

Annals of Mathematics

Lacunary Recurrence Formulas for The Numbers of Bernoulli and Euler

Author(s): D. H. Lehmer

Source: *Annals of Mathematics*, Second Series, Vol. 36, No. 3 (Jul., 1935), pp. 637-649

Published by: [Annals of Mathematics](#)

Stable URL: <http://www.jstor.org/stable/1968647>

Accessed: 09/03/2014 16:18

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



Annals of Mathematics is collaborating with JSTOR to digitize, preserve and extend access to *Annals of Mathematics*.

<http://www.jstor.org>

LACUNARY RECURRENCE FORMULAS FOR THE NUMBERS OF BERNOULLI AND EULER

BY D. H. LEHMER

(Received February 24, 1934)

Recurrence relations for the computation of the numbers of Bernoulli have been the subject of a great many papers. Nevertheless, only two extensive calculations have been carried out. Adams¹ has calculated the first 62 (non-zero) Bernoulli numbers, while Cerébrenikoff² has given the first 92. Both these intrepid calculators used recurrence formulas of the most primitive sort, in spite of the fact that several formulas had already been given, which would have saved them many hundreds of hours.

It is customary to give recurrences whose coefficients are neatly expressed in terms of familiar functions, sometimes at the expense of considerable labor in calculating their actual value. In the recurrence relations in this paper the coefficients are designed for ease of calculation at the expense of compactness of expression.

The reader may question the utility of tabulating more than 92 Bernoulli numbers and hence the need of giving formulas for extending their calculation. It is true that for ordinary purposes of analysis, for example in the asymptotic series of the Euler Maclaurin summation formula, a dozen Bernoulli numbers are sufficient. There are other problems, however, which depend upon more subtle properties of the Bernoulli numbers, such as their divisibility by a given prime. Examples of such problems are the second case of Fermat's Last Theorem and the Riemann Zeta-function hypothesis. Our knowledge as to the divisibility properties of the Bernoulli numbers is still quite primitive and it would be highly desirable to add more to it, even if the knowledge thus gained be purely empirical.

The method which we use applies not only to Bernoulli numbers B , but with equal ease to the numbers E , R , and G of Euler, Lucas and Genocchi.³ These four sets of numbers may thus be considered together to make a symmetrical theory. Moreover R and G may be simply expressed in terms of B so that we thus obtain three sets of recurrences for Bernoulli numbers. The coefficients of the power series for trigonometric functions of higher order may also be dealt with by the same method as we indicate briefly in §4.

In contrast to the recurrences usually given for Bernoulli numbers, in which

¹ Journal für Math. 85 (1878), 269–272. Collected Papers, v. 1.

² Akademiia Nauk. Math. Phys. Kl: Mémoires. Ser. 8. v. 16, no. 10 (1905), and v. 19, no. 4 (1906).

³ See Lucas: *Théorie des Nombres*, Paris, 1890, chapters 13 and especially 14. For elaborations of Lucas' theory of Bernoulli and allied numbers see the following papers by E. T. Bell: Trans. Amer. Math. Soc. 24 (1922), 89–112; 28 (1926), 129–148; 31 (1929), 405–421.

B_n is made to depend upon all⁴ the preceding (non-zero) B 's the recurrences given in this paper have gaps so that B_n can be computed from those preceding B 's whose subscripts are congruent to n with respect to a modulus m the length of each gap. Recurrences with arbitrarily large gaps seem to have been considered first by van den Berg.⁵ Twelve years later the same results (in less explicit form, from the point of view of application) were obtained by R. Haussner,⁶ who expressed the coefficients of the recurrences with large gaps in terms of the hypergeometric function. Both van den Berg and Haussner based their method on the power series expansion of the product of $\sin \omega x$, where ω runs over the n^{th} roots of unity (a product suggested by Kronecker⁷). Both treatments, especially that of van den Berg, are unnecessarily long and complicated. A more straightforward discussion in finite terms has been given by Nielsen,⁸ who gives only the simpler results, however. Mention should be made of the recurrences given by Ramanujan⁹ for small gaps. In the present paper we obtain the recurrences of van den Berg and Haussner in a form for practical application, as a part of a more general discussion by a natural finite method, simpler than that of Nielsen.

1. The present development is based on a certain sum¹⁰ $\sigma_n(p, q, r, s, t)$, which is defined for positive integral values of p, q, r, s , and for non-negative¹¹ values of n and t .

$$(1) \quad \sigma_n = \sigma_n(p, q, r, s, t) = \sum_{\eta} (\eta_1 \eta_2 \cdots \eta_{s-1})^t (1 + \eta_1 \epsilon + \eta_2 \epsilon^2 + \cdots + \eta_{s-1} \epsilon^{s-1})^n$$

where $\epsilon = e^{2\pi i / pr}$ and each η_r takes on the values $1, \eta, \eta^2, \dots, \eta^{pq-1}$ where $\eta = e^{2\pi i / pq}$, and the sum extends over all $(pq)^{s-1}$ possible combinations of these values. For $s = r$ we have the following

THEOREM A. $\sigma_{k pq - r t}(p, q, r, r, t) = 0$ for every k for which kq is not a multiple of r .

Proof. The $(pq)^{r-1}$ terms of $\sigma_n(p, q, r, r, t)$ may be grouped in two different ways into pq sets of $(pq)^{r-2}$ terms each, by assigning definite values to either η_1 or η_{r-1} . We have in this way two partitions of σ as follows:

$$(2) \quad \sigma = \sum_{\nu=0}^{pq-1} S_{\nu} = \sum_{\nu=0}^{pq-1} S'_{\nu}$$

⁴ Recurrences for B_n involving only B_{ν} for $n/2 \leq \nu < n$ have been given by Stern: *Journal für Math.* 84, 216; Radike, *ibid* 89, 259; Saalschütz, *Vorlesungen über die Bernoullischen Zahlen* (Berlin 1893), p. 30 and Lucas *loc. cit* p. 240.

⁵ "Over Periodieke Terugloopende Betrekkingen tusschen de Coëfficiënten in de Ontwikkeling van Functien," *Verslagen en Mededeelingen der Koninklijke Akademie van Wetenschappen.* Amsterdam (1881), 16, 74-176.

⁶ *Göttinger Nachrichten* (1893), 777-809.

⁷ *Journal de Math.* (2), 1 (1856), 385.

⁸ *Traité élémentaire des nombres de Bernoulli*, Paris 1922, 195-225.

⁹ *Journal of the Indian Math. Soc.* 3, 1911, 219-234. *Collected Papers*, p. 1-14.

¹⁰ This sum is a natural generalization of the two sums of Kronecker, *loc. cit*.

¹¹ For $n = 0$, σ_n may involve 0^0 . This is taken as unity.

where in S_r , η_1 is fixed at η^r , and in S'_r , η_{r-1} is fixed at η^r . We now consider

$$\begin{aligned} \epsilon^n S'_r &= \sum (\eta_1 \eta_2 \cdots \eta_{r-2} \eta^r)^t (\eta^r \epsilon^r + \epsilon + \eta_1 \epsilon^2 + \cdots + \eta_{r-2} \epsilon^{r-1})^n \\ &= \eta^{rt+n(q+r)} \sum (\eta_1 \eta_2 \cdots \eta_{r-2})^t (1 + \eta^{-(q+\nu)} \epsilon + \eta_1 \eta^{-(q+\nu)} \epsilon^2 \\ &\qquad\qquad\qquad + \cdots + \eta_{r-2} \eta^{-(q+\nu)} \epsilon^{r-1})^n \end{aligned}$$

since $\epsilon^r = \eta^q$. If now we introduce into the first parenthesis the factor $\eta^{-(r-1)(q+\nu)}$, this parenthesis will contain the product of the η 's in the second parenthesis. Carrying out the summation indicated, we obtain precisely $S_{pq-(q+\nu)}$. Hence

$$\begin{aligned} (3) \quad \epsilon^n S'_r &= \eta^{rt+n(q+\nu)+(r-1)(q+\nu)t} S_{pq-(q+\nu)} \\ &= \eta^{q(n+rt-t)+\nu(n+rt)} S_{pq-(q+\nu)}. \end{aligned}$$

Setting $n = kpq - rt$ we observe that $n + rt$ is a multiple of pq , so that the power of η in (3) is not a function of ν , and, in fact, is $\eta^{-qt} = \epsilon^{-rt}$. Summing (3) for $\nu = 0, 1, 2, \dots, pq - 1$ we thus obtain in view of (2), for $n = kpq - rt$,

$$\epsilon^{kpq-rt} \sigma_{kpq-rt} = \epsilon^{-rt} \sigma_{kpq-rt}.$$

That is

$$((\epsilon^p)^{kq} - 1) \sigma_{kpq-rt} = 0.$$

Since ϵ^p is a primitive r^{th} root of unity, the theorem follows.

2. The Numbers B_n, G_n, R_n , and E_n . These four sets of numbers may be defined by

$$\begin{aligned} (4) \quad B_0 &= 1, & B_1 &= -1/2, & (B + 1)^n - B^n &= 0 & (n > 1), \\ G_0 &= 0, & G_1 &= 1, & (G + 1)^n + G^n &= 0 & (n > 1), \\ R_0 &= 1/2, & R_1 &= 0, & (R + 1)^n - (R - 1)^n &= 0, & (n > 1), \\ E_0 &= 1, & E_1 &= 0, & (E + 1)^n + (E - 1)^n &= 0, & (n > 1), \end{aligned}$$

in which, after expansion, the exponents are degraded to subscripts. Equivalent definitions in terms of generating functions are

$$\begin{aligned} (5) \quad e^{Bz} &= \frac{z}{e^z - 1} & \text{or} & & \cos 2Bz &= z \cot z \\ e^{Gz} &= \frac{2z}{e^z + 1} & \text{or} & & \cos 2Gz &= 2z \tan z \\ e^{Rz} &= \frac{ze^z}{e^{2z} - 1} & \text{or} & & \cos 2Rz &= z \csc 2z \\ e^{Ez} &= \frac{2e^z}{e^{2z} + 1} & \text{or} & & \cos Ez &= \sec z. \end{aligned}$$

The numbers $B_n, R_n, G_n,$ and E_n vanish when n is odd with the exceptions $B_1 = -1/2, G_1 = 1.$ We subjoin a small table of the values of these numbers when n is even.

n	B_n	G_n	R_n	E_n
2	+1/6	-1	-1/6	-1
4	-1/30	+1	+7/30	+5
6	+1/42	-3	-31/42	-61
8	-1/30	+17	+127/30	+1385
10	+5/66	-155	-2555/66	-50521

If we multiply each of the equations (4) by $a_m \binom{m}{n} x^{m-n}$ and sum over n from 0 to $m,$ (taking into special consideration the cases $n = 0, 1,$) and then sum over $m,$ we get the fundamental relations

$$(6.1) \quad f(B + x + 1) - f(B + x) = f'(x),$$

$$(6.2) \quad f(G + x + 1) + f(G + x) = 2f'(x),$$

$$(6.3) \quad f(R + x + 1) - f(R + x - 1) = f'(x),$$

$$(6.4) \quad f(E + x + 1) + f(E + x - 1) = 2f(x),$$

where $f(x) = a_0 + a_1x + a_2x^2 + \dots$

If in (6.3) and (6.4) we set $f(x) = x^n,$ while in (6.1) and (6.2) we set $f(x) = (2x - 1)^n,$ and replace $2x$ by x in these latter results, we get

$$(7) \quad \begin{aligned} (2B + (1 + x))^n - (2B - (1 - x))^n &= 2n(x - 1)^{n-1}, \\ (2G + (1 + x))^n + (2G - (1 - x))^n &= 4n(x - 1)^{n-1} \\ (R + (1 + x))^n - (R - (1 - x))^n &= nx^{n-1} \\ (E + (1 + x))^n + (E - (1 - x))^n &= 2x^n. \end{aligned}$$

Before expanding the binomials in (7) according to increasing powers of the umbral letters $B, G, R,$ and $E,$ we recall that all odd powers vanish, except B^1 and $G^1.$ The terms arising from these symbols we transpose. After expanding we may let x range over values x_j to be determined later, and then sum over these values. We thus obtain

$$(8.1) \quad \sum_{\nu=0}^{\lfloor n/2 \rfloor} 2^{2\nu} B_{2\nu} \binom{n}{2\nu} \sum_j (1 + x_j)^{n-2\nu} - (-1)^n (1 - x_j)^{n-2\nu} \\ = n \sum_j (1 + x_j)^{n-1} - (-1)^n (1 - x_j)^{n-1}$$

$$(8.2) \quad \sum_{\nu=0}^{\lfloor n/2 \rfloor} 2^{2\nu} G_{2\nu} \binom{n}{2\nu} \sum_i (1+x_i)^{n-2\nu} + (-1)^n (1-x_i)^{n-2\nu} = -2n \sum_i (1+x_i)^{n-1} + (-1)^n (1-x_i)^{n-1}$$

$$(8.3) \quad \sum_{\nu=0}^{\lfloor n/2 \rfloor} R_{2\nu} \binom{n}{2\nu} \sum_i (1+x_i)^{n-2\nu} - (-1)^n (1-x_i)^{n-2\nu} = n \sum_i x_i^{n-1}$$

$$(8.4) \quad \sum_{\nu=0}^{\lfloor n/2 \rfloor} E_{2\nu} \binom{n}{2\nu} \sum_i (1+x_i)^{n-2\nu} + (-1)^n (1-x_i)^{n-2\nu} = 2 \sum_i x_i^n$$

We now seek to determine the x_i with a view to applying Theorem A. It is simplest to consider first the equations (8.2) and (8.4). In these equations we make n even, say $n = 2m$. In the definition of $\sigma_n(p, q, r, r, t)$ set $pq = 2$ (that is $\eta = -1$). We now observe that if the set x_i coincides with the set

$$\epsilon + \eta_2 \epsilon^2 + \eta_3 \epsilon^3 + \dots + \eta_{r-1} \epsilon^{r-1},$$

then

$$(9) \quad \sum_i (1+x_i)^\mu + (1-x_i)^\mu = \sigma_\mu(p, q, r, r, 0) \\ \sum_i x_i^{2m} = e^{4\pi i m / pr} \sigma_{2m}(p, q, r, r-1, 0)$$

where $p = 1, q = 2$ or $p = 2, q = 1$. Substituting (9) into (8.2) and (8.4) and using Theorem A with $t = 0$, we have at once the general lacunary recurrences for the numbers of Genocchi and Euler. First with $p = 2, q = 1$ we have

$$(10.2) \quad \sum_{\lambda=0}^{\lfloor m/r \rfloor} 2^{2m-2\lambda r} G_{2m-2\lambda r} \binom{2m}{2\lambda r} \sigma_{2\lambda r}(2, 1, r, r, 0) = -4m \sigma_{2m-1}(2, 1, r, r, 0)$$

$$(10.4) \quad \sum_{\lambda=0}^{\lfloor m/r \rfloor} E_{2m-2\lambda r} \binom{2m}{2\lambda r} \sigma_{2\lambda r}(2, 1, r, r, 0) = 2e^{2\pi i m / r} \sigma_{2m}(2, 1, r, r-1, 0).$$

If r is odd, $\sigma_n(2, 1, r, r, 0) = \sigma_n(1, 2, r, r, 0)$. Hence (10.2) and (10.4) remain unaltered if r is odd and 2, 1 is replaced by 1, 2. For $r = 2h$ however, different formulas are obtained from Theorem A, when $p = 1$, and $q = 2$. These are

$$(11.2) \quad \sum_{\lambda=0}^{\lfloor m/h \rfloor} 2^{2m-2\lambda h} G_{2m-2\lambda h} \binom{2m}{2\lambda h} \sigma_{2\lambda h}(1, 2, 2h, 2h, 0) = -4_r h \sigma_{2m-1}(1, 2, 2h, 2h, 0)$$

$$(11.4) \quad \sum_{\lambda=0}^{\lfloor m/h \rfloor} E_{2m-2\lambda h} \binom{2m}{2\lambda h} \sigma_{2\lambda h}(1, 2, 2h, 2h, 0) = 2e^{2\pi i m / h} \sigma_{2m}(1, 2, 2h, 2h-1, 0).$$

We now return to the numbers B and R . If we rewrite (8.1) and (8.3), substituting y_j for x_j and if we multiply the equations in y_j by $(-1)^n$ and add them to the equations in x_j , we obtain at once

$$\begin{aligned}
 (12.1) \quad \sum_{\nu=0}^{\lfloor n/2 \rfloor} 2^{2\nu} B_{2\nu} \binom{n}{2\nu} \sum_j (1+x_j)^{n-2\nu} - (1-y_j)^{n-2\nu} - (-1)^n(1-x_j)^{n-2\nu} \\
 + (-1)^n(1+y_j)^{n-2\nu} = n \sum_j (1+x_j)^{n-1} - (1-y_j)^{n-1} \\
 - (-1)^n(1-x_j)^{n-1} + (-1)^n(1+y_j)^{n-1}
 \end{aligned}$$

$$\begin{aligned}
 (12.3) \quad \sum_{\nu=0}^{\lfloor n/2 \rfloor} R_{2\nu} \binom{n}{2\nu} \sum_j (1+x_j)^{n-2\nu} - (1-y_j)^{n-2\nu} - (-1)^n(1-x_j)^{n-2\nu} \\
 + (-1)^n(1+y_j)^{n-2\nu} = n \sum_j x_j^{n-1} + (-1)^n y_j^{n-1}.
 \end{aligned}$$

Again we set $pq = 2$ in (1), and choose n in (12.1) and (12.3) of the same parity as r in (1). Finally we choose for x_j the set of 2^{r-3} quantities

$$x_j = \epsilon + \eta_2 \epsilon^2 + \eta_3 \epsilon^3 + \dots + \eta_{r-1} \epsilon^{r-1} \quad \text{where } \eta_2 \eta_3 \dots \eta_{r-1} = +1,$$

and for y_j the 2^{r-3} quantities

$$y_j = -\epsilon + \eta_2 \epsilon^2 + \eta_3 \epsilon^3 + \dots + \eta_{r-1} \epsilon^{r-1} \quad \text{where } \eta_2 \eta_3 \dots \eta_{r-1} = (-1)^{r-1}.$$

With these choices it is seen that

$$\sum_j (1+x_j)^\mu - (1-y_j)^\mu - (-1)^n(1-x_j)^\mu + (-1)^n(1+y_j)^\mu = \sigma_\mu(p, q, r, r, 1)$$

and

$$\sum_j x_j^{n-1} + (-1)^n y_j^{n-1} = e^{2\pi i(n-1)/pr} \sigma_{n-1}(p, q, r, r-1, 1).$$

Applying Theorem A and writing $n - r = 2m$, we have for $p = 2, q = 1$

$$\begin{aligned}
 (13.1) \quad \sum_{\lambda=0}^{\lfloor m/r \rfloor} 2^{2m-2\lambda r} B_{2m-2\lambda r} \binom{2m+r}{2\lambda r+r} \sigma_{2\lambda r+r}(2, 1, r, r, 1) \\
 = (2m+r) \sigma_{2m+r-1}(2, 1, r, r, 1),
 \end{aligned}$$

$$\begin{aligned}
 (13.3) \quad \sum_{\lambda=0}^{\lfloor m/r \rfloor} R_{2m-2\lambda r} \binom{2m+r}{2\lambda r+r} \sigma_{2\lambda r+r}(2, 1, r, r, 1) \\
 = (2m+r) e^{\pi i(2m+r-1)/r} \sigma_{2m+r-1}(2, 1, r, r-1, 1).
 \end{aligned}$$

If r is odd we may again interchange p and q without altering (13.1) or (13.3). For $r = 2h$, however, Theorem A gives us

$$(14.1) \quad \sum_{\lambda=0}^{\lfloor m/h \rfloor} 2^{2m-2\lambda h} B_{2m-2\lambda h} \binom{2m+2h}{2\lambda h+2h} \sigma_{2\lambda h+2h}(1, 2, 2h, 2h, 1) = 2(m+h)\sigma_{2m+2h-1}(1, 2, 2h, 2h, 1)$$

$$(14.3) \quad \sum_{\lambda=0}^{\lfloor m/h \rfloor} R_{2m-2\lambda h} \binom{2m+2h}{2\lambda h+2h} \sigma_{2\lambda h+2h}(1, 2, 2h, 2h, 1) = e^{\pi i(2m-1)/h} 2(m+h)\sigma_{2m+2h-1}(1, 2, 2h, 2h-1, 1).$$

3. Explicit recurrences. In order to obtain practical recurrences we have only to evaluate or to give some effective method for calculating the various σ 's that appear in equations 10 to 14. For small values of r or h this is simple enough. But it is very desirable to have large gaps, which means large values of r and h . The labor of calculating the σ 's increases very rapidly, so that a practical limit to the size of the gaps is soon reached. The most practical recurrences have gaps of 12. We subjoin a few explicit recurrences which may be derived from equations 10 to 14, by evaluating the σ 's and simplifying.

From (10) and (13) we obtain for $r = 2$ the following recurrences with gaps of 4:

$$(15) \quad \begin{aligned} \sum_{\lambda=0}^{\lfloor m/2 \rfloor} (-1)^\lambda 2^{m-2\lambda} B_{2m-4\lambda} \binom{2m+2}{4\lambda+2} &= (-1)^{\lfloor \frac{m}{2} \rfloor} \frac{m+1}{2}, \\ \sum_{\lambda=0}^{\lfloor m/2 \rfloor} (-1)^\lambda 2^{m-2\lambda} G_{2m-4\lambda} \binom{2m}{4\lambda} &= (-1)^{\lfloor \frac{m+2}{2} \rfloor} 2m, \\ \sum_{\lambda=0}^{\lfloor m/2 \rfloor} (-1)^\lambda 2^{2\lambda} R_{2m-4\lambda} \binom{2m+2}{4\lambda+2} &= (-1)^m \frac{m+1}{2}, \\ \sum_{\lambda=0}^{\lfloor m/2 \rfloor} (-1)^\lambda 2^{2\lambda} E_{2m-4\lambda} \binom{2m}{4\lambda} &= (-1)^m. \end{aligned}$$

In order to obtain recurrences with gaps of 4 from equations (11) and (14) we set $h = 2$, and get

$$\begin{aligned} \sum_{\lambda=0}^{\lfloor m/2 \rfloor} B_{2m-4\lambda} \binom{2m+4}{4\lambda+4} ((-1)^\lambda 2^{2\lambda+1} + 1) &= \frac{m+2}{2} \left((-1)^{\lfloor \frac{m}{2} \rfloor} 2^{m+1} + 1 \right) \\ 2G_{2m} + \sum_{\lambda=1}^{\lfloor m/2 \rfloor} G_{2m-4\lambda} \binom{2m}{4\lambda} ((-1)^\lambda 2^{2\lambda-1} + 1) &= -m \left((-1)^{\lfloor \frac{m}{2} \rfloor} 2^{m-1} + 1 \right) \end{aligned}$$

$$\sum_{\lambda=0}^{\lfloor m/2 \rfloor} R_{2m-4\lambda} \binom{2m+4}{4\lambda+4} 2^{4\lambda+4} ((-1)^\lambda 2^{2\lambda+1} + 1) = (m+2)(c_m + 1)$$

$$2E_{2m} + \sum_{\lambda=1}^{\lfloor m/2 \rfloor} E_{2m-4\lambda} \binom{2m}{4\lambda} 2^{4\lambda} ((-1)^\lambda 2^{2\lambda-1} + 1) = c'_m + 1,$$

where $c_0 = 11$, $c_1 = -41$, and $c_n = -6c_{n-1} - 25c_{n-2}$
 $c'_0 = 1$, $c'_1 = -3$, and $c'_n = -6c'_{n-1} - 25c'_{n-2}$.

It is evident that these last recurrences are more complicated than (15). The same is true and to a greater extent for larger gaps. We shall not give space to further examples of equations (11) and (14).

Recurrences with gaps of 6 are obtained from (10) and (13) putting $r = 3$, and are as follows:

$$\sum_{\lambda=0}^{\lfloor m/3 \rfloor} B_{2m-6\lambda} \binom{2m+3}{6\lambda+3} = \begin{cases} -\frac{2m+3}{6} & \text{if } m = 3k - 1. \\ \frac{2m+3}{3} & \text{otherwise.} \end{cases}$$

$$4G_{2m} + 3 \sum_{\lambda=1}^{\lfloor m/3 \rfloor} G_{2m-6\lambda} \binom{2m}{6\lambda} = \begin{cases} 2m & \text{if } m = 3k - 1. \\ -4m & \text{otherwise.} \end{cases}$$

$$\sum_{\lambda=0}^{\lfloor m/3 \rfloor} R_{2m-6\lambda} \binom{2m+3}{6\lambda+3} 2^{6\lambda} = \frac{2m+3}{24} ((-1)^m 3^{m+1} + 1).$$

$$E_{2m} + 3 \sum_{\lambda=1}^{\lfloor m/3 \rfloor} E_{2m-6\lambda} \binom{2m}{6\lambda} 2^{6\lambda} = \frac{1}{2} ((-1)^m 3^m + 1).$$

For $r > 3$, $\sigma_n(2, 1, r, s, t)$ can no longer be expressed explicitly without introducing powers of irrationalities.¹² The practical method of evaluating these expressions is to resort to linear recurring series. Hence in the recurrences with larger gaps, the σ 's are given as recurring series, which, after all, are nearly as easy to work with as powers of an integer. Thus for $r = 4$, the following recurring series are used.

n	α_n	α'_n	f_n	f'_n	g_n	g'_n
0	2	2	1	1	-1	1
1	1	0	-7	-3	-5	1
2	1	3	41	17	-1	-7

¹² Thus for g_n we have $-2g_n = (5 + \sqrt{-2})(1 - \sqrt{-8})^{n-1} + (5 - \sqrt{-2})(1 + \sqrt{-8})^{n-1}$.

3	-4	10	-239	-99	43	-23
4	-34	14	1393	577	95	17
5	-29	-12	-8119	-3363	-197	241
6	-41	-99	47321	19601	-1249	329
7	140	-338	-275807	-114243	-725	-1511

α and α' satisfy the recurrence $U_n = -34U_{n-4} - U_{n-8}$

f and f' satisfy the recurrence $U_n = -6 U_{n-1} - U_{n-2}$

g and g' satisfy the recurrence $U_n = 2 U_{n-1} - 9U_{n-2}$.

With these definitions we have the following recurrences with gaps of 8:

$$\sum_{\lambda=0}^{\lfloor m/4 \rfloor} B_{2m-8\lambda} \binom{2m+4}{8\lambda+4} 2^{m+1-2\lfloor \frac{m+1}{4} \rfloor - 2\lambda} \alpha'_{4\lambda+2} = (-1)^{\lfloor \frac{m}{2} \rfloor} (m+2) \alpha'_{m-2}$$

$$\sum_{\lambda=0}^{\lfloor m/4 \rfloor} G_{2m-8\lambda} \binom{2m}{8\lambda} 2^{m-2\lfloor \frac{m+3}{4} \rfloor - 2\lambda} \alpha_{4\lambda} = (-1)^{\lfloor \frac{m+2}{4} \rfloor} m \alpha_m$$

$$\sum_{\lambda=0}^{\lfloor m/4 \rfloor} R_{2m-8\lambda} \binom{2m+4}{8\lambda+4} 2^{6\lambda+4} \alpha'_{4\lambda+2} = -(m+2)(f_{m+1} + g_{m+1})$$

$$\sum_{\lambda=0}^{\lfloor m/4 \rfloor} E_{2m-8\lambda} \binom{2m}{8\lambda} 2^{6\lambda} \alpha_{4\lambda} = f'_m + g'_m.$$

The recurrences with gaps of 12 are given explicitly in terms of the following 8 recurring series:

n	β_n	β'_n	n	u_n	v_n	w_n
0	1	2	0	2	2	2
1	5	5	1	-6	2	10
2	26	7	2	10	-70	90
3	97	-26	3	306	-1078	1234
4	265	-265	4	-4846	-2446	18290
5	362	-1351				
6	-1351	-5042	n	u'_n	v'_n	w'_n
7	-13775	-13775	0	-1	1	0
8	-70226	-18817	1	11	1	12
9	-262087	70226	2	-101	-179	200

10	-716035	716035	3	559	-1259	2660
11	-978122	3650401	4	119	7129	35472

β and β' satisfy $U_n = -2702U_{n-6} - U_{n-12}$

u and u' satisfy $U_n = -12 U_{n-1} - 62 U_{n-2} + 36 U_{n-3} - 169 U_{n-4}$

v and v' satisfy $U_n = 4 U_{n-1} - 78 U_{n-2} - 428U_{n-3} - 1369U_{n-4}$

w and w' satisfy $U_n = 20 U_{n-1} - 110U_{n-2} + 356U_{n-3} - 25 U_{n-4}$.

We have then the following recurrences

$$\begin{aligned}
 & \sum_{\lambda=0}^{\lfloor m/6 \rfloor} B_{2m-12\lambda} \binom{2m+6}{12\lambda+6} (\beta_{6\lambda+2} + (-1)^\lambda 2^{6\lambda+2}) \\
 (15a) \quad & = \begin{cases} -\frac{m+3}{6} \left(\beta_{m+2} + (-1)^{\lfloor \frac{m}{2} \rfloor} 2^{m+2} - (-1)^{\frac{m+1}{3}} 3 \right) & \text{if } m \equiv 2 \pmod{3} \\ \frac{m+3}{3} \left(\beta_{m+2} + (-1)^{\lfloor \frac{m}{2} \rfloor} 2^{m+2} \right) & \text{if } m \not\equiv 2 \pmod{3} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 8 G_{2m} + 3 \sum_{\lambda=1}^{\lfloor m/6 \rfloor} G_{2m-12\lambda} \binom{2m}{12\lambda} (1 + \beta_{6\lambda} + (-1)^\lambda 2^{6\lambda-1}) \\
 = \theta m \left(1 + \beta'_{m-1} + (-1)^{\lfloor \frac{m}{2} \rfloor} 2^{m-1} \right),
 \end{aligned}$$

where θ is equal to 1 or -2 according as $m \equiv 2 \pmod{3}$ or not.

$$\begin{aligned}
 3 \sum_{\lambda=0}^{\lfloor m/6 \rfloor} R_{2m-12\lambda} \binom{2m+6}{12\lambda+6} 2^{12\lambda+5} (\beta_{6\lambda+2} + (-1)^\lambda 2^{6\lambda+2}) \\
 = (-1)^{m-1} (m+3) (u'_{m+2} + v'_{m+2} - w'_{m+2}).
 \end{aligned}$$

$$\begin{aligned}
 8 E_{2m} + 3 \sum_{\lambda=1}^{\lfloor m/6 \rfloor} E_{2m-12\lambda} \binom{2m}{12\lambda} 2^{12\lambda-1} (1 + \beta_{6\lambda} + (-1)^\lambda 2^{6\lambda-1}) \\
 = (-1)^m (3^m + (-1)^m + u_m + v_m + w_m).
 \end{aligned}$$

Recurrences with gaps of 14 or more are not so practical as those given above. We have worked out those with gaps of 16 (the next simplest case after 12), but these involve 8th order recurring series whose scales of relation have very large coefficients exceeding 10¹⁵ in most cases.

Replacing G_n and R_n by

$$G_n = 2(1 - 2^n)B_n$$

$$R_n = (1 - 2^{n-1})B_n$$

in the above recurrences, we obtain further formulas for B_n .

In connection with the calculation of B_n , it is well to point out that Adams' method of using the von Staudt-Clausen theorem to eliminate fractions, can be adapted to any recurrence for B_n whatever, having integer coefficients. In fact, we make the substitution¹³

$$(16) \quad B_{2n} = A_n - \sum 1/p$$

where the sum extends to all primes p for which $p - 1$ divides $2n$. Then A_n is an integer and may be considered as the unknown part of B_{2n} . Now if we substitute (16) in any recurrence, we may combine and transpose to the right all terms arising from $1/p$ for each p involved. In this way it is clear that we obtain a series of fractions whose denominators are distinct primes and whose sum is an integer. Hence these fractions are actually integers¹⁴ and the recurrences reduce to operations with whole numbers.

To illustrate this process let us consider the computation of B_{196} using (15a). For brevity we write

$$(-1)^\lambda \beta_{6\lambda+2} + (-1)^\lambda 2^{6\lambda+2} = c_\lambda,$$

$$c_0 = 30, c_1 = 70482, c_2 = 189767010, \dots$$

Setting $m = 98$ in (15a) we obtain at once

$$\sum_{\lambda=0}^{16} (-1)^\lambda B_{196-12\lambda} \binom{202}{12\lambda+6} c_\lambda = -\frac{101}{6} (\beta_{100} - 2^{100} + 3).$$

Writing

$$\binom{202}{12\lambda+6} c_\lambda = T_\lambda$$

and using (16) we have

$$(17) \quad \sum_{\lambda=0}^{16} (-1)^\lambda A_{98-6\lambda} T_\lambda = -\frac{101}{6} (\beta_{100} - 2^{100} + 3) + \sum_p 1/p \sum_\nu T_\nu (-1)^\nu,$$

where $p = 2, 3, 5, \dots$ are primes and ν are those solutions of the congruence

$$(18) \quad 196 - 12\nu \equiv 0 \pmod{p-1}$$

for which $0 \leq \nu \leq 16$. From (18) we see that $p \leq 197$ and that if $p > 3$, p is of the form $6x + 5$. It is easy to verify that $\beta_{100} - 2^{100} + 3$ is a multiple of 6.

¹³ Adams (loc. cit.) wrote the equivalent of $I_n = (-1)^{n-1} (A_n - 1)$ so that $I_\nu = 0$ for $\nu \leq 6$. For lacunary recurrences this artifice is of little use.

¹⁴ This fact leads at once to interesting congruences modulo p involving the coefficients of the recurrence.

Hence each of the terms $1/p \sum T_r(-1)^r$ is an integer. In fact, the actual terms are as follows:

$$\sum_p 1/p \sum T_r(-1)^r = \frac{31}{30} \sum_{r=0}^{16} T_r(-1)^r + (-T_{13} + T_8 - T_3)/11$$

$$- (T_{15} + T_{11} + T_7 + T_3)/17 - T_9/23 + (T_{14} - T_7 + T_0)/29$$

$$- (T_{13} + T_3)/41 - T_1/47 + T_{12}/53 - T_9/89 + T_8/101 - T_7/113$$

$$- T_5/137 + T_4/149 + T_2/173 + T_0/197.$$

The fractional part of B_{196} by (16) is $183883/171390$. Calculating A_{98} from (17) with the aid of the tables of Adams and Cerébrenikoff, and subtracting the above fractional part, we obtain the following value for B_{196} :

Numerator								Denominator	
-62753	13511	04611	93672	55310	66998	93713	60315	}	171390.
30541	53311	89530	55906	39107	01782	46402	41378		
48048	46255	54578	57614	21158	35788	96086	55345		
32214	56098	29255	49798	68376	27052	31316	61171		
66687	49347	22145	80056	71217	06735	79434	16524		
98443	87718	31113							

This is the first instance of an isolated entry in the table of Bernoulli numbers.

4. Application of Theorem A to other sets of numbers. The four sequences of numbers B, G, R, E are special cases, for $h = 2$, of a set of h^2 sequences of rational numbers which are the coefficients of the power series developments of the h reciprocals and the $h(h - 1)$ ratios of the h functions of Olivier¹⁵

$$\varphi_\nu(z) = \frac{z^\nu}{\nu!} + \frac{z^{\nu+h}}{(\nu+h)!} + \frac{z^{\nu+2h}}{(\nu+2h)!} + \dots \quad (\nu = 0, 1, 2, \dots, h-1).$$

The detailed account of these h^2 sequences will appear elsewhere. We illustrate here merely the application of Theorem A to one of the 9 sequences associated with $h = 3$. This sequence of integers (the counterpart of E) we designate by W and define as follows:

$$W_0 = 1, \quad (W + 1)^n + (W + \omega)^n + (W + \omega^2)^n = 0,$$

where $\omega = e^{2\pi i/3}$. It follows as in (6.4) that

$$(19) \quad f(W + x + 1) + f(W + x + \omega) + f(W + x + \omega^2) = 3f(x).$$

¹⁵ Journ. für Math. 2 (1827), 243-251.

Setting $x = 0$ and $f(t) = e^{zt}$ we have as the counterpart of (5.4),

$$\frac{3}{e^z + e^{\omega z} + e^{\omega^2 z}} = e^{Wz} = \frac{1}{\varphi_0(z)},$$

from which we see that W_n is zero if n is not a multiple of 3. The first few non-zero values of W are

$$W_0 = 1, \quad W_3 = -1, \quad W_6 = 19, \quad W_9 = -1513, \quad W_{12} = 315523.$$

Next set $f(t) = t^{3k}$ in (19) and let x range over a set x_j . Expanding and summing we obtain as in (8.4)

$$\sum_{\lambda=0}^k W_{3k-3\lambda} \binom{3k}{3\lambda} \sum (1+x_j)^{3\lambda} + (1+\omega x_j)^{3\lambda} + (1+\omega^2 x_j)^{3\lambda} = 3 \sum x_j^{3k}.$$

In order to apply Theorem A we let x_j run through the values

$$x_j = \epsilon + \eta_2 \epsilon^2 + \eta_3 \epsilon^3 + \dots + \eta_{r-1} \epsilon^{r-1},$$

where each $\eta_r = 1, \omega, \omega^2$. We have then the general lacunary recurrence

$$\sum_{\lambda=0}^{\lfloor k/r \rfloor} W_{3k-3\lambda r} \binom{3k}{3\lambda r} \sigma_{3\lambda r}(p, q, r, r, 0) = 3e^{6k\pi i/pr} \sigma_{3k}(p, q, r, r-1, 0),$$

where $pq = 3$. Setting $r = 2$ we have the following example with $p = 3, q = 1$:

$$W_{3k} + 2 \sum_{\lambda=1}^{\lfloor k/2 \rfloor} W_{3k-6\lambda} \binom{3k}{6\lambda} (-1)^\lambda 3^{3\lambda-1} = (-1)^k.$$

PRINCETON, NEW JERSEY.