

SURVEY PAPER

THE MELLIN INTEGRAL TRANSFORM IN FRACTIONAL CALCULUS

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*Dedicated to Professor Francesco Mainardi
on the occasion of his 70th anniversary*

Abstract

In Fractional Calculus (FC), the Laplace and the Fourier integral transforms are traditionally employed for solving different problems. In this paper, we demonstrate the role of the Mellin integral transform in FC. We note that the Laplace integral transform, the sin- and cos-Fourier transforms, and the FC operators can all be represented as Mellin convolution type integral transforms. Moreover, the special functions of FC are all particular cases of the Fox H -function that is defined as an inverse Mellin transform of a quotient of some products of the Gamma functions.

In this paper, several known and some new applications of the Mellin integral transform to different problems in FC are exemplarily presented. The Mellin integral transform is employed to derive the inversion formulas for the FC operators and to evaluate some FC related integrals and in particular, the Laplace transforms and the sin- and cos-Fourier transforms of some special functions of FC. We show how to use the Mellin integral transform to prove the Post-Widder formula (and to obtain its new modification), to derive some new Leibniz type rules for the FC operators, and to get new completely monotone functions from the known ones.

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Key Words and Phrases: fractional calculus, Mellin transform, Laplace transform, Fourier transform, H -function, Mittag-Leffler function, Wright function, Erdélyi-Kober derivatives and integrals, Post-Widder formula, Leibniz rules for FC operators, completely monotone functions

1. Introduction

Like in the classical Calculus and in (integer order) differential equations, the Laplace and the Fourier integral transforms are routinely employed in Fractional Calculus (FC) in general, and especially in fractional order differential equations. In some cases, this is inevitable. Thus the Riesz and the Riesz-Feller fractional derivatives are defined as pseudo-differential operators in terms of the Fourier and the inverse Fourier transforms (see e.g. [6], [9] or [25]). Another prominent example is given by the Laplace transform formulas for the Caputo and the Riemann-Liouville fractional derivatives that are routinely used in solving of fractional differential equations ([24], [25], [31]).

The aim of this paper is to demonstrate the role of the Mellin integral transform in FC and to show that applying the Mellin transform can essentially simplify some of the FC operations and derivations. Until now, the Mellin integral transform was only sporadically employed in the FC publications. We mention here e.g. the papers [11] and [25], where the Mellin integral transform was used to get a representation of the Green function for the space-time fractional diffusion equation in terms of the Mellin-Barnes integrals (Fox H -function) and to analyze its properties. In [19], [20], the mixed operators of the Erdélyi-Kober type were shown to be generating operators for the integral transforms of Mellin convolution type. In particular, the hyper-Bessel differential operator is a generating operator for the Obrechhoff transform and for the related generalized Hankel transform, see details in [1], [4], [15, Ch.3], [20], [23], etc. Leibniz type rules for several FC operators were deduced in [34], [36] by applying the technique of the Mellin integral transform. Of course, the Mellin integral transform was employed in connection with the FC special functions, like the Mittag-Leffler and the Wright functions and their generalizations. These functions are particular cases of the Fox H -function that can be interpreted as an inverse Mellin transform (see e.g. [15], [16], [18], [20], [22], [25], to refer to only few of many publications). Finally, we mention the book [36], where a theory of the integral transforms of the Mellin convolution type and some of its applications, also in connection with the FC operators, was presented. Some of the examples we deal with in this paper are motivated by [36].

In this paper, we first emphasize that many of the FC operators like the Riemann-Liouville and the Erdélyi-Kober derivatives and integrals can be interpreted as Mellin convolution type integral transforms. Using this

interpretation, the inversion formulas as well as some important formulas for the compositions of the FC operators can be easily deduced. Moreover, the Laplace integral transform and the sin- and cos-Fourier transforms are all of Mellin convolution type, too. This fact allows us, among many other things, to evaluate the Laplace and the sin- and cos-Fourier transforms of the FC special functions in a unified manner. Further applications of the Mellin integral transform in FC we deal with in this paper are evaluation of some FC related integrals and derivation of important formulas like the Post-Widder inversion formula for the Laplace transform and the Leibniz type rules for the FC operators. By means of the Mellin integral transform, we can even easily prove that some of the special functions of FC are completely monotone. Note that the above mentioned list of applications of the Mellin transform in FC is far from being complete, with many others contained in our works and in the works of different authors. By this survey paper we hope to make the Mellin transform more familiar to the FC researchers and to activate its usage in FC. Because the primary aim of this paper is to demonstrate how to use the Mellin transform for FC problems, we often do manipulations with integrals, limits, series, etc. without a rigorous mathematical justification for the following reasons: On the one hand, once a final formula is deduced, the necessary justification can be often easily found in the suitable spaces of functions. On the other hand, all manipulations with the Mellin integral transform we present in this paper are valid in the special functional spaces $\mathfrak{M}_{c,\gamma}^{-1}(L)$ that we do not introduce to make the paper understandable for FC people working in applications, too (see [36] for more details).

The remainder of this paper is organized as follows. In Section **2**, basic definitions and properties of the Mellin integral transform are given. In particular, we specify the Mellin integral transforms of some important FC special functions. Section **3** is devoted to representations of the FC operators in the form of Mellin convolution operators. These representations are explored for inversion of the FC integral operators and for construction of their compositions. In Section **4**, a rather general method for evaluation of some improper integrals is discussed. In particular, this method is applied for evaluation of the Laplace and the sin- and cos-Fourier transforms of some special functions of FC. In Section **5**, a new method for derivation of the Post-Widder formula by means of the Mellin transform is presented. With this method, a new real-line inversion formula of the Post-Widder type is deduced. Section **6** presents a very powerful method for derivation of the Leibniz type rules for the FC operators and for other operators of

Mellin convolution type. Finally, in Section 7, the Mellin integral transform is employed to give a new proof that some of the FC special functions are completely monotone functions.

2. Basic definitions and properties of the Mellin transform

This section is devoted to a presentation of some basic facts from the theory of the Mellin integral transform that are used in the further discussions. For more information regarding the Mellin integral transform including its properties and particular cases we refer the interested reader to e.g. [2], [3], [5], [15], [20], [28], [30], [32], and [35].

The *Mellin integral transform* of a sufficiently well-behaved function f is defined as

$$\mathfrak{M}\{f(t); s\} = f^*(s) = \int_0^{+\infty} f(t)t^{s-1}dt, \quad (2.1)$$

and the *inverse Mellin integral transform* as

$$f(t) = \mathfrak{M}^{-1}\{f^*(s); t\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)t^{-s}ds, \quad t > 0, \quad \gamma = \Re(s), \quad (2.2)$$

where the integral is understood in the sense of the Cauchy principal value.

It is worth mentioning that the Mellin integral transform can be obtained from the Fourier integral transform by the variables substitution $t = e^x$ and by rotation of the complex plane by a right angle:

$$\mathfrak{M}\{f(t); s\} = \int_0^{+\infty} f(t)t^{s-1}dt = \int_{-\infty}^{+\infty} f(e^x)e^{ix(-is)}dx = \mathfrak{F}\{f(e^x); -is\},$$

where $\mathfrak{F}\{f(x); \kappa\}$ denotes the Fourier transform of the function f at the point κ . Accordingly, the inverse Mellin transform and the convolution for the Mellin transform can be obtained by the same substitutions from the inverse Fourier transform and the convolution for the Fourier transform.

The integral in the right-hand side of (2.1) is well defined e.g. for the functions $f \in L^c(\epsilon, E)$, $0 < \epsilon < E < \infty$ continuous in the intervals $(0, \epsilon]$, $[E, +\infty)$ and satisfying the estimates $|f(t)| \leq M t^{-\gamma_1}$ for $0 < t < \epsilon$, $|f(t)| \leq M t^{-\gamma_2}$ for $t > E$, where M is a constant and $\gamma_1 < \gamma_2$. If these conditions hold true, the Mellin transform $f^*(s)$ exists and is an analytical function in the vertical strip $\gamma_1 < \Re(s) < \gamma_2$.

If a function f is piecewise differentiable, $f(t)t^{\gamma-1} \in L^c(0, +\infty)$, and its Mellin integral transform $f^*(s)$ is given by (2.1) then the formula (2.2) holds true at all points, where the function f is continuous.

The *Mellin convolution*

$$(f \underset{*}{\mathfrak{M}} g)(x) = \int_0^{+\infty} f(x/t)g(t)\frac{dt}{t} \quad (2.3)$$

plays a very essential role in the further discussions. It is well known (see e.g. [35]) that if $f(t)t^{\gamma-1} \in L(0, \infty)$ and $g(t)t^{\gamma-1} \in L(0, \infty)$ then the Mellin convolution $h = (f * g)$ given by (2.3) is well defined, satisfies the important property

$$\mathfrak{M} \left\{ (f * g)(x); s \right\} = \mathfrak{M} \{f(t); s\} \cdot \mathfrak{M} \{g(t); s\}, \tag{2.4}$$

and $h(x)x^{\gamma-1} \in L(0, \infty)$. Moreover, the Parseval equality

$$\int_0^{+\infty} f(x/t)g(t)\frac{dt}{t} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)g^*(s)x^{-s}ds \tag{2.5}$$

holds true.

In the further discussions, we often use some of the elementary properties of the Mellin integral transform that are summarized in the remainder of this section.

Denoting by $\overset{\mathfrak{M}}{\leftrightarrow}$ the juxtaposition of a function f with its Mellin transform f^* , the main rules are:

$$f(at) \overset{\mathfrak{M}}{\leftrightarrow} a^{-s}f^*(s), \quad a > 0, \tag{2.6}$$

$$t^\alpha f(t) \overset{\mathfrak{M}}{\leftrightarrow} f^*(s + \alpha), \tag{2.7}$$

$$f(t^\alpha) \overset{\mathfrak{M}}{\leftrightarrow} \frac{1}{|\alpha|} f^*(s/\alpha), \quad \alpha \neq 0, \tag{2.8}$$

$$f^{(n)}(t) \overset{\mathfrak{M}}{\leftrightarrow} \frac{\Gamma(n + 1 - s)}{\Gamma(1 - s)} f^*(s - n) \tag{2.9}$$

if $\lim_{t \rightarrow 0} t^{s-k-1} f^{(k)}(t) = 0, \quad k = 0, 1, \dots, n - 1,$

$$\prod_{j=1}^n \left(\alpha + j + \frac{1}{\beta} t \frac{d}{dt} \right) f(t) \overset{\mathfrak{M}}{\leftrightarrow} \frac{\Gamma(1 + \alpha + n - s/\beta)}{\Gamma(1 + \alpha - s/\beta)} f^*(s), \tag{2.10}$$

$$\prod_{j=0}^{n-1} \left(\alpha + j - \frac{1}{\beta} t \frac{d}{dt} \right) f(t) \overset{\mathfrak{M}}{\leftrightarrow} \frac{\Gamma(\alpha + n + s/\beta)}{\Gamma(\alpha + s/\beta)} f^*(s). \tag{2.11}$$

Let us mention that the variables substitution $t = 1/\tau$ in the Parseval equality (2.5) along with the properties (2.7), (2.8) of the Mellin integral transform leads to a very useful representation

$$\int_0^{+\infty} f(xt)g(t)dt = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)g^*(1-s)x^{-s}ds. \tag{2.12}$$

The Mellin transforms of the elementary and many of the special functions can be found e.g. in [5], [28], [30], and [32]. We specify here the Mellin

transform formulas that are used in the further discussions, namely:

$$e^{-t^\alpha} \stackrel{\mathfrak{M}}{\Leftrightarrow} \frac{1}{|\alpha|} \Gamma(s/\alpha) \text{ if } \Re(s/\alpha) > 0, \quad (2.13)$$

$$\frac{(1-t^\beta)_+^{\alpha-1}}{\Gamma(\alpha)} \stackrel{\mathfrak{M}}{\Leftrightarrow} \frac{\Gamma(s/\beta)}{|\beta|\Gamma(s/\beta+\alpha)} \text{ if } \Re(\alpha) > 0, \Re(s/\beta) > 0, \quad (2.14)$$

$$\frac{(t^\beta-1)_+^{\alpha-1}}{\Gamma(\alpha)} \stackrel{\mathfrak{M}}{\Leftrightarrow} \frac{\Gamma(1-\alpha-s/\beta)}{|\beta|\Gamma(1-s/\beta)} \text{ if } 0 < \Re(\alpha) < 1 - \Re(s/\beta), \quad (2.15)$$

$$\frac{\sin(2\sqrt{t})}{\sqrt{\pi}} \stackrel{\mathfrak{M}}{\Leftrightarrow} \frac{\Gamma(s+1/2)}{\Gamma(1-s)} \text{ if } |\Re(s)| < 1/2, \quad (2.16)$$

$$\frac{\cos(2\sqrt{t})}{\sqrt{\pi}} \stackrel{\mathfrak{M}}{\Leftrightarrow} \frac{\Gamma(s)}{\Gamma(1/2-s)} \text{ if } 0 < \Re(s) < 1/2, \quad (2.17)$$

$$E_{\alpha,\beta}(-t) \stackrel{\mathfrak{M}}{\Leftrightarrow} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)} \text{ if } 0 < \Re(s) < 1, 0 < \alpha < 2 \quad (2.18)$$

or $0 < \Re(s) < \min\{1, \Re(\beta)/2\}$, $\alpha = 2$,

$$W_{\lambda,\mu}(-t) \stackrel{\mathfrak{M}}{\Leftrightarrow} \frac{\Gamma(s)}{\Gamma(\mu-\lambda s)} \text{ if } 0 < \Re(s), \lambda < 1 \text{ or} \quad (2.19)$$

$0 < \Re(s) < \Re(\mu)/2 - 1/4$, $\lambda = 1$,

$$H_{p,q}^{m,n} \left(\begin{matrix} (\alpha, a)_p \\ (\beta, b)_q \end{matrix} \middle| t \right) \stackrel{\mathfrak{M}}{\Leftrightarrow} \frac{\prod_{j=1}^m \Gamma(\beta_j + b_j s) \prod_{j=1}^n \Gamma(1 - \alpha_j - a_j s)}{\prod_{j=n+1}^p \Gamma(\alpha_j + a_j s) \prod_{j=m+1}^q \Gamma(1 - \beta_j - b_j s)} \quad (2.20)$$

if $-\min_{1 \leq j \leq m} \Re(\beta_j)/b_j < \Re(s) < \min_{1 \leq j \leq n} (1 - \Re(\alpha_j))/a_j$

and

1) $a^* > 0$ or

2) $a^* = 0$, $\delta \Re(s) < \frac{q-p}{2} - 1 + \Re \left(\sum_{j=1}^p \alpha_j - \sum_{j=1}^q \beta_j \right)$,

where a^* and δ are defined as in the formula (2.24).

In the formulas given above, we used the notations

$$t_+^\alpha := \begin{cases} t^\alpha, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

for the truncated power function,

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0$$

for the Mittag-Leffler function, and

$$W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)}, \quad \lambda > -1$$

for the Wright function. By $H_{p,q}^{m,n}$ the Fox H -function is denoted. Because all elementary functions and most of the special functions including the so-called special functions of FC are known to be particular cases of the Fox H -function, and the Mellin transform (2.20) of the H -function is represented as a quotient of products of Gamma functions, the Mellin transforms of most of the known functions have the same form, too (see e.g. (2.13)-(2.20)). It is exactly this fact that makes the Mellin transform technique extremely powerful in general and in applications to the FC problems.

For the reader's convenience, the Fox H -function is introduced and shortly discussed in the rest of this section. It is defined by means of a contour integral of Mellin-Barnes type ([8], [14], [15], [26], [29], [32], [33], [36])

$$H_{p,q}^{m,n} \left(\begin{matrix} (\alpha_1, a_1), \dots, (\alpha_p, a_p) \\ (\beta_1, b_1), \dots, (\beta_q, b_q) \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_L \Phi(s) z^{-s} ds, \quad (2.21)$$

where $z \neq 0$, $0 \leq m \leq q$, $0 \leq n \leq p$, $\alpha_i \in \mathcal{C}$, $a_i > 0$, $1 \leq i \leq p$, $\beta_i \in \mathcal{C}$, $b_i > 0$, $1 \leq i \leq q$,

$$\Phi(s) = \frac{\prod_{i=1}^m \Gamma(\beta_i + b_i s) \prod_{i=1}^n \Gamma(1 - \alpha_i - a_i s)}{\prod_{i=n+1}^p \Gamma(\alpha_i + a_i s) \prod_{i=m+1}^q \Gamma(1 - \beta_i - b_i s)}, \quad (2.22)$$

an empty product, if it occurs, is taken to be one, and the infinite contour L that separates the poles of $\prod_{i=1}^m \Gamma(\beta_i + b_i s)$ from the poles of $\prod_{i=1}^n \Gamma(1 - \alpha_i - a_i s)$ can be of the following three types:

- 1) $L = L_{i\infty}$ if $a^* > 0$, $|\arg z| < a^* \pi/2$ or $a^* \geq 0$, $|\arg z| = a^* \pi/2$, $\gamma \delta < -\Re \mu$;
- 2) $L = L_{-\infty}$ if $\delta > 0$, $0 < |z| < \infty$ or $\delta = 0$, $0 < |z| < \beta$
or $\delta = 0$, $|z| = \beta$, $a^* \geq 0$, $\Re \mu < 0$;
- 3) $L = L_{+\infty}$ if $\delta < 0$, $0 < |z| < \infty$ or $\delta = 0$, $|z| > \beta$
or $\delta = 0$, $|z| = \beta$, $a^* \geq 0$, $\Re \mu < 0$.

In the conditions 1) – 3), μ , γ , and β are defined by

$$\mu = \frac{p-q}{2} + 1 + \sum_{i=1}^q \beta_i - \sum_{i=1}^p \alpha_i, \quad \gamma = \lim_{s \rightarrow L_{i\infty}} \Re s, \quad \beta = \prod_{i=1}^p a_i^{-a_i} \prod_{i=1}^q b_i^{b_i}, \quad (2.23)$$

and a^* , δ are given by

$$a^* = \sum_{i=1}^n a_i - \sum_{i=n+1}^p a_i + \sum_{i=1}^m b_i - \sum_{i=m+1}^q b_i, \quad \delta = \sum_{i=1}^q b_i - \sum_{i=1}^p a_i. \quad (2.24)$$

The contours $L_{i\infty}$, $L_{-\infty}$, $L_{+\infty}$ are defined as follows:

The contour $L = L_{-\infty}$ lies in a horizontal strip $\Im(s) \leq C$ and goes from the point $-\infty + iy_1$ to the point $-\infty + iy_2$, $y_1 < y_2$;

The contour $L = L_{+\infty}$ lies in a horizontal strip $\Im(s) \leq C$ and goes from the point $+\infty + iy_1$ to the point $+\infty + iy_2$, $y_1 < y_2$;

The contour $L = L_{i\infty}$ lies in a vertical strip $\Re(s) \leq C$ and goes from the point $\gamma - i\infty$ to the point $\gamma + i\infty$.

In particular, if the inequality $\max_{1 \leq i \leq m} \Re(-\beta_i/b_i) < \min_{1 \leq i \leq n} \Re((1 - \alpha_i)/a_i)$ is fulfilled, the contour $L_{i\infty}$ is a straight line $\Re(s) = \gamma$, where γ satisfies the inequalities $\max_{1 \leq i \leq m} \Re(-\beta_i/b_i) < \gamma < \min_{1 \leq i \leq n} \Re((1 - \alpha_i)/a_i)$ and the formula (2.21) is nothing else than the inverse Mellin transform of the function f^* defined by the formula (2.22). In the case, the contour L is either $L_{-\infty}$ or $L_{+\infty}$, the integral in the right-hand side of (2.21) can be evaluated by means of the Cauchy residue theorem, the Jordan lemma, and the well known formula

$$\operatorname{Res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!}, \quad n = 0, 1, 2, \dots$$

that leads to a series representation of the H -function as a linear combination of generalized hypergeometric series (see e.g. [28] or [32]).

For some values of the parameters of the H -function, several conditions out of the conditions given in 1) – 3) can be simultaneously satisfied. In this case, the corresponding integrals are equal each to other due to the Cauchy integral theorem.

In fact, the conditions specified in 1) – 3) are nothing else than the convergence conditions of the integral from the right-hand side of (2.21). They can be easily obtained from the known asymptotical formulas for the Gamma function (see e.g. [28] or [36]):

$$\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s} (1 + O(s^{-1})), \quad |\arg(s)| < \pi, \quad |s| \rightarrow \infty, \quad (2.25)$$

$$\frac{\Gamma(s + \alpha)}{\Gamma(s + \beta)} = s^{\alpha-\beta} (1 + O(s^{-1})), \quad |\arg(s)| < \pi, \quad \alpha, \beta \in \mathcal{C}, \quad |s| \rightarrow \infty, \quad (2.26)$$

$$|\Gamma(x + iy)| = \sqrt{2\pi} |y|^{x-\frac{1}{2}} e^{-\pi|y|/2} (1 + O(|y|^{-1})), \quad x, y \in \mathbb{R}, \quad |y| \rightarrow \infty. \quad (2.27)$$

More details and results regarding the H -function can be found e.g. in the books [14], [15], [29], [32], [33], or [36].

For other properties of the Mellin transform and applications of the Mellin transform technique, not mentioned here, we refer the interested reader to the valuable works [2], [3], [27].

3. FC operators as Mellin convolution type transforms

We start with the Erdélyi-Kober fractional integrals that are defined as follows (see e.g. [15], [17], [36]):

$$(I_{\beta}^{\gamma, \delta} f)(x) = \frac{\beta}{\Gamma(\delta)} x^{-\beta(\gamma+\delta)} \int_0^x (x^\beta - t^\beta)^{\delta-1} t^{\beta(\gamma+1)-1} f(t) dt, \quad \beta > 0, \Re(\delta) > 0, \tag{3.1}$$

$$(K_{\beta}^{\tau, \alpha} f)(x) = \frac{\beta}{\Gamma(\alpha)} x^{\beta\tau} \int_x^{\infty} (t^\beta - x^\beta)^{\alpha-1} t^{-\beta(\tau+\alpha-1)-1} f(t) dt, \quad \beta > 0, \Re(\alpha) > 0. \tag{3.2}$$

When $\delta = 0$ or $\alpha = 0$, respectively, these operators are defined as the identity operator. For $\beta = 1$, the Erdélyi-Kober fractional integrals (3.1), (3.2) are reduced to the Riemann-Liouville fractional integrals with the power weights:

$$(x^{-\gamma-\delta} I_{0+}^{\delta} x^{\gamma} f)(x) = \frac{1}{\Gamma(\delta)} x^{-\gamma-\delta} \int_0^x (x-t)^{\delta-1} t^{\gamma} f(t) dt, \quad \Re(\delta) > 0, \tag{3.3}$$

$$(x^{\tau} I_{-}^{\alpha} x^{-\tau-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} x^{\tau} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\tau-\alpha} f(t) dt, \quad \Re(\alpha) > 0. \tag{3.4}$$

The operators (3.1) and (3.2) can be represented in the form of Mellin convolutions (2.3):

$$(I_{\beta}^{\gamma, \delta} f)(x) = (k_1 \overset{\mathfrak{M}}{*} f)(x), \quad (K_{\beta}^{\tau, \alpha} f)(x) = (k_2 \overset{\mathfrak{M}}{*} f)(x), \tag{3.5}$$

where

$$k_1(x) = \frac{\beta}{\Gamma(\delta)} x^{-\beta(\gamma+\delta)} (x^\beta - 1)_+^{\delta-1}, \quad k_2(x) = \frac{\beta}{\Gamma(\alpha)} x^{\beta\tau} (1 - x^\beta)_+^{\alpha-1}. \tag{3.6}$$

The Parseval equality (2.5) for the Mellin transform along with the formulas (2.7), (2.14), and (2.15) readily leads to the useful representations ([15], [36])

$$(I_{\beta}^{\gamma, \delta} f)(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1 + \gamma - s/\beta)}{\Gamma(1 + \gamma + \delta - s/\beta)} f^*(s) x^{-s} ds, \tag{3.7}$$

$$(K_{\beta}^{\tau, \alpha} f)(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\tau + s/\beta)}{\Gamma(\tau + \alpha + s/\beta)} f^*(s) x^{-s} ds \tag{3.8}$$

of the Erdélyi-Kober fractional integrals.

Now, we show how to use these representations to get the inversion formulas and formulas for compositions, say of two Erdélyi-Kober fractional integrals. Similarly, compositions of their finite number $m \geq 2$ have been extensively studied in [15], [36], and called there as multiple Erdélyi-Kober operators.

First we notice that the Parseval equality (2.5) can be applied for a composition of several operators of Mellin convolution type, say, with the kernels k_1 and k_2 , in the form

$$(K_1 f)(x) = \int_0^{+\infty} k_1(x/t)g(t)\frac{dt}{t}, \quad (K_2 f)(x) = \int_0^{+\infty} k_2(x/t)g(t)\frac{dt}{t},$$

and we get the representation

$$(K_2 \circ K_1)(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} k_2^*(s)k_1^*(s)f^*(s)x^{-s} ds \quad (3.9)$$

for the composition $K_2 \circ K_1$.

In particular, the known semigroup properties of the Erdélyi-Kober fractional integrals (see e.g. [15])

$$I_{\beta}^{\gamma+\delta, \alpha} \circ I_{\beta}^{\gamma, \delta} = I_{\beta}^{\gamma, \delta+\alpha}, \quad K_{\beta}^{\tau+\alpha, \delta} \circ K_{\beta}^{\tau, \alpha} = K_{\beta}^{\tau, \alpha+\delta}$$

immediately follow from the representations (3.7), (3.8), and (3.9). Say, in the case of the first formula we get

$$\begin{aligned} (I_{\beta}^{\gamma+\delta, \alpha} \circ I_{\beta}^{\gamma, \delta} f)(x) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1+\gamma+\delta-s/\beta)}{\Gamma(1+\gamma+\delta+\alpha-s/\beta)} \frac{\Gamma(1+\gamma-s/\beta)}{\Gamma(1+\gamma+\delta-s/\beta)} \times \\ & f^*(s)x^{-s} ds = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1+\gamma-s/\beta)}{\Gamma(1+\gamma+\delta+\alpha-s/\beta)} f^*(s)x^{-s} ds = (I_{\beta}^{\gamma, \delta+\alpha} f)(x). \end{aligned}$$

To obtain the inverse operator for, say, $I_{\beta}^{\gamma, \delta}$, let us denote $(I_{\beta}^{\gamma, \delta} f)(x)$ by $g(x)$. Then it follows from (3.7) that

$$g^*(s) = \frac{\Gamma(1+\gamma-s/\beta)}{\Gamma(1+\gamma+\delta-s/\beta)} f^*(s)$$

and we get the inversion formula in the Mellin transform space

$$f^*(s) = \frac{\Gamma(1+\gamma+\delta-s/\beta)}{\Gamma(1+\gamma-s/\beta)} g^*(s) \quad (3.10)$$

that can be represented in the form

$$f^*(s) = \frac{\Gamma(1+\gamma+\delta-s/\beta)}{\Gamma(1+\gamma+n-s/\beta)} \frac{\Gamma(1+\gamma+n-s/\beta)}{\Gamma(1+\gamma-s/\beta)} g^*(s) \quad (3.11)$$

with $n \in \mathbb{N}$, $n-1 < \delta \leq n$.

Now we use the representation (3.9) and the formulas (2.10) and (3.7) to get the inversion of the Mellin transform (3.11) in the form

$$f(x) = (D_{\beta}^{\gamma, \delta} g)(x) := \prod_{j=1}^n \left(\gamma + j + \frac{1}{\beta} x \frac{d}{dx} \right) (I_{\beta}^{\gamma+\delta, n-\delta} g)(x), \quad n \in \mathbb{N}, \quad n-1 < \delta \leq n. \quad (3.12)$$

The operator $D_{\beta}^{\gamma,\delta}$ is called the Erdélyi-Kober fractional derivative ([15], [36]). The same method works for inversion of the Erdélyi-Kober fractional integral $K_{\beta}^{\tau,\alpha}$. In this case, we apply the formulas (2.11) and (3.8) and get

$$f(x) = (P_{\beta}^{\tau,\alpha}g)(x) := \prod_{j=0}^{n-1} \left(\tau + j - \frac{1}{\beta}x \frac{d}{dx}\right) (K_{\beta}^{\tau+\alpha, n-\alpha}g)(x) \tag{3.13}$$

with $n \in \mathbb{N}$, $n-1 < \alpha \leq n$. Of course, like in the case of the Erdélyi-Kober fractional integrals, the Erdélyi-Kober fractional derivatives can be represented in form of the Mellin-Barnes integrals

$$(D_{\beta}^{\gamma,\delta}f)(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1 + \gamma + \delta - s/\beta)}{\Gamma(1 + \gamma - s/\beta)} f^*(s)x^{-s} ds, \tag{3.14}$$

$$(P_{\beta}^{\tau,\alpha}f)(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\tau + \alpha + s/\beta)}{\Gamma(\tau + s/\beta)} f^*(s)x^{-s} ds. \tag{3.15}$$

Finally, we demonstrate how to use the Mellin transform technique to invert the Riemann-Liouville integral

$$(I_{0+}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \Re(\alpha) > 0 \tag{3.16}$$

that can be represented as a Mellin convolution operator

$$g(x) = (I_{0+}^{\alpha}f)(x) = (k(t) * t^{\alpha} f(t))(x), \quad k(t) = \frac{1}{\Gamma(\alpha)} (t-1)_{+}^{\alpha-1}. \tag{3.17}$$

The convolution formula (2.4) along with the formulas (2.7) and (2.15) leads to the equation

$$g^*(s) = \frac{\Gamma(1 - \alpha - s)}{\Gamma(1 - s)} f^*(s + \alpha) \tag{3.18}$$

that can be solved for f^* :

$$f^*(s) = \frac{\Gamma(1 + \alpha - s)}{\Gamma(1 - s)} g^*(s - \alpha). \tag{3.19}$$

Let us represent the right-hand side of the last equality in the form

$$\frac{\Gamma(1 + \alpha - s)}{\Gamma(1 - s)} g^*(s - \alpha) = \frac{\Gamma(1 + n - s)}{\Gamma(1 - s)} h^*(s - n) \tag{3.20}$$

with

$$h^*(s) := \frac{\Gamma(1 + \alpha - n - s)}{\Gamma(1 - s)} g^*(s + n - \alpha), \quad n \in \mathbb{N}, \quad n-1 < \alpha \leq n. \tag{3.21}$$

The formula (2.9) and the representations (3.9), (3.17), (3.19)-(3.21) lead then to the well known inversion formula

$$f(x) = (D_{0+}^{\alpha}g)(x) := \frac{d^n}{dx^n} (I_{0+}^{n-\alpha}g)(x), \quad n \in \mathbb{N}, \quad n-1 < \alpha \leq n$$

for the Riemann-Liouville integral.

4. Evaluation of improper integrals containing the FC functions

A very general method of evaluation of integrals of Mellin convolution type containing special functions of hypergeometric type was suggested in [28]. Here we demonstrate this method to evaluate some FC integrals. The idea is a very simple one, namely, to use the Mellin convolution formula (2.4), the Parseval equality (2.5) and its modification (see (2.12)):

$$\int_0^{+\infty} f(t)g(xt)dt = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(1-s)g^*(s)x^{-s}ds. \quad (4.1)$$

In this case, both f and g are the functions of hypergeometric type, their Mellin transforms are some quotients of products of Gamma functions (or some linear combinations of such quotients) and thus the improper integral from the left-hand side of (4.1) can be represented as a Mellin-Barnes integral or the Fox H -function from the right-hand side of (4.1). In the reference book [32], an extensive list of particular cases of the H -function is given, so that in many cases the H -functions can be written in form of more simple elementary and special functions. In the general case, series representations of the H -function as linear combinations of some hypergeometric functions can be determined in explicit form. In this section, we illustrate this method of evaluation of integrals on some examples related to FC.

For example, let us evaluate the integral (see e.g. [31])

$$I_1(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} E_{\alpha,\beta}(-t^\alpha) t^{\beta-1} dt, \quad 0 < \alpha \leq 2, \quad 0 < \nu.$$

This integral can be represented as the Mellin convolution

$$I_1(x) = (f \overset{\mathfrak{M}}{*} g)(x), \quad f(t) = \frac{1}{\Gamma(\nu)} (t-1)_+^{\nu-1}, \quad g(t) = t^{\beta+\nu-1} E_{\alpha,\beta}(-t^\alpha). \quad (4.2)$$

The formulas (2.7), (2.8), and (2.18) give us (under suitable restrictions on the parameters) the Mellin transform correspondence

$$t^\gamma E_{\alpha,\beta}(-t^\alpha) \overset{\mathfrak{M}}{\leftrightarrow} \frac{1}{\alpha} \frac{\Gamma((s+\gamma)/\alpha)\Gamma(1-(s+\gamma)/\alpha)}{\Gamma(\beta-\gamma-s)}. \quad (4.3)$$

In particular, we get

$$t^{\beta+\nu-1} E_{\alpha,\beta}(-t^\alpha) \overset{\mathfrak{M}}{\leftrightarrow} \frac{1}{\alpha} \frac{\Gamma((s+\beta+\nu-1)/\alpha)\Gamma(1-(s+\beta+\nu-1)/\alpha)}{\Gamma(1-\nu-s)}$$

that together with the formulas (2.8), (4.2) and the Mellin convolution formula (2.4) leads to the representation

$$I_1^*(s) = \frac{1}{\alpha} \frac{\Gamma((s+\beta+\nu-1)/\alpha)\Gamma(1-(s+\beta+\nu-1)/\alpha)}{\Gamma(1-\nu-s)} \frac{\Gamma(1-\nu-s)}{\Gamma(1-s)}$$

$$= \frac{1}{\alpha} \frac{\Gamma((s + \beta + \nu - 1)/\alpha)\Gamma(1 - (s + \beta + \nu - 1)/\alpha)}{\Gamma(1 - s)}.$$

Finally, the last formula and the Mellin transform correspondence (4.3) allows us to represent the integral I_1 in the form

$$I_1(x) = x^{\beta+\nu-1} E_{\alpha, \beta+\nu}(-x^\alpha).$$

Of course, in the case of the integral $I_1(x)$ we could use the series representation of the Mittag-Leffler function and fractional order term-by-term integration of the series to get the same result.

Let us consider a more complicated example ([12])

$$I_2(x) = \int_0^\infty E_\alpha(-t^\alpha) W_{\lambda, \mu}(-xt) dt, \quad 0 < \alpha \leq 2, \quad 0 < \lambda \leq 1$$

with the Mittag-Leffler function $E_\alpha(z) := E_{\alpha, 1}(z)$ and the Wright function $W_{\lambda, \mu}(z)$. Using the Parseval equality (4.1) and the Mellin transform formulas (2.19) for the Wright function and (4.3) for the Mittag-Leffler function, we get

$$\begin{aligned} I_2(x) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{\alpha} \frac{\Gamma((1-s)/\alpha)\Gamma(1 - (1-s)/\alpha)}{\Gamma(s)} \frac{\Gamma(s)}{\Gamma(\mu - \lambda s)} x^{-s} ds \\ &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{\alpha} \frac{\Gamma((1-s)/\alpha)\Gamma(1 - (1-s)/\alpha)}{\Gamma(\mu - \lambda s)} x^{-s} ds. \end{aligned}$$

The variables substitution $1 - (1 - s)/\alpha = p$ in the last integral gives us the representation

$$I_2(x) = x^{\alpha-1} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-p)\Gamma(p)}{\Gamma(\mu + \alpha - 1 - \lambda\alpha p)} (x^\alpha)^{-p} dp$$

that along with the relation (2.18) leads to the final result

$$I_2(x) = x^{\alpha-1} E_{\lambda\alpha, \mu+\alpha-1}(-x^\alpha).$$

In the case $\alpha = 1$, the Mittag-Leffler function E_α is reduced to the exponential function and the integral I_2 can be interpreted as the Laplace transform of the Wright function $W_{\lambda, \mu}$ with $0 < \lambda \leq 1$. Thus we obtain the well known result (see e.g. [12])

$$\mathfrak{L}\{W_{\lambda, \mu}(-t); p\} = \frac{1}{p} E_{\lambda, \mu}\left(-\frac{1}{p}\right),$$

where

$$\mathfrak{L}\{f(t); p\} := \int_0^\infty f(t) e^{-pt} dt, \quad \Re(p) > c_f, \tag{4.4}$$

is the Laplace transform of the function f .

Let us mention that the Laplace transform (4.4) coincides with the left-hand side of the Parseval equality (4.1) with the function $g(t) = \exp(-t)$ and can be represented in the form

$$\mathfrak{L}\{f(t); p\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(1-s)\Gamma(s)p^{-s} ds, \quad (4.5)$$

in view of the Mellin transform formula (2.13). This representation along with the Mellin transform formulas for the FC special functions (e.g. (2.19) for the Wright function and (4.3) for the Mittag-Leffler function) allows us to easily determine the Laplace transforms of the FC special functions. Say, let us evaluate the Laplace transform of the function $f(t) = t^{\mu-1}W_{\lambda,\mu}(-t^\lambda)$ with $-1 < \lambda < 0$ (see e.g. [12]) that we denote by $\tilde{f}(p)$. The Mellin transform correspondence

$$t^{\mu-1}W_{\lambda,\mu}(-t^\lambda) \overset{\mathfrak{M}}{\leftrightarrow} -\frac{1}{\lambda} \frac{\Gamma((s+\mu-1)/\lambda)}{\Gamma(1-s)} \quad (4.6)$$

follows from the formulas (2.7), (2.8), and (2.19). Applying the formula (4.5) we get

$$\begin{aligned} \tilde{f}(p) &= -\frac{1}{\lambda} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma((\mu-s)/\lambda)}{\Gamma(s)} \Gamma(s) p^{-s} ds \\ &= -\frac{1}{\lambda} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma((\mu-s)/\lambda) p^{-s} ds. \end{aligned}$$

The Mellin transform formula (2.13) along with the formulas (2.7), (2.8) leads to the final result:

$$\mathfrak{L}\{t^{\mu-1}W_{\lambda,\mu}(-t^\lambda); p\} = p^{-\mu} \exp(-p^{-\lambda}), \quad -1 < \lambda < 0.$$

Because the Mellin transform formulas for the sin- and cos-functions are known (see (2.16) and (2.17)), the same technique we apply for the Laplace transform allows us easily to treat the sin- and cos-Fourier transforms

$$\begin{aligned} I_c(x) &= \frac{1}{\pi} \int_0^\infty f(t) \cos(xt) dt, \quad x > 0, \\ I_s(x) &= \frac{1}{\pi} \int_0^\infty f(t) \sin(xt) dt, \quad x > 0 \end{aligned}$$

of the special functions of FC. In this case we prefer to employ the Parseval equality

$$\int_0^{+\infty} f(t)g(xt)dt = \frac{1}{x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)g^*(1-s)x^s ds \quad (4.7)$$

that is obtained from (4.1) via the variables substitution $1-s = s_1$ and then denoting s_1 by s . Using (4.7) and the formulas (2.6), (2.8), (2.16),

and (2.17), the integrals I_c and I_s can be rewritten in the form

$$I_c(x) = \frac{1}{\sqrt{\pi x}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s) \frac{\Gamma(1/2 - s/2)}{2^s \Gamma(s/2)} x^s ds, \quad x > 0, \quad 0 < \gamma < 1, \tag{4.8}$$

$$I_s(x) = \frac{1}{\sqrt{\pi x}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s) \frac{\Gamma(1 - s/2)}{2^s \Gamma(1/2 + s/2)} x^s ds, \quad x > 0, \quad 0 < \gamma < 2. \tag{4.9}$$

In particular, in [25] the representations (4.8) and (4.9) were applied to represent the fundamental solution to the space-time-fractional diffusion-wave equation in form of a Mellin-Barnes integral. In [11], [21], and [25], the fundamental solution G_α to the fractional wave equation of the order α , $1 < \alpha < 2$ was represented as the cos-Fourier transform

$$G_\alpha(x, t) = \frac{1}{\pi} \int_0^\infty \cos(\kappa x) E_\alpha(-\kappa^\alpha t^\alpha) d\kappa, \quad x \in \mathbb{R}, \quad t > 0. \tag{4.10}$$

Now, we demonstrate how to apply the formula (4.8) to represent G_α in terms of elementary functions for every α , $1 < \alpha < 2$. Indeed, for $x = 0$ the integral from the right-hand side of (4.10) is reduced to the Mellin integral transform of the Mittag-Leffler function E_α at the point $s = \frac{1}{\alpha}$. It converges under the conditions $\alpha > 1$ and its value is given by the formula (4.3):

$$\begin{aligned} \frac{1}{\pi} \int_0^\infty E_\alpha(-\kappa^\alpha t^\alpha) d\kappa &= \frac{1}{\pi \alpha t} \int_0^\infty E_\alpha(-u) u^{\frac{1}{\alpha}-1} du \tag{4.11} \\ &= \frac{1}{\pi \alpha t} \frac{\Gamma(\frac{1}{\alpha})\Gamma(1 - \frac{1}{\alpha})}{\Gamma(1 - \alpha\frac{1}{\alpha})} = 0, \quad t > 0, \end{aligned}$$

because the Gamma function has a pole at the point $z = 0$: $1/\Gamma(0) = 0$.

Since G_α is an even function, we consider the integral from the right-hand side of (4.10) just in the case $x = |x| > 0$. The formulas (2.6), (4.3), and (4.8) lead to the representation

$$G_\alpha(x, t) = \frac{1}{\sqrt{\pi \alpha x}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha}) \Gamma(1 - \frac{s}{\alpha}) \Gamma(\frac{1}{2} - \frac{s}{2})}{\Gamma(1 - s) 2^s \Gamma(\frac{s}{2})} \left(\frac{t}{x}\right)^{-s} ds \tag{4.12}$$

of the fundamental solution G_α in terms of a Mellin-Barnes integral. The representation (4.12) can be simplified to the form

$$G_\alpha(x, t) = \frac{1}{\alpha x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha})\Gamma(1 - \frac{s}{\alpha})}{\Gamma(1 - \frac{s}{2})\Gamma(\frac{s}{2})} \left(\frac{t}{x}\right)^{-s} ds \tag{4.13}$$

by using the duplication formula for the Gamma function $\Gamma(1 - s)$.

Now let us represent the Mellin-Barnes integral (4.13) in the form of some convergent series that can be summated in explicit form in terms of some elementary functions. The general theory of the Mellin-Barnes

integrals presented e.g. in [28] (see also [26]) says that the integral in (4.13) is convergent under the condition $0 < \alpha < 2$. For $0 < t < x$, the contour of integration in the integral (4.13) can be transformed to the loop $L_{-\infty}$ starting and ending at infinity and encircling all poles $s_k = -\alpha k$, $k = 0, 1, 2, \dots$ of the function $\Gamma(s/\alpha)$. Taking into account the relation

$$\operatorname{res}_{s=-k}\Gamma(s) = \frac{(-1)^k}{k!}, \quad k = 0, 1, 2, \dots,$$

the residue theorem provides us with the desired series representation:

$$G_\alpha(x, t) = \frac{1}{\alpha x} \sum_{k=0}^{\infty} \frac{\alpha(-1)^k}{k!} \frac{\Gamma(1+k)}{\Gamma(-\frac{\alpha}{2}k) \Gamma(1-\frac{\alpha}{2}k)} \left(\frac{t}{x}\right)^{\alpha k} \quad (4.14)$$

that can be transformed to the form

$$G_\alpha(x, t) = -\frac{1}{\pi x} \sum_{k=1}^{\infty} \sin(\alpha\pi k/2) \left(-\frac{t^\alpha}{x^\alpha}\right)^k \quad (4.15)$$

by using the reflection formula for the Gamma function.

Now we use the summation formula

$$\sum_{k=1}^{\infty} r^k \sin(ka) = \Im \left(\sum_{k=1}^{\infty} r^k e^{ika} \right) = \Im \left(\frac{r e^{ia}}{1 - r e^{ia}} \right) = \frac{r \sin a}{1 - 2r \cos a + r^2} \quad (4.16)$$

that is valid for $a \in \mathbb{R}$, $|r| < 1$ to summate the series in (4.15) and obtain the nice representation

$$G_\alpha(x, t) = \frac{1}{\pi t^{2\alpha} + 2x^\alpha t^\alpha \cos(\pi\alpha/2) + x^{2\alpha}} \quad (4.17)$$

for the Green function G_α that is valid for $0 < t < x$.

In the case $0 < x < t$ we can transform the contour of integration in (4.13) to the loop $L_{+\infty}$ encircling all poles $s_k = \alpha(1+k)$, $k = 0, 1, 2, \dots$ of the function $\Gamma(1 - \frac{s}{\alpha})$. Applying the residue theorem we arrive at the representation

$$G_\alpha(x, t) = \frac{1}{\alpha x} \sum_{k=0}^{\infty} \frac{\alpha(-1)^k}{k!} \frac{\Gamma(1+k)}{\Gamma(\frac{\alpha}{2}(k+1)) \Gamma(1-\frac{\alpha}{2}(k+1))} \left(\frac{x}{t}\right)^{\alpha(k+1)} \quad (4.18)$$

that can be transformed to the form

$$G_\alpha(x, t) = -\frac{1}{\pi x} \sum_{k=1}^{\infty} \sin(\alpha\pi k/2) \left(-\frac{x^\alpha}{t^\alpha}\right)^k \quad (4.19)$$

by using the reflection formula for the Gamma function. The formula (4.16) applied to the series from the right-hand side of (4.19) again leads to the representation (4.17), this time for $0 < x < t$. Finally, the validity of the formula (4.17) for $0 < x = t$ follows from the principle of analytic continuation for the Mellin-Barnes integrals. Thus the fundamental solution G_α

for the fractional wave equation given by the improper integral (4.10) can be represented in the form

$$G_\alpha(x, t) = \frac{1}{\pi} \frac{|x|^{\alpha-1} t^\alpha \sin(\pi\alpha/2)}{t^{2\alpha} + 2|x|^\alpha t^\alpha \cos(\pi\alpha/2) + |x|^{2\alpha}}, \quad t > 0, \quad x \in \mathbb{R} \quad (4.20)$$

for $1 < \alpha < 2$.

5. The Post-Widder formula

In this section, the Mellin transform technique is applied to deduce the Post-Widder inversion formula for the Laplace transform and its new modification. Because the Mellin transforms of many special functions and the Mellin convolution type integral transforms with these functions in the kernel are related to quotients of some products of Gamma functions, it is no wonder that the known formulas for the Gamma function play an important role both in the theory of the special functions of hypergeometric type and in the theory of integral transforms with these functions in the kernel.

In this section we demonstrate how the known formula

$$\Gamma(s) = \lim_{n \rightarrow \infty} \frac{n! n^s}{s(s+1) \cdots (s+n)}, \quad s \neq 0, -1, -2, \dots \quad (5.1)$$

for the Gamma function leads to the Post-Widder formula ([13])

$$f(t) := \mathcal{L}^{-1} \{F(p); t\} = \lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} F^{(n)} \left(\frac{n}{t}\right) \quad (5.2)$$

for the inverse Laplace transform of $F(p) := \mathcal{L} \{f(t); p\}$.

For our aims, we need a simple consequence of the formula (5.1) in form

$$\frac{1}{\Gamma(1-s)} = \lim_{n \rightarrow \infty} \frac{(1-s)(2-s) \cdots (n-s)}{n! n^{-s}}, \quad s \in \mathcal{C}, \quad (5.3)$$

and the representation (4.5) of the Laplace transform

$$F(p) := \mathcal{L} \{f(t); p\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(1-s) \Gamma(s) p^{-s} ds, \quad (5.4)$$

from that we get the formula

$$F^*(s) = f^*(1-s) \Gamma(s) \quad (5.5)$$

for the Mellin transforms of f and its Laplace transform F . It follows from (5.5) that

$$f^*(s) = \frac{F^*(1-s)}{\Gamma(1-s)}. \quad (5.6)$$

We use now the formulas (2.6), (2.7), and (2.9) to get the Mellin transform correspondence

$$\frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} F^{(n)} \left(\frac{n}{t}\right) \overset{\mathfrak{M}}{\leftrightarrow} \frac{(1-s)(2-s) \cdots (n-s)}{n! n^{-s}} F(1-s),$$

and thus the correspondence

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} F^{(n)}\left(\frac{n}{t}\right) \overset{\mathfrak{M}}{\leftrightarrow} \lim_{n \rightarrow \infty} \frac{(1-s) \cdots (n-s)}{n!n^{-s}} F(1-s) = \frac{F^*(1-s)}{\Gamma(1-s)}$$

according to the formula (5.3). Applying the formula (5.6), we arrive at the Widder-Post formula (5.2). Note that the convolutional approach to derive it has been a base yet in the pioneering book [13].

Now we illustrate how the Mellin transform technique can be employed to deduce a new formula of the Post-Widder type (5.2). Again, we use the formulas (5.3) and (5.6) that we combine to the representation

$$f^*(s) = \frac{F^*(1-s)}{\Gamma(1-s)} = \lim_{n \rightarrow \infty} \frac{(1-s)(2-s) \cdots (n-s)}{n!n^{-s}} F^*(1-s). \tag{5.7}$$

Introducing a function H that satisfies the relation

$$H^*(s-n) = \frac{F^*(1-s)}{n^{-s}}, \tag{5.8}$$

by formulas (2.9) and (5.7) we get the representation

$$f(t) = \lim_{n \rightarrow \infty} \frac{1}{n!} \frac{d^n}{dt^n} H(t). \tag{5.9}$$

Now let us determine the function H . From the equation (5.8) we first obtain the formula

$$H^*(s) = \frac{F^*(1-n-s)}{n^{-n-s}}$$

and then the representation

$$H(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} H^*(s)t^{-s} ds = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F^*(1-n-s)}{n^{-n-s}} t^{-s} ds.$$

Variables substitution $1-n-s = s_1$ leads then to the formula

$$H(t) = \frac{nt^{n-1}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F^*(s_1) \left(\frac{n}{t}\right)^{-s_1} ds_1 = nt^{n-1} F\left(\frac{n}{t}\right).$$

Combining this representation with the formula (5.9), we obtain a new Post-Widder type formula for the inverse Laplace transform, namely

$$f(t) = \lim_{n \rightarrow \infty} \frac{1}{(n-1)!} \frac{d^n}{dt^n} \left(t^{n-1} F\left(\frac{n}{t}\right) \right). \tag{5.10}$$

Real inversion formulas of Post-Widder type for convolutional type generalizations of the Laplace transform, as the Obrechhoff and the generalized Obrechhoff transforms, using Mellin transform techniques, have been obtained in other papers of ours as [1], [4], [20], etc.

6. Leibniz type rules for the FC operators

In the theory of the special functions, some summation theorems for the hypergeometric functions are known. Most of these summation theorems can be interpreted as representations of some quotients of products of Gamma functions in terms of infinite series. One of the most popular summation theorem is the one for the Gauss hypergeometric function ${}_2F_1$, saying that

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad \Re(c - a - b) > 0, \quad (6.1)$$

where the Gauss function is defined as the series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!} \quad (6.2)$$

for $|z| \leq 1$ and $\Re(c - a - b) > 0$, and as an analytic continuation of this series for other values of z . The expression $(a)_n := \Gamma(a + n)/\Gamma(a)$ stays for the Pochhammer symbol.

For the our aims, we rewrite the summation formula (6.1) in the form

$$\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c + n)n!}, \quad \Re(c - a - b) > 0, \quad (6.3)$$

and show that this representation of the quotient of the products of Gamma functions leads to the Leibniz type formula

$$(D_{\beta}^{\gamma, \delta} f(t) \cdot g(t))(x) = \sum_{n=0}^{\infty} \binom{\delta}{n} (D_{\beta}^{\alpha_1 + n, \delta - n} f)(x) \prod_{j=0}^{n-1} \left(\frac{1}{\beta} x \frac{d}{dx} - \alpha_2 - j\right) g(x), \quad (6.4)$$

for the Erdélyi-Kober fractional derivatives, where α_1, α_2 are arbitrary real numbers satisfying the relation $\alpha_1 - \alpha_2 = \gamma$, $\binom{\delta}{n} := \frac{\Gamma(\delta + 1)}{n! \Gamma(\delta - n + 1)}$ is the generalized binomial coefficient and $D_{\beta}^{\gamma, \alpha}$ stays for the fractional Erdélyi-Kober derivative (3.12) if $\alpha > 0$, for the Erdélyi-Kober integral (3.1) of order $-\alpha$ if $\alpha < 0$, and for the identity operator if $\alpha = 0$.

To start with, let us write down a representation of the Mellin transform of a product of two functions, say, f and g , with the known Mellin transforms f^* and g^* . Because of the Mellin inversion formulas

$$f(x) = \frac{1}{2\pi i} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} f^*(s_1) x^{-s_1} ds_1, \quad g(x) = \frac{1}{2\pi i} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} g^*(s_2) x^{-s_2} ds_2,$$

we first get the representation

$$f(x) \cdot g(x) = \frac{1}{(2\pi i)^2} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} f^*(s_1) g^*(s_2) x^{-s_1 - s_2} ds_1 ds_2$$

that after the variables substitution $s_1 + s_2 = s$, $s_1 = s_1$ can be rewritten as the inverse Mellin transform

$$f(x) \cdot g(x) = \frac{1}{2\pi i} \int_{\gamma_3 - i\infty}^{\gamma_3 + i\infty} \left(\frac{1}{2\pi i} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} f^*(s_1)g^*(s - s_1) ds_1 \right) x^{-s} ds.$$

Applying the Mellin transform to the last formula we get the desired representation for the Mellin transform of the product $f \cdot g$ in the form

$$(f \cdot g)^*(s) = \frac{1}{2\pi i} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} f^*(s_1)g^*(s - s_1) ds_1. \tag{6.5}$$

Now we remember the representation (3.14) of the Erdelyi-Kober fractional derivative in form of a Mellin-Barnes integral and apply this representation to the function $f \cdot g$ with the Mellin transform given by (6.5) to get the formula

$$\begin{aligned} (D_\beta^{\gamma, \delta} f(t) \cdot g(t))(x) &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(1 + \gamma + \delta - s/\beta)}{\Gamma(1 + \gamma - s/\beta)} \\ &\times \left(\frac{1}{2\pi i} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} f^*(s_1)g^*(s - s_1) ds_1 \right) x^{-s} ds. \end{aligned}$$

The variables substitution $s - s_1 = s_2$, $s_1 = s_1$ in the integrals on the right-hand side of the last formula leads to the representation

$$\begin{aligned} (D_\beta^{\gamma, \delta} f(t) \cdot g(t))(x) &= \frac{1}{(2\pi i)^2} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} \frac{\Gamma(1 + \gamma + \delta - (s_1 + s_2)/\beta)}{\Gamma(1 + \gamma - (s_1 + s_2)/\beta)} \\ &\times f^*(s_1)g^*(s_2)x^{-s_1 - s_2} ds_1 ds_2 \end{aligned} \tag{6.6}$$

that plays the key role in derivation of the Leibniz type formula (6.4). Indeed, let us substitute the values $a = -\delta$, $b = \alpha_2 + s_2/\beta$, and $c = 1 + \alpha_1 - s_1/\beta$ into the formula (6.3). Then we get the representation

$$\begin{aligned} &\frac{\Gamma(1 + \gamma + \delta - (s_1 + s_2)/\beta)}{\Gamma(1 + \gamma - (s_1 + s_2)/\beta)} \\ &= \sum_{n=0}^{\infty} \frac{(-\delta)_n}{n!} \frac{\Gamma(1 + \alpha_1 + \delta - s_1/\beta)}{\Gamma(1 + \alpha_1 + n - s_1/\beta)} \frac{\Gamma(\alpha_2 + n + s_2/\beta)}{\Gamma(\alpha_2 + s_2/\beta)} \end{aligned}$$

and substitute it into the formula (6.6). Interchanging the order of integration and summation in (6.6), we arrive at the representation

$$\begin{aligned} (D_\beta^{\gamma, \delta} f(t) \cdot g(t))(x) &= \sum_{n=0}^{\infty} \frac{(-\delta)_n}{n!} \frac{1}{2\pi i} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} \frac{\Gamma(\alpha_2 + n + s_2/\beta)}{\Gamma(\alpha_2 + s_2/\beta)} g^*(s_2)x^{-s_2} ds_2 \\ &\times \frac{1}{2\pi i} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \frac{\Gamma(1 + \alpha_1 + \delta - s_1/\beta)}{\Gamma(1 + \alpha_1 + n - s_1/\beta)} f^*(s_1)x^{-s_1} ds_1. \end{aligned}$$

The formula $\frac{(-\delta)_n}{n!} = (-1)^n \binom{\delta}{n}$ along with the representations (3.7), (3.14), (3.14) for the Erdélyi-Kober fractional integrals and derivatives finally leads to the Leibniz type formula (6.4) for the Erdélyi-Kober fractional derivatives.

Other known representations of the quotients of products of Gamma functions, like the Dougall formula

$$\sum_{n=-\infty}^{+\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(d+n)} = \frac{\pi^2}{\sin(\pi a)\sin(\pi b)} \frac{\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(d-a)\Gamma(c-b)\Gamma(d-b)},$$

valid for $\Re(a+b-c-d) < -1$, $a, b \notin \mathbb{Z}$, or the integral representation

$$\int_{-\infty}^{+\infty} G(u, a, b, c) du = \frac{\Gamma(a+b+c+d-3)}{\Gamma(a+c-1)\Gamma(a+d-1)\Gamma(b+c-1)\Gamma(b+d-1)}$$

with

$$G(u, a, b, c) = \frac{1}{\Gamma(a+u)\Gamma(b+u)\Gamma(c-u)\Gamma(d-u)}$$

valid for $\Re(a+b+c+d) > 3$, lead to the modified Leibniz type formulas and their integral analogues both for the Erdélyi-Kober fractional derivatives and for other operators of Mellin convolution type. For the details, we refer the interested reader to [34] or [36].

7. Completely monotone FC functions

The completely monotone functions are known to play an important role in different branches of mathematics and especially in the probability theory ([7]). Also, the famous Bernstein theorem is well known. In FC, completely monotone functions are used e.g. to show that fundamental solutions to some fractional differential equations can be interpreted as some probability densities (see e.g. [10], [25] and references therein). In this section we demonstrate how the Mellin integral transform can be employed to deduce some new completely monotone functions based on the known ones.

DEFINITION 7.1. A function $f : (0, \infty) \rightarrow \mathbb{R}$ is called a completely monotone function if it is of class C^∞ and $(-1)^n f^{(n)}(x) \geq 0$ for all $n \in \mathbb{N}$ and $x > 0$.

The basic property of the completely monotone functions that we need in this section is the following one: A function $f : (0, \infty) \rightarrow \mathbb{R}$ is completely monotone if and only if it can be represented as the Laplace transform of a non-negative measure (non-negative function or generalized function). Because the Laplace transform is a Mellin convolution type transform, the

technique of the Mellin transform can be applied for investigation of completely monotone functions.

Let the representation

$$f(x) = \int_0^{\infty} e^{-xt} F(t) dt, \quad x > 0 \quad (7.1)$$

hold true for a non-negative function F with a known Mellin transform. Then the function f is completely monotone and its Mellin transform is given by the formula (see (4.5))

$$f^*(s) = \Gamma(s)F^*(1-s) \quad (7.2)$$

that leads to the formula

$$F^*(s) = \frac{f^*(1-s)}{\Gamma(1-s)}. \quad (7.3)$$

But if the function $F(t)$, $t > 0$ is non-negative, then the function $G(t) := t^\gamma F(t^{-\beta})$ is non-negative for any $\gamma, \beta \in \mathbb{R}$, too. Thus the function g of the form

$$g(x) = \int_0^{\infty} e^{-xt} G(t) dt, \quad x > 0 \quad (7.4)$$

is completely monotone and it follows from the relation (7.2) that

$$g^*(s) = \Gamma(s)G^*(1-s). \quad (7.5)$$

Using the formulas (2.7), (2.8), the Mellin transform G^* can be written in the form

$$G^*(s) = \frac{1}{|\beta|} F^* \left(-\frac{\gamma}{\beta} - \frac{s}{\beta} \right)$$

and thus from (7.5) we get the Mellin transform formula

$$g^*(s) = \frac{1}{|\beta|} \Gamma(s) F^* \left(\frac{s}{\beta} - \frac{1+\gamma}{\beta} \right).$$

The complete monotone function g given by (7.4) can be then evaluated as the Mellin-Barnes type integral

$$g(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{|\beta|} \Gamma(s) F^* \left(\frac{s}{\beta} - \frac{1+\gamma}{\beta} \right) x^{-s} ds, \quad (7.6)$$

provided that the Mellin transform F^* of the complete monotone function f is known. In many cases f is a particular case of the Fox H -function and thus F^* is represented in form of a quotient of products of Gamma functions, that means that the new complete monotone function g is a particular case of the H -function, too.

Let us consider a simple example. It is known that the exponential function $f(x) = \exp(-x^\alpha)$, $0 < \alpha < 1$ is complete monotone with the Mellin transform given by the formula (2.13). The function F^* from (7.3) has then the form

$$F^*(s) = \frac{f^*(1-s)}{\Gamma(1-s)} = \frac{1}{\alpha} \frac{\Gamma\left(\frac{1}{\alpha} - \frac{s}{\alpha}\right)}{\Gamma(1-s)}.$$

It follows from the arguments presented above that the function

$$g(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{\alpha|\beta|} \frac{\Gamma(s)\Gamma\left(\frac{\beta+\gamma+1}{\alpha\beta} - \frac{s}{\alpha\beta}\right)}{\Gamma\left(\frac{\beta+\gamma+1}{\beta} - \frac{s}{\beta}\right)} x^{-s} ds \tag{7.7}$$

is complete monotone, too. The function g given by (7.7) is evidently a particular case of the Fox H -function. In particular, in the case $\beta > \frac{1}{\alpha} - 1$ it can be represented as the convergent series

$$g(x) = \frac{1}{\alpha|\beta|} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\beta+\gamma+1}{\alpha\beta} + \frac{n}{\alpha\beta}\right)}{n! \Gamma\left(\frac{\beta+\gamma+1}{\beta} + \frac{n}{\beta}\right)} (-x)^n. \tag{7.8}$$

We can easily recognize it as a particular case of the generalized Wright function defined by the series

$${}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1) \dots (b_q, B_q) \end{matrix} ; z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i + A_i k) z^k}{\prod_{i=1}^q \Gamma(b_i + B_i k) k!} \tag{7.9}$$

for the z -values where the series converges, and by the analytic continuation of this series for other z -values. Thus, we have proved that the generalized Wright function

$$g(x) = {}_1\Psi_1 \left[\begin{matrix} \left(\frac{\beta+\gamma+1}{\alpha\beta}, \frac{1}{\alpha\beta}\right) \\ \left(\frac{\beta+\gamma+1}{\beta}, \frac{1}{\beta}\right) \end{matrix} ; -x \right] \tag{7.10}$$

is complete monotone under the conditions $0 < \alpha < 1, \frac{1}{\alpha} - 1 < \beta$. In particular, let us take the parameter values $\beta = \frac{1}{\alpha}, \gamma = -\frac{1}{\alpha}$. Then the series (7.8) takes the form

$$g(x) = \sum_{n=0}^{\infty} \frac{\Gamma(1+n)}{n! \Gamma(\alpha + \alpha n)} (-x)^n = \sum_{n=0}^{\infty} \frac{(-x)^n}{\Gamma(\alpha + \alpha n)} \tag{7.11}$$

that defines the Mittag-Leffler function $E_{\alpha,\alpha}(-x)$, known to be completely monotone for $0 < \alpha < 1$. Taking other known completely monotone functions and applying the procedure described above, other new completely monotone functions can be easily obtained.

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