} \sum_{j=0}^k \binom{k}{j} B_{k-j} \int_{\mathbb{Z}_p} x^j d\mu(x) \\
 &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} \sum_{j=0}^k \binom{k}{j} B_{k-j} B_j.
 \end{aligned} \tag{2.1}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$B_m = \frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^k \binom{m+1}{k} \binom{k}{j} B_{k-j} B_j. \tag{2.2}$$

Let us apply (1.9) to the bosonic p -adic integral of (1.16).

$$\begin{aligned}
 \int_{\mathbb{Z}_p} x^m d\mu(x) &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} \int_{\mathbb{Z}_p} B_k(x) d\mu(x) \\
 &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \int_{\mathbb{Z}_p} B_k(1-x) d\mu(x) \\
 &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} B_{k-j} \int_{\mathbb{Z}_p} (1-x)^j d\mu(x) \\
 &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} B_{k-j} (-1)^j B_j(-1).
 \end{aligned} \tag{2.3}$$

Then, we can express (2.3) in three different ways.

By (1.13), (2.3) can be written as

$$\begin{aligned}
 \int_{\mathbb{Z}_p} x^m d\mu(x) &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} B_{k-j} (j + B_j + \delta_{1,j}) \\
 &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \left(kB_{k-1}(1) + kB_{k-1} + \sum_{j=0}^k \binom{k}{j} B_{k-j} B_j \right) \\
 &= - \sum_{k=0}^{m-1} \binom{m}{k} (B_k + (-1)^k B_k) + \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} B_{k-j} B_j \\
 &= - \sum_{k=0}^{m-1} \binom{m}{k} (B_k + B_k + \delta_{1,k}) + \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} B_{k-j} B_j \\
 &= -2(B_m(1) - B_m) - (m - \delta_{1,m}) + \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} B_{k-j} B_j \\
 &= -\delta_{1,m} - m + \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} B_{k-j} B_j.
 \end{aligned} \tag{2.4}$$

Thus, we have the following theorem.

Theorem 2.2. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$B_m = -\delta_{1,m} - m + \frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^k \binom{m+1}{k} \binom{k}{j} (-1)^k B_{k-j} B_j. \tag{2.5}$$

Corollary 2.3. *Let m be an integer ≥ 2 . Then one has the following:*

$$B_m + m = \frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^k \binom{m+1}{k} \binom{k}{j} (-1)^k B_{k-j} B_j. \tag{2.6}$$

Especially, for an odd integer m with $m \geq 3$, we obtain the following corollary.

Corollary 2.4. *Let m be an odd integer with $m \geq 3$. Then one has the following:*

$$m(m+1) = \sum_{k=0}^m \sum_{j=0}^k \binom{m+1}{k} \binom{k}{j} (-1)^k B_{k-j} B_j. \tag{2.7}$$

By (1.13), (2.3) can be written as

$$\begin{aligned}
 \int_{\mathbb{Z}_p} x^m d\mu(x) &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} B_{k-j} (j + (-1)^j B_j) \\
 &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \left(kB_{k-1}(1) + \sum_{j=0}^k \binom{k}{j} (-1)^j B_{k-j} B_j \right) \\
 &= -\sum_{k=0}^{m-1} \binom{m}{k} B_k + \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} (-1)^j B_{k-j} B_j \\
 &= -B_m(1) + B_m + \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} (-1)^j B_{k-j} B_j.
 \end{aligned} \tag{2.8}$$

By (1.10), (2.8) can be written as

$$\int_{\mathbb{Z}_p} x^m d\mu(x) = (-1)^{m+1} B_m + B_m + \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} (-1)^j B_{k-j} B_j. \tag{2.9}$$

So, we get the following theorem.

Theorem 2.5. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$B_m = \frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^k \binom{m+1}{k} \binom{k}{j} (-1)^{m+k+j} B_{k-j} B_j. \tag{2.10}$$

By (1.10), (2.8) can also be written as

$$\int_{\mathbb{Z}_p} x^m d\mu(x) = -\delta_{1,m} + \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} (-1)^k \sum_{j=0}^k \binom{k}{j} (-1)^j B_{k-j} B_j. \tag{2.11}$$

Thus, we have the following theorem.

Theorem 2.6. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$B_m = -\delta_{1,m} + \frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^k \binom{m+1}{k} \binom{k}{j} (-1)^{k+j} B_{k-j} B_j. \tag{2.12}$$

3. Identities Involving Euler Numbers

Taking the fermionic p -adic integral on both sides of (1.32), we have the following:

$$\begin{aligned}
 \int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} \left(E_m(x) + \frac{1}{2} \sum_{k=0}^{m-1} \binom{m}{k} E_k(x) \right) d\mu_{-1}(x) \\
 &= \sum_{l=0}^m \binom{m}{l} E_{m-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) + \frac{1}{2} \sum_{k=0}^{m-1} \binom{m}{k} \sum_{j=0}^k \binom{k}{j} E_{k-j} \int_{\mathbb{Z}_p} x^j d\mu_{-1}(x) \quad (3.1) \\
 &= \sum_{l=0}^m \binom{m}{l} E_{m-l} E_l + \frac{1}{2} \sum_{k=0}^{m-1} \binom{m}{k} \sum_{j=0}^k \binom{k}{j} E_{k-j} E_j.
 \end{aligned}$$

So, we obtain the following theorem.

Theorem 3.1. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$E_m = \sum_{l=0}^m \binom{m}{l} E_{m-l} E_l + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} E_{k-j} E_j. \quad (3.2)$$

Let us apply (1.25) to the fermionic p -adic integral of (1.32).

$$\begin{aligned}
 \int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) &= (-1)^m \int_{\mathbb{Z}_p} E_m(1-x) d\mu_{-1}(x) + \frac{1}{2} \sum_{k=0}^{m-1} \binom{m}{k} (-1)^k \int_{\mathbb{Z}_p} E_k(1-x) d\mu_{-1}(x) \\
 &= (-1)^m \sum_{k=0}^m \binom{m}{k} E_{m-k} (-1)^k E_k(-1) \quad (3.3) \\
 &\quad + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^k E_{k-j} (-1)^j E_j(-1).
 \end{aligned}$$

Then, we can express (3.3) in two different ways.

By (1.29), (3.3) can be written as

$$\begin{aligned}
\int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) &= (-1)^m \sum_{k=0}^m \binom{m}{k} E_{m-k} (2 + E_k - 2\delta_{0,k}) \\
&\quad + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^k E_{k-j} (2 + E_j - 2\delta_{0,j}) \\
&= 2E_m + (-1)^m \sum_{k=0}^m \binom{m}{k} E_{m-k} E_k + 2(-1)^{m+1} E_m + \sum_{k=0}^{m-1} \binom{m}{k} E_k \\
&\quad + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^k E_{k-j} E_j + \sum_{k=0}^{m-1} \binom{m}{k} (-1)^{k+1} E_k \\
&= 2E_m + (-1)^m \sum_{k=0}^m \binom{m}{k} E_{m-k} E_k + 2(-1)^{m+1} E_m + E_m(1) - E_m \\
&\quad + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^k E_{k-j} E_j + (-1)^{m+1} (E_m(-1) - E_m) \\
&= -2 + 2\delta_{0,m} + (-1)^m \sum_{k=0}^m \binom{m}{k} E_{m-k} E_k + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^k E_{k-j} E_j.
\end{aligned} \tag{3.4}$$

Thus, we get the following theorem.

Theorem 3.2. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$E_m = -2 + 2\delta_{0,m} + (-1)^m \sum_{k=0}^m \binom{m}{k} E_{m-k} E_k + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^k E_{k-j} E_j. \tag{3.5}$$

Corollary 3.3. *Let $m \in \mathbb{Z}_{> 0}$. Then one has the following:*

$$E_m + 2 = (-1)^m \sum_{k=0}^m \binom{m}{k} E_{m-k} E_k + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^k E_{k-j} E_j. \tag{3.6}$$

By (1.29), (3.3) can be written as

$$\begin{aligned}
 \int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) &= (-1)^m \sum_{k=0}^m \binom{m}{k} E_{m-k} (2 - (-1)^k E_k) \\
 &\quad + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^k E_{k-j} (2 - (-1)^j E_j) \\
 &= 2E_m + (-1)^{m+1} \sum_{k=0}^m \binom{m}{k} (-1)^k E_{m-k} E_k \\
 &\quad + \sum_{k=0}^{m-1} \binom{m}{k} E_k - \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^k (-1)^j E_{k-j} E_j \\
 &= 2E_m + (-1)^{m+1} \sum_{k=0}^m \binom{m}{k} (-1)^k E_{m-k} E_k \\
 &\quad + E_m(1) - E_m - \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^k (-1)^j E_{k-j} E_j \\
 &= 2\delta_{0,m} + (-1)^{m+1} \sum_{k=0}^m \binom{m}{k} (-1)^k E_{m-k} E_k \\
 &\quad - \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^k (-1)^j E_{k-j} E_j.
 \end{aligned} \tag{3.7}$$

So, we have the following theorem.

Theorem 3.4. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$E_m = 2\delta_{0,m} + (-1)^{m+1} \sum_{k=0}^m \binom{m}{k} (-1)^k E_{m-k} E_k - \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^{k+j} E_{k-j} E_j. \tag{3.8}$$

Corollary 3.5. *Let $m \in \mathbb{Z}_{> 1}$. Then one has the following:*

$$E_m = (-1)^{m+1} \sum_{k=0}^m \binom{m}{k} (-1)^k E_{m-k} E_k - \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^k \binom{m}{k} \binom{k}{j} (-1)^{k+j} E_{k-j} E_j. \tag{3.9}$$

4. Identities Involving Bernoulli and Euler Numbers

By (1.16) and (1.32), we have the following:

$$\begin{aligned}
 \int_{\mathbb{Z}_p} x^{m+n} d\mu(x) &= \int_{\mathbb{Z}_p} \left(\frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k(x) \right) \left(E_n(x) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l(x) \right) d\mu(x) \\
 &= \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} \int_{\mathbb{Z}_p} B_k(x) E_n(x) d\mu(x) \\
 &\quad + \frac{1}{2(m+1)} \sum_{k=0}^m \binom{m+1}{k} \sum_{l=0}^{n-1} \binom{n}{l} \int_{\mathbb{Z}_p} B_k(x) E_l(x) d\mu(x) \\
 &= \frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^k \sum_{l=0}^n \binom{m+1}{k} \binom{k}{j} \binom{n}{l} B_{k-j} E_{n-l} B_{j+l} \\
 &\quad + \frac{1}{2(m+1)} \sum_{k=0}^m \sum_{l=0}^{n-1} \sum_{j=0}^k \sum_{i=0}^l \binom{m+1}{k} \binom{n}{l} \binom{k}{j} \binom{l}{i} B_{k-j} E_{l-i} B_{j+i}.
 \end{aligned} \tag{4.1}$$

Therefore, we get the following theorem.

Theorem 4.1. *Let $m, n \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$\begin{aligned}
 B_{m+n} &= \frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^k \sum_{l=0}^n \binom{m+1}{k} \binom{k}{j} \binom{n}{l} B_{k-j} E_{n-l} B_{j+l} \\
 &\quad + \frac{1}{2(m+1)} \sum_{k=0}^m \sum_{l=0}^{n-1} \sum_{j=0}^k \sum_{i=0}^l \binom{m+1}{k} \binom{n}{l} \binom{k}{j} \binom{l}{i} B_{k-j} E_{l-i} B_{j+i}.
 \end{aligned} \tag{4.2}$$

By (1.16) and (1.33), we have the following:

$$\begin{aligned}
 \int_{\mathbb{Z}_p} x^m B_{k,n}(x) d\mu(x) &= \int_{\mathbb{Z}_p} \frac{1}{m+1} \sum_{l=0}^m \binom{m+1}{l} B_l(x) B_{k,n}(x) d\mu(x) \\
 &= \frac{1}{m+1} \binom{n}{k} \sum_{l=0}^m \sum_{i=0}^l \binom{m+1}{l} \binom{l}{i} B_{l-i} \int_{\mathbb{Z}_p} x^{i+k} (1-x)^{n-k} d\mu(x) \\
 &= \frac{1}{m+1} \binom{n}{k} \sum_{l=0}^m \sum_{i=0}^l \sum_{j=0}^{n-k} \binom{m+1}{l} \binom{l}{i} \binom{n-k}{j} (-1)^j B_{l-i} \int_{\mathbb{Z}_p} x^{i+k+j} d\mu(x) \\
 &= \frac{1}{m+1} \binom{n}{k} \sum_{l=0}^m \sum_{i=0}^l \sum_{j=0}^{n-k} \binom{m+1}{l} \binom{l}{i} \binom{n-k}{j} (-1)^j B_{l-i} B_{i+k+j}.
 \end{aligned} \tag{4.3}$$

By (1.33), we have the following:

$$\begin{aligned} \int_{\mathbb{Z}_p} x^m B_{k,n}(x) d\mu(x) &= \binom{n}{k} \int_{\mathbb{Z}_p} x^{m+k} (1-x)^{n-k} d\mu(x) \\ &= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \int_{\mathbb{Z}_p} x^{m+k+j} d\mu(x) \\ &= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j B_{m+k+j}. \end{aligned} \tag{4.4}$$

By (4.3) and (4.4), we obtain the following theorem.

Theorem 4.2. *Let $m, n, k \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j B_{m+k+j} = \frac{1}{m+1} \sum_{l=0}^m \sum_{i=0}^l \sum_{j=0}^{n-k} \binom{m+1}{l} \binom{l}{i} \binom{n-k}{j} (-1)^j B_{l-i} B_{i+k+j}. \tag{4.5}$$

Especially, one has the following:

$$(m+1)B_{m+n} = \sum_{l=0}^m \sum_{i=0}^l \binom{m+1}{l} \binom{l}{i} B_{l-i} B_{i+n}. \tag{4.6}$$

By (4.2) and (4.6), we have the following theorem. Note that (4.8) in the following was obtained in [6].

Theorem 4.3. *Let $m, n \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$B_{m+n} = \sum_{l=0}^n \binom{n}{l} E_{n-l} B_{m+l} + \frac{1}{2} \sum_{l=0}^{n-1} \sum_{i=0}^l \binom{n}{l} \binom{l}{i} E_{l-i} B_{m+i}. \tag{4.7}$$

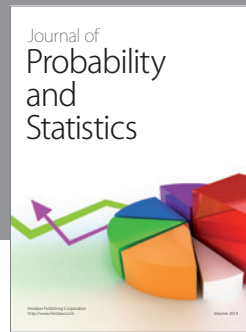
In particular, we have the following:

$$B_n = \sum_{l=0}^n \binom{n}{l} E_{n-1} B_l + \frac{1}{2} \sum_{l=0}^{n-1} \sum_{i=0}^l \binom{n}{l} \binom{l}{i} E_{l-i} B_i. \tag{4.8}$$

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