BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **34**(2) (2011), 279–286

On Chebyshev's Polynomials and Certain Combinatorial Identities

[1]CHAN-LYE LEE AND [2]K. B. WONG
Institute of Mathematical Sciences, University of Malaya
50603 Kuala Lumpur, Malaysia
[1]chanlye77@yahoo.com, [2]kbwong@um.edu.my

Abstract. Let $T_n(x)$ and $U_n(x)$ be the Chebyshev's polynomial of the first kind and second kind of degree n, respectively. For $n \geq 1$, $U_{2n-1}(x) = 2T_n(x)U_{n-1}(x)$ and $U_{2n}(x) = (-1)^n A_n(x)A_n(-x)$, where $A_n(x) = 2^n \prod_{i=1}^{n}(x - \cos i\theta)$, $\theta = 2\pi/(2n+1)$. In this paper, we will study the polynomial $A_n(x)$. Let $A_n(x) = \sum_{m=0}^{n} a_{n,m}x^m$. We prove that $a_{n,m} = (-1)^k 2^m \binom{l}{k}$, where $k = \lfloor \frac{n-m}{2} \rfloor$ and $l = \lfloor \frac{n+m}{2} \rfloor$. We also completely factorize $A_n(x)$ into irreducible factors over $\mathbb{Z}$ and obtain a condition for determining when $A_r(x)$ is divisible by $A_s(x)$. Furthermore we determine the greatest common divisor of $A_r(x)$ and $A_s(x)$ and also greatest common divisor of $A_r(x)$ and the Chebyshev's polynomials. Finally we prove certain combinatorial identities that arise from the polynomial $A_n(x)$.

2010 Mathematics Subject Classification: 11R09, 13A05, 05A19

Keywords and phrases: Chebyshev's polynomials, irreducibility, greatest common divisor, combinatorial identities.

1. Introduction

Chebyshev's polynomials are of great importance in many area of mathematics, particularly approximation theory. Interesting properties of the Chebyshev's polynomials can be found in [9] and [10]. Certain algebraic properties of Chebyshev's polynomials have been studied by Bang [1], Carlitz [3], and Rankin [7]. In 1984, Hsiao [5] gave a complete factorization of Chebyshev's polynomials of the first kind into irreducible factors over the ring of integer $\mathbb{Z}$. Using Hsiao's method, Rivlin [9] extended it to complete factorization of Chebyshev's polynomials of the second kind. Certain decomposition properties of Chebyshev's polynomials including factorization and divisibility have been studied by Rayes, Trevisan, and Wang [8].

The *Chebyshev's polynomials of the first kind* $T_n(x)$ can be defined inductively as follow:

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad \text{and}$$

Communicated by Ang Miin Huey.
Received: June 10, 2009; *Revised:* November 17, 2009.

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \qquad n = 2, 3, \ldots \tag{1.1}$$

Alternatively, it may be defined as

$$T_n(x) = \cos n(\arccos x),$$

where $0 \leq \arccos x \leq \pi$. The roots of $T_n(x)$ are

$$\cos \frac{(2k-1)\pi}{2n}, \qquad k = 1, 2, \ldots, n.$$

The *Chebyshev's polynomials of the second kind* $U_n(x)$ is defined inductively as follow:

$$U_0(x) = 1, \qquad U_1(x) = 2x, \qquad \text{and}$$
$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \qquad n = 2, 3, \ldots \tag{1.2}$$

Alternatively, it may be defined as

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)},$$

where $0 \leq \arccos x \leq \pi$. The roots of $U_n(x)$ are

$$\cos \frac{k\pi}{n+1}, \qquad k = 1, 2, \ldots, n.$$

Note that the leading coefficients of $T_n(x)$ and $U_n(x)$ are 2^{n-1} and 2^n, respectively, for $n \geq 1$. By looking at the roots of $U_{2n-1}(x)$, we see that

$$U_{2n-1}(x) = 2T_n(x)U_{n-1}(x), \quad n = 1, 2, \ldots. \tag{1.3}$$

For $U_{2n}(x)$, the roots are $\cos(k\pi/(2n+1))$, where $k = 1, 2, \ldots, 2n$. Note that for $1 \leq i \leq n$, $\cos((2i-1)\pi/(2n+1)) = -\cos(2(n-i+1)\pi/(2n+1))$. Therefore

$$U_{2n}(x) = (-1)^n A_n(x) A_n(-x), \quad n = 1, 2, \ldots, \tag{1.4}$$

where $A_n(x) = 2^n \prod_{i=1}^{n} (x - \cos(2i\pi/(2n+1)))$.

In this paper, we will study the polynomial $A_n(x)$. We will completely factorize $A_n(x)$ into irreducible factors over $\mathbb{Z}$ and prove certain combinatorial identities that arise from the polynomial $A_n(x)$.

2. Properties of $A_n(x)$

Let $\theta = 2\pi/(2n+1)$. The θ will be fixed throughout the paper.

Let us look at the polynomial $T_{n+1}(x) - T_n(x)$. Note that $T_{n+1}(1) - T_n(1) = 0$ and for $i = 1, 2, \ldots, n$,

$$\begin{aligned} T_{n+1}(\cos i\theta) - T_n(\cos i\theta) &= -2\sin\left(\frac{(2n+1)i\theta}{2}\right)\sin\left(\frac{i\theta}{2}\right) \\ &= -2\sin(i\pi)\sin\left(\frac{i\theta}{2}\right) = 0. \end{aligned}$$

This implies Lemma 2.1.

Lemma 2.1. $(x-1)A_n(x) = T_{n+1}(x) - T_n(x)$ *for* $n = 1, 2, \ldots$.

For the sake of completeness, we define $A_0(x) = 1$. This leads (1.4) and Lemma 2.1 to be true even for $n = 0$. Now by Lemma 2.1 and (1.1), $A_1(x) = 2x + 1$. Furthermore one can deduce Lemma 2.2.

Lemma 2.2. *$A_n(x) = 2xA_{n-1}(x) - A_{n-2}(x)$ for $n = 2, 3, \ldots$.*

Recall that a polynomial $p(x) \in \mathbb{Z}[x]$ is said to *divide* $h(x) \in \mathbb{Z}[x]$ or is a *divisor* of $h(x)$ if $h(x) = p(x)l(x)$ for some $l(x) \in \mathbb{Z}[x]$. A polynomial $h(x) \in \mathbb{Z}[x]$ is said to be *irreducible* if the only divisors of $h(x)$ are ± 1 and $\pm h(x)$.

A number $\zeta \in \mathbb{C}$ is said to be an *algebraic number* if there is a $p(x) \in \mathbb{Z}[x]$ with $p(\zeta) = 0$. Furthermore if $p(x)$ is irreducible and of degree k, we say ζ is *algebraic of degree k*. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_i \in \mathbb{Z}$ for all i. If $a_n = 1$, we say ζ is an *algebraic integer*. So an algebraic integer is an algebraic number.

Given any $r(x), s(x) \in \mathbb{Z}[x]$, the *greatest common divisor* of $r(x)$ and $s(x)$ will be denoted by $\gcd(r(x), s(x))$. Note that the leading coefficient of the greatest common divisor will be chosen to be positive. Consider a fixed integer $n \geq 1$. Let l_h denote the number of elements in

$$S_h = \{i \ : \ \gcd(i, 2n+1) = h, 1 \leq i \leq n\}.$$

Clearly $l_h = \phi((2n+1)/h)/2$, where ϕ is the *Euler's totient function.* Properties of ϕ can be found in [4, p. 52]. Now let

$$F_h(x) = 2^{l_h} \prod_{\substack{1 \leq i \leq n \\ \gcd(i,2n+1)=h}} (x - \cos i\theta).$$

Theorem 2.1. *For $n \geq 1$,*

$$A_n(x) = \prod_h F_h(x),$$

where $h \leq n$ runs through all positive divisors of $2n+1$. All the F_h are irreducible over $\mathbb{Z}$.

Proof. Clearly $A_n(x) = \prod_h F_h(x)$. So it is sufficient to show that F_h are irreducible over $\mathbb{Z}$. By Lehmer's Theorem [6, Theorem 1], if $\gcd(i, 2n+1) = 1$ then $2\cos(i\theta)$ is an algebraic integer of degree $\phi(2n+1)/2$. Following the proof of Lehmer's Theorem, we see that all $2\cos(i\theta)$ with $\gcd(i, 2n+1) = 1$ are the roots of the same irreducible polynomial, say $Q(x)$. Note that $Q(2x)$ is also irreducible and $F_1(x) = Q(2x)$. Now if $\gcd(i, 2n+1) = h$ then $\gcd(i/h, (2n+1)/h) = 1$ and $2\cos(i\theta/h)$ is an algebraic integer of degree $\phi((2n+1)/h)/2$. As in the previous paragraph, F_h is irreducible. ∎

An immediate consequence of Theorem 2.1 is the following corollary.

Corollary 2.1. *For all $n \in \mathbb{N}$,*

(a) *$F_1(x)$ is the irreducible factor of $A_n(x)$ of the largest degree $= \phi(2n+1)/2$.*
(b) *The number of irreducible factors of $A_n(x)$ equal to the number of divisors $h \leq n$ of $2n+1$.*

Corollary 2.2. *$A_n(x)$ is irreducible if and only if $n = (p-1)/2$ for some prime p.*

Proof. If $n = (p-1)/2$ for some prime p, then by (b) of Corollary 2.1, $A_n(x)$ is irreducible. Suppose $A_n(x)$ is irreducible. If $2n+1$ is not a prime, then $2n+1 = rs$ for some $r, s \in \mathbb{N}$, $r > s > 1$. This implies that $2n+1 > s^2$ and $s \leq n$. But then by (b) of Corollary 2.1, the number of irreducible factors of $A_n(x)$ is at least 2, a contradiction. Hence $2n+1$ is a prime. ∎

Let $\psi_m(x)$ be the minimal polynomial of $\cos(2\pi/m)$. If $m = 2n+1$, then (see [12, Theorem])

$$T_{n+1}(x) - T_n(x) = 2^n \prod_{d|m} \psi_d(x).$$

Therefore by Lemma 2.1, $A_n(x) = \left(2^n \prod_{d|m} \psi_d(x)\right)/(x-1) = 2^n \prod_{d|m, d\neq 1} \psi_d(x)$. When m is a prime, $A_n(x) = 2^n \psi_m(x)$. The polynomial $\psi_m(x)$ when m is a prime has been studied by Beslin and de Angelis [2], and Surowski and McCombs [11].

Let $A_n(x) = \sum_{m=0}^{n} a_{n,m} x^m$. Given any real number $x \in \mathbb{R}$, we shall denote the greatest integer less than or equal to x by $\lfloor x \rfloor$, and we shall denote the smallest integer greater than or equal to x by $\lceil x \rceil$. As usual, the *binomial coefficient* $\binom{r}{t}$ is the coefficient of x^t in the polynomial expansion of $(1+x)^r$. Recall that $A_0(x) = 1$ and $A_1(x) = 2x+1$. By Lemma 2.2, $A_2(x) = 4x^2 + 2x - 1$.

Theorem 2.2. *Let* $k = \left\lfloor \frac{n-m}{2} \right\rfloor$ *and* $l = \left\lfloor \frac{n+m}{2} \right\rfloor$. *Then*

$$a_{n,m} = (-1)^k 2^m \binom{l}{k} \qquad \text{for } 0 \leq m \leq n.$$

Proof. It can be verified that $a_{n,m} = (-1)^k 2^m \binom{l}{k}$ for all $0 \leq m \leq n$ where $n = 0, 1, 2$. Let $n \geq 3$. Assume that the formula holds for $a_{n',m'}$, for all $0 \leq m' \leq n'$ with $1 \leq n' < n$. Now $A_n(x) = 2xA_{n-1}(x) - A_{n-2}(x)$ (Lemma 2.2) implies that

$$\begin{aligned} a_{n,0} &= -a_{n-2,0}, \\ a_{n,m} &= 2a_{n-1,m-1} - a_{n-2,m} \ \text{ for } 1 \leq m \leq n-2, \\ a_{n,m} &= 2a_{n-1,m-1} \ \text{ for } n-1 \leq m \leq n. \end{aligned}$$

Therefore $a_{n,0} = (-1)^{\lfloor n/2 \rfloor}$, $a_{n,n-1} = 2^{n-1}$, $a_{n,n} = 2^n$ and for all $1 \leq m \leq n-2$,

$$\begin{aligned} a_{n,m} &= 2a_{n-1,m-1} - a_{n-2,m} \\ &= (-1)^k 2^m \binom{l'}{k} + (-1)^k 2^m \binom{l'}{k-1} \\ &= (-1)^k 2^m \binom{l}{k}, \end{aligned}$$

where $l' = \lfloor (n+m-2)/2 \rfloor$. Here we make use of the facts that $\lfloor (t-2)/2 \rfloor = \lfloor t/2 \rfloor - 1$ for all $t \in \mathbb{Z}$, $\binom{r}{s} + \binom{r}{s-1} = \binom{r+1}{s}$ for all $r, s \in \mathbb{N}$, and induction hypothesis. Hence the proof is complete. ∎

Note that $a_{n,0} = (-1)^{\lfloor n/2 \rfloor}$ and $a_{n,n} = 2^n$. Recall that a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ is said to be *primitive* if $a_n > 0$ and $\gcd(a_n, a_{n-1}, \ldots, a_1, a_0) = 1$. Therefore $A_n(x)$ is primitive.

Corollary 2.3. *$A_n(x)$ is primitive for all integer $n \geq 0$.*

Theorem 2.3. *Let $r \geq s$ be two positive integers. Then $A_s(x)$ divides $A_r(x)$ if and only if $r = (2l+1)s + l$ for some integer $l \geq 0$.*

Proof. Suppose $r = (2l+1)s + l$ for some integer $l \geq 0$. Then the roots of $A_r(x)$ are

$$\cos\left(\frac{2i\pi}{2r+1}\right) = \cos\left(\frac{2i\pi}{(2l+1)(2s+1)}\right) \qquad \text{for } i = 1, 2, \ldots, r.$$

By taking $i_j = (2l+1)j$ for $j = 1, 2, \ldots, s$, we see that $\cos(2j\pi/(2s+1))$ are roots of $A_r(x)$. Note that $\cos(2j\pi/(2s+1))$ are roots of $A_s(x)$. So together with the division algorithm, we have $A_r(x) = H(x)A_s(x)$ for some $H(x) \in \mathbb{Q}[x]$. By Corollary 2.3, $A_r(x)$ and $A_s(x)$ are primitive. Using a standard argument as in [4, Proof of Theorem 237 on p. 205], we may assume that $H(x) \in \mathbb{Z}[x]$. Hence $A_s(x)$ divides $A_r(x)$.

Suppose $A_s(x)$ divides $A_r(x)$. Then $A_s(-x)$ divides $A_r(-x)$. By (1.4), $U_{2s}(x)$ divides $U_{2r}(x)$. Then by [8, Theorem 3], $2r = (l'+1)2s + l'$ for some integer $l' \geq 0$. Clearly, $l' = 2l$ for some integer l. Hence $r = (2l+1)s + l$. ∎

Corollary 2.4. *Let r, s be two nonnegative integers and $\gcd(2r+1, 2s+1) = t$. Then $\gcd(A_r(x), A_s(x)) = A_{(t-1)/2}(x)$.*

Proof. Let $\gcd(A_r(x), A_s(x)) = g(x)$. By Theorem 2.3, $A_{(t-1)/2}(x)$ divides $A_r(x)$ and $A_s(x)$. If $g(x)$ is of degree $(t-1)/2$, then $g(x) = A_{(t-1)/2}(x)$, and we are done. Suppose the degree of $g(x)$ is greater than $(t-1)/2$. Note that $g(-x)$ divides $A_r(-x)$ and $A_s(-x)$. This implies that $g(x)g(-x)$ divides $A_r(x)A_r(-x)$ and $A_s(x)A_s(-x)$. By (1.4), we see that $g(x)g(-x)$ divides $U_{2r}(x)$ and $U_{2s}(x)$. Now $\gcd(U_{2r}(x), U_{2s}(x)) = U_{t-1}(x)$ (see [8, Theorem 4]). But then $g(x)g(-x)$ divides $U_{t-1}(x)$, a contradiction, for the degree of $g(x)g(-x)$ is greater than $t-1$. Hence $\gcd(A_r(x), A_s(x)) = A_{(t-1)/2}(x)$. ∎

Theorem 2.4. *Let r, s be two nonnegative integers. Then* $\gcd(A_r(x), A_s(-x)) = 1$.

Proof. If either $r = 0$ or $s = 0$, we are done. So we may assume $r \geq s \geq 1$. Suppose $\gcd(A_r(x), A_s(-x)) \neq 1$. Then $-\cos(2i'\pi/(2s+1))$ is a root of $A_r(x)$ for some $1 \leq i' \leq s$. This implies that $\cos(2i'\pi/(2s+1)) + \cos(2i\pi/(2r+1)) = 0$ for some $1 \leq i \leq r$. Therefore

$$(2.1) \qquad 2\cos\left(\frac{(2r+1)i' + (2s+1)i}{(2s+1)(2r+1)}\pi\right)\cos\left(\frac{(2r+1)i' - (2s+1)i}{(2s+1)(2r+1)}\pi\right) = 0.$$

Note that the first term in (2.1) is zero if and only if $2((2r+1)i' + (2s+1)i) = (2s+1)(2r+1)t$ for some odd t. But this is impossible. Now the second term in (2.1) is zero if and only if $2((2r+1)i' - (2s+1)i) = (2s+1)(2r+1)t_1$ for some odd t_1, which is again impossible. Hence $\gcd(A_r(x), A_s(-x)) = 1$. ∎

Corollary 2.5. *For any nonnegative integers r, s,*

(a) $\gcd(A_r(x), A_r(-x)) = 1$.
(b) $\gcd(U_r(x), A_s(x)) = A_{(t-1)/2}$, *where* $t = \gcd(r+1, 2s+1)$.
(c) $\gcd(T_r(x), A_s(x)) = 1$.

Proof. (a) follows from Theorem 2.4.

(b) By [8, Theorem 4], $\gcd(U_r(x), U_{2s}(x)) = U_{t-1}$ where $t = \gcd(r+1, 2s+1)$. Note that $\gcd(A_s(x), A_{(t-1)/2}(-x)) = 1$ and $A_{(t-1)/2}(x)$ divides $A_s(x)$ (see Theorem

2.4 and Theorem 2.3). Recall that $U_{t-1}(x) = (-1)^{(t-1)/2}A_{(t-1)/2}(x)A_{(t-1)/2}(-x)$ (see (1.4)). Therefore $\gcd(U_r(x), A_s(x)) = A_{(t-1)/2}$.

(c) By (1.3), $U_{2r-1}(x) = 2T_r(x)U_{r-1}(x)$. By part (b), $\gcd(U_{2r-1}(x), A_s(x)) = A_{(t-1)/2}(x)$, where $t = \gcd(2r, 2s+1)$, and $\gcd(U_{r-1}(x), A_s(x)) = A_{(t'-1)/2}(x)$, where $t' = \gcd(r, 2s+1)$. Note that $t' = t$. Let $\gcd(T_r(x), A_s(x)) = d(x)$. Then $d(x)$ divides $U_{2r-1}(x)$ and thus $A_{(t-1)/2}(x)$. In fact $d(x)$ divides $U_{2r-1}(x)/A_{(t-1)/2}(x)$. Since all the roots of $U_{2r-1}(x)$ are distinct, we conclude that $\gcd(T_r(x), A_s(x)) = 1$. ∎

3. Certain combinatorial identities

Now if $P(x) = \sum_{i=0}^{n} c_i x^i$ is a polynomial of degree n with roots r_i(not necessarily distinct), $i = 1, 2, \ldots, n$, then $P(x) = c_n \prod_{i=1}^{n}(x - r_i)$. By expanding $\prod_{i=1}^{n}(x - r_i)$ and comparing the coefficient of x^{n-m}, we have the following Vieta's formula.

Proposition 3.1. [Vieta's formula]

$$\sum_{1\le\alpha_1<\alpha_2<\ldots<\alpha_m\le n} r_{\alpha_1} r_{\alpha_2}\ldots r_{\alpha_m} = (-1)^m \frac{c_{n-m}}{c_n}.$$

By Theorem 2.2 and Proposition 3.1, we have Corollary 3.1.

Corollary 3.1.

$$\sum_{1\le r_1<r_2<\ldots<r_m\le n} \cos\ r_1\theta \cos\ r_2\theta \ldots \cos\ r_m\theta = (-1)^{\overline{m}} \frac{\binom{n-\overline{m}}{\underline{m}}}{2^m},$$

where $\overline{m} = \lceil m/2 \rceil$ *and* $\underline{m} = \lfloor m/2 \rfloor$.

Recall that

$$T_n(x) = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2}, \quad x \in \mathbb{R}. \tag{3.1}$$

By Lemma 2.1, we have Proposition 3.2.

Proposition 3.2.

$$A_n(x) = \frac{(x+\sqrt{x^2-1})^n(x-1+\sqrt{x^2-1}) + (x-\sqrt{x^2-1})^n(x-1-\sqrt{x^2-1})}{2(x-1)}.$$

Corollary 3.2 follows from Theorem 2.2 and Proposition 3.2 (take limit $x \to \pm 1$).

Corollary 3.2.

$$A_n(1) = \sum_{m=0}^{n} (-1)^{\lfloor \frac{n-m}{2} \rfloor} 2^m \binom{\lfloor \frac{n+m}{2} \rfloor}{\lfloor \frac{n-m}{2} \rfloor} = 2n+1 \qquad \text{and}$$

$$A_n(-1) = \sum_{m=0}^{n} (-1)^{m+\lfloor \frac{n-m}{2} \rfloor} 2^m \binom{\lfloor \frac{n+m}{2} \rfloor}{\lfloor \frac{n-m}{2} \rfloor} = (-1)^n.$$

Now

$$\begin{aligned} A_n(-1) &= (-2)^n \prod_{i=1}^{n}(1+\cos i\theta) \\ &= (-2)^n \left(1 + \sum_{m=1}^{n} \sum_{1\le r_1<r_2<\ldots<r_m\le n} \cos\ r_1\theta \cos\ r_2\theta \ldots \cos\ r_m\theta\right). \end{aligned}$$

Then using Corollary 3.2, we deduce Corollary 3.3.

Corollary 3.3.

$$\sum_{m=1}^{n} \sum_{1\le r_1<r_2<\ldots<r_m\le n} \cos\ r_1\theta \cos\ r_2\theta \ldots \cos\ r_m\theta = \frac{1-2^n}{2^n}.$$

For $\lfloor n/2\rfloor + 1 \le i \le n$, we have $n+1 \le 2i \le 2n$ and $1 \le 2n+1-2i \le n$. This implies that

$$\{2i\ :\ 1 \le i \le \lfloor n/2\rfloor\} \cup \{2n+1-2i\ :\ \lfloor n/2\rfloor + 1 \le i \le n\} = \{1, 2, \ldots, n\}.$$

Now $\cos((2n+1-2i)\theta) = \cos(2i\theta)$. Therefore

$$A_n(x) = 2^n \prod_{i=1}^{n}(x - \cos i\theta) = 2^n \prod_{i=1}^{n}(x - \cos 2i\theta). \tag{3.2}$$

Let $B_n(x) = A_n(2x-1)$. Then Lemma 3.1 follows from (3.2).

Lemma 3.1. *The roots of $B_n(x)$ are $\cos^2 i\theta$, $i = 1, 2, \ldots, n$.*

Note that $B_0(x) = 1$, $B_1(x) = 4x - 1$ and by Lemma 2.2, we have the following recurrence relation for $B_n(x)$.

Lemma 3.2. $B_n(x) = 2(2x-1)B_{n-1}(x) - B_{n-2}(x)$ *for* $n = 2, 3, \ldots$.

Let $B_n(x) = \sum_{m=0}^{n} b_{n,m}x^m$. By mathematical induction and Lemma 3.2 (similar to the proof of Theorem 2.2), one can determine $b_{n,m}$.

Theorem 3.1.

$$b_{n,m} = (-1)^{n-m}4^m \binom{m+n}{2m} \qquad \text{for } 0 \le m \le n.$$

Note that $b_{n,0} = (-1)^n$ and $b_{n,n} = 4^n$. So $B_n(x)$ is primitive.

Corollary 3.4. *$B_n(x)$ is primitive for all integer $n \ge 0$.*

Now Corollary 3.5 follows from Theorem 3.1 and Proposition 3.1, and Corollary 3.6 follows from Proposition 3.2.

Corollary 3.5.

$$\sum_{1\le r_1<r_2<\ldots<r_m\le n} \cos^2\ r_1\theta \cos^2\ r_2\theta \ldots \cos^2\ r_m\theta = (-1)^{n-m}4^{-m}\binom{2n-m}{m}.$$

Corollary 3.6.

$$B_n(x) = \frac{(h(x))^n(h(x)-1) + (g(x))^n(g(x)-1)}{4(x-1)},$$

where $h(x) = 2x - 1 + \sqrt{(2x-1)^2 - 1}$ *and* $g(x) = 2x - 1 - \sqrt{(2x-1)^2 - 1}$.

As in Corollary 3.2, Corollary 3.7 follows from Theorem 3.1 and Corollary 3.6 by taking limit $x \to \pm 1$.

Corollary 3.7.

$$B_n(1) = \sum_{m=0}^{n} (-1)^{n-m} 4^m \binom{m+n}{2m} = 2n+1 \qquad \text{and}$$

$$B_n(-1) = (-1)^n \sum_{m=0}^{n} 4^m \binom{m+n}{2m} = -\frac{(h(-1))^n(h(-1)-1) + (g(-1))^n(g(-1)-1)}{8},$$

where $h(x)$ and $g(x)$ are as in Corollary 3.6.

As in Corollary 3.3, we can deduce Corollary 3.8.

Corollary 3.8.

$$\sum_{m=1}^{n} \sum_{1 \le r_1 < r_2 < \ldots < r_m \le n} \cos^2\ r_1\theta \cos^2\ r_2\theta \ldots \cos^2\ r_m\theta = (-1)^n \frac{B_n(-1)}{4^n} - 1.$$

Recall that $U_{2n}(x) = (-1)^n A_n(x) A_n(-x)$, (see (1.4)). So

$$U_{2n}(x) = 4^n \prod_{i=1}^{n} (x^2 - \cos^2 i\theta) = B_n(x^2) \quad \text{and}$$

$$B_n(x) = (-1)^n A_n(x^{1/2}) A_n(-x^{1/2}), \quad n = 0, 1, \ldots. \tag{3.3}$$

By Theorem 2.2, Theorem 3.1 and (3.3), we have the following corollary.

Corollary 3.9.

$$\sum_{i=0}^{2m} \binom{\lfloor \frac{n+i}{2} \rfloor}{i} \binom{m + \lfloor \frac{n-i}{2} \rfloor}{2m-i} = \binom{m+n}{2m}, \text{ for all } 0 \le m \le n.$$

References

[1] T. Bang, Congruence properties of Tchebycheff polynomials, *Math. Scand.* **2** (1954), 327–333.
[2] S. Beslin and V. De Angelis, The minimal polynomials of $\sin\left(\frac{2\pi}{p}\right)$ and $\cos\left(\frac{2\pi}{p}\right)$, *Math. Mag.* **77** (2004), no. 2, 146–149.
[3] L. Carlitz, Quadratic residues and Tchebycheff polynomials, *Portugal. Math.* **18** (1959), 193–198.
[4] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, fifth edition, Oxford Univ. Press, New York, 1979.
[5] H. J. Hsiao, On factorization of Chebyshev's polynomials of the first kind, *Bull. Inst. Math. Acad. Sinica* **12** (1984), no. 1, 89–94.
[6] D. H. Lehmer, Questions, discussions, and notes: A note on trigonometric algebraic numbers, *Amer. Math. Monthly* **40** (1933), no. 3, 165–166.
[7] R. A. Rankin, Chebyshev polynomials and the modulary group of level p, *Math. Scand.* **2** (1954), 315–326.
[8] M. O. Rayes, V. Trevisan and P. S. Wang, Factorization properties of Chebyshev polynomials, *Comput. Math. Appl.* **50** (2005), no. 8-9, 1231–1240.
[9] T. J. Rivlin, *Chebyshev Polynomials*, second edition, Pure and Applied Mathematics (New York), Wiley, New York, 1990.
[10] M. A. Snyder, *Chebyshev Methods in Numerical Approximation*, Prentice Hall, Englewood Cliffs, NJ, 1966.
[11] D. Surowski and P. McCombs, Homogeneous polynomials and the minimal polynomial of $\cos(2\pi/n)$, *Missouri J. Math. Sci.* **15** (2003), no. 1, 4–14.
[12] W. Watkins and J. Zeitlin, The minimal polynomial of $\cos(2\pi/n)$, *Amer. Math. Monthly* **100** (1993), no. 5, 471–474.