Congruences for higher-order Euler numbers

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Abstract: In this paper, we prove some congruences for higher-order Euler numbers.

Key words: Higher-order Euler numbers; Euler numbers, congruences.

1. Introduction and results. For an integer k, the Euler number $E_{2n}^{(k)}$ of order k (the index k may be negative) is defined by (see [2, 5])

$$(\sec x)^k = \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n}}{(2n)!}, \tag{1.1}$$

or equivalently

$$\left(\frac{2}{e^x + e^{-x}}\right)^k = \sum_{n=0}^{\infty} E_{2n}^{(k)} \frac{x^{2n}}{(2n)!}. \tag{1.2}$$

The numbers $E_{2n}^{(1)} = E_{2n}$ are the ordinary Euler numbers. By (1.1) or (1.2), we can get

$$E_{2n}^{(k)} = (2n)! \sum_{\substack{v_1+v_2+\cdots+v_k=n \\ v_1\geq 0, v_2\geq 0,\ldots,v_k\geq 0}} \frac{E_{2v_1}E_{2v_2}\cdots E_{2v_k}}{(2v_1)!(2v_2)!\cdots(2v_k)!} \tag{1.3}$$

when k is positive.

The Euler numbers E_{2n} satisfy the recurrence relation

$$E_0 = 1, \quad E_{2n} = -\sum_{k=0}^{n-1} \binom{2n}{2k} E_{2k}. \tag{1.4}$$

By induction, all the Euler numbers $E_0, E_2, E_4, \ldots$ are integers. By (1.3), we know $E_{2n}^{(k)}$ is an integer.

Recently, several researchers have studied the congruences for Euler numbers. For example:

In [7], Wenpeng Zhang obtained an interesting congruence for Euler numbers,

$$E_{p-1} \equiv 1 + (-1)^{(p+1)/2} \pmod{p}. \tag{1.5}$$

where p is any odd prime.

In [4], Guodong Liu obtained an congruence for Euler numbers,

$$\sum_{k=1}^{(p-1)/2} E_{2n+2k} \equiv -1 \pmod{p}. \tag{1.6}$$

where $n \geq 0$ is integer and p is any odd prime.

The following conjecture is on Euler numbers (see [1] B45).

Conjecture. *For any prime* $p \equiv 1 \pmod 4$, *the congruence* $E_{(p-1)/2} \not\equiv 0 \pmod p$ *holds*.

In [3], Guodong Liu proved the conjecture for $p \equiv 5 \pmod 8$. In [6], Pingzhi Yuan, using a result of [3] and the class number formula for the quadratic field with negative discriminant, proved the above conjecture.

The main purpose of this paper is to prove some congruences for higher-order Euler numbers. That is, we shall prove the following main conclusion.

Theorem 1. *Let* $n \geq 0, r \geq 3$ *be integers*, p *be any odd prime. If* $r \equiv 2k+1 \pmod p$, *where* $1 \leq k \leq (p-1)/2$. *Then*

$$\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(r)} \equiv 0 \pmod{p}. \tag{1.7}$$

Theorem 2. *Let* $n \geq 0, r \geq 2$ *be integers*, p *be any odd prime. If* $r \equiv 2k+2 \pmod p$, *where* $0 \leq k \leq (p-1)/2$. *Then*

$$\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(r)} \equiv \frac{(-1)^{(p+1)/2}}{2^{2k}} \binom{2k}{k} \pmod{p}. \tag{1.8}$$

Taking $r = 2$ in Theorem 2, we may immediately deduce the following

Corollary 1. *Let* $n \geq 0$ *be any integers*, p *be any odd prime. Then we have*

$$\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(2)} \equiv (-1)^{(p+1)/2} \pmod{p}. \tag{1.9}$$

Setting $p = 3, 5, 7, 11$ in Corollary 1, we can get

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(1.10) $E_{2n+2}^{(2)} \equiv 1 \pmod 3$.

(1.11) $E_{2n+2}^{(2)} + E_{2n+4}^{(2)} \equiv -1 \pmod 5$.

(1.12) $E_{2n+2}^{(2)} + E_{2n+4}^{(2)} + E_{2n+6}^{(2)} \equiv 1 \pmod 7$.

(1.13)
$$E_{2n+2}^{(2)} + E_{2n+4}^{(2)} + E_{2n+6}^{(2)} + E_{2n+8}^{(2)} + E_{2n+10}^{(2)} \equiv 1 \pmod{11}.$$

2. Some lemmas.

Lemma 1. *Let $n \geq 1$, $k \geq 1$ be integers. Then we have*

$$E_{2n}^{(k)} \equiv 0 \pmod k. \tag{2.1}$$

Proof. By (1.1), we have

$$\sum_{n=1}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n-1}}{(2n-1)!} = k(\sec x)^k \tan x. \tag{2.2}$$

By $(\sec x)^2 = \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(2)} (x^{2n}/((2n)!)) = \sum_{n=1}^{\infty} (-1)^{n-1} E_{2n-2}^{(2)} (x^{2n-2}/((2n-2)!))$, we get

$$\tan x = \sum_{n=1}^{\infty} (-1)^{n-1} E_{2n-2}^{(2)} \frac{x^{2n-1}}{(2n-1)!}. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n-1}}{(2n-1)!} &= k \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n}}{(2n)!} \times \sum_{n=1}^{\infty} (-1)^{n-1} E_{2n-2}^{(2)} \frac{x^{2n-1}}{(2n-1)!} \\ &= k \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{i=0}^{n-1} \binom{2n-1}{2i} \times E_{2i}^{(k)} E_{2n-2i-2}^{(2)} \frac{x^{2n-1}}{(2n-1)!}. \end{aligned} \tag{2.4}$$

Comparing the coefficients of x^{2n-1} on both sides of (2.4), we get

$$E_{2n}^{(k)} = -k \sum_{i=0}^{n-1} \binom{2n-1}{2i} E_{2i}^{(k)} E_{2n-2i-2}^{(2)} \equiv 0 \pmod k. \tag{2.5}$$

This completes the proof of Lemma 1. □

Lemma 2. *Let $n \geq 0, k \geq 1, m \geq 1$ be integers. Then we have*

$$E_{2n}^{(k+m)} \equiv E_{2n}^{(k)} \pmod m. \tag{2.6}$$

Proof. By (1.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k+m)} \frac{x^{2n}}{(2n)!} &= (\sec x)^{k+m} \\ &= (\sec x)^k (\sec x)^m \\ &= \left(\sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n}}{(2n)!} \right) \left(\sum_{n=0}^{\infty} (-1)^n E_{2n}^{(m)} \frac{x^{2n}}{(2n)!} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{n} \binom{2n}{2j} E_{2j}^{(k)} E_{2n-2j}^{(m)} \frac{x^{2n}}{(2n)!}. \end{aligned} \tag{2.7}$$

Comparing the coefficients of x^{2n} on both sides of (2.7), we get

$$\begin{aligned} E_{2n}^{(k+m)} &= \sum_{j=0}^{n} \binom{2n}{2j} E_{2n-2j}^{(k)} E_{2j}^{(m)} \\ &= E_{2n}^{(k)} + \sum_{j=1}^{n} \binom{2n}{2j} E_{2n-2j}^{(k)} E_{2j}^{(m)}. \end{aligned} \tag{2.8}$$

By (2.8) and Lemma 1, we have

$$E_{2n}^{(k+m)} \equiv E_{2n}^{(k)} \pmod m. \tag{2.9}$$

This completes the proof of Lemma 2. □

Lemma 3. *Let $n \geq 1, k \geq 1, m \geq 0$ be integers. Then we have*

$$E_{2n}^{(k)} \equiv \frac{1}{2^m} \sum_{i=0}^{m} \binom{m}{i} (m-2i)^{2n} \pmod{(m+k)}. \tag{2.10}$$

Proof. By (1.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n}}{(2n)!} &= (\sec x)^k \\ &= (\sec x)^{m+k} (\sec x)^{-m} \\ &= \left(\sum_{n=0}^{\infty} (-1)^n E_{2n}^{(m+k)} \frac{x^{2n}}{(2n)!} \right) \times \left(\sum_{n=0}^{\infty} (-1)^n E_{2n}^{(-m)} \frac{x^{2n}}{(2n)!} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{n} \binom{2n}{2j} E_{2j}^{(m+k)} E_{2n-2j}^{(-m)} \frac{x^{2n}}{(2n)!}. \end{aligned} \tag{2.11}$$

Comparing the coefficients of x^{2n} on both sides of (2.11), we get

$$E_{2n}^{(k)} = \sum_{j=0}^{n} \binom{2n}{2j} E_{2j}^{(m+k)} E_{2n-2j}^{(-m)} \tag{2.12}$$
$$= E_{2n}^{(-m)} + \sum_{j=1}^{n} \binom{2n}{2j} E_{2j}^{(m+k)} E_{2n-2j}^{(-m)}.$$

By (2.12) and Lemma 1, we have

$$E_{2n}^{(k)} \equiv E_{2n}^{(-m)} \pmod{(m+k)}. \tag{2.13}$$

On the other hand, by (1.2), we have

(2.14)
$$\sum_{n=0}^{\infty} E_{2n}^{(-m)} \frac{x^{2n}}{(2n)!} = \left(\frac{e^x + e^{-x}}{2}\right)^m$$
$$= 2^{-m} \sum_{i=0}^{m} \binom{m}{i} e^{(m-2i)x}$$
$$= 2^{-m} \sum_{i=0}^{m} \binom{m}{i} \sum_{n=0}^{\infty} (m-2i)^n \frac{x^n}{n!}.$$

Comparing the coefficients of x^{2n} on both sides of (2.14), we get

$$E_{2n}^{(-m)} = 2^{-m} \sum_{i=0}^{m} \binom{m}{i} (m-2i)^{2n}. \tag{2.15}$$

By (2.13) and (2.15), we immediately obtain (2.10). This completes the proof of Lemma 3. □

3. Proof of the theorems.

Proof of Theorem 1. By Lemma 2 and Lemma 3, we have

$$\begin{aligned}
\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(r)} &\equiv \sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(2k+1)} \\
&\equiv \frac{1}{2^{p-2k-1}} \sum_{j=0}^{p-2k-1} \binom{p-2k-1}{j} \\
&\quad \times \sum_{i=1}^{(p-1)/2} (p-2k-1-2j)^{2n+2i} \\
&\equiv \frac{1}{2^{p-2k-1}} \sum_{j=0}^{p-2k-1} \binom{p-2k-1}{j} \\
&\quad \times (p-2k-1-2j)^{2n+2} \\
&\quad \times \frac{(p-2k-1-2j)^{p-1}-1}{(p-2k-1-2j)^2-1} \equiv 0 \pmod{p}.
\end{aligned} \tag{3.1}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. By Lemma 2 and Lemma 3, we have

(3.2)
$$\begin{aligned}
&\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(r)} \equiv \sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(2k+2)} \\
&\equiv \frac{1}{2^{p-2k-2}} \sum_{j=0}^{p-2k-2} \binom{p-2k-2}{j} \\
&\quad \times \sum_{i=1}^{(p-1)/2} (p-2k-2-2j)^{2n+2i} \\
&= \frac{1}{2^{p-2k-2}} \sum_{\substack{j=0 \\ p-2k-2-2j=\pm 1}}^{p-2k-2} \binom{p-2k-2}{j} \\
&\quad \times \sum_{i=1}^{(p-1)/2} (p-2k-2-2j)^{2n+2i} \\
&\qquad + \frac{1}{2^{p-2k-2}} \sum_{\substack{j=0 \\ p-2k-2-2j\neq \pm 1}}^{p-2k-2} \binom{p-2k-2}{j} \\
&\quad \times \sum_{i=1}^{(p-1)/2} (p-2k-2-2j)^{2n+2i} \\
&= \frac{p-1}{2^{p-2k-1}} \\
&\quad \times \left(\binom{p-2k-2}{(p-2k-1)/2} + \binom{p-2k-2}{(p-2k-3)/2}\right) \\
&\quad + \frac{1}{2^{p-2k-2}} \sum_{\substack{j=0 \\ p-2k-2-2j\neq \pm 1}}^{p-2k-2} \binom{p-2k-2}{j} \\
&\quad \times (p-2k-2-2j)^{2n+2} \\
&\quad \times \frac{(p-2k-2-2j)^{p-1}-1}{(p-2k-2-2j)^2-1} \\
&\equiv -2^{2k} \left(\binom{p-2k-2}{(p-2k-1)/2} + \binom{p-2k-2}{(p-2k-3)/2}\right) \\
&\equiv -2^{2k} \binom{p-2k-1}{(p-2k-1)/2} \\
&\equiv \frac{(-1)^{(p+1)/2}}{2^{2k}} \binom{2k}{k} \pmod{p}.
\end{aligned}$$

This completes the proof of Theorem 2. □

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