



# Some Relationships Between the Apostol-Bernoulli and Apostol-Euler Polynomials

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**Abstract**—Recently, Srivastava and Pintér [1] investigated several interesting properties and relationships involving the classical as well as the generalized (or higher-order) Bernoulli and Euler polynomials. They also showed (among other things) that the *main* relationship (proven earlier by Cheon [2]) can easily be put in a much more general setting. The main object of the present sequel to these earlier works is to derive several general properties and relationships involving the Apostol-Bernoulli and Apostol-Euler polynomials. Some of these general results can indeed be suitably specialized in order to deduce the corresponding properties and relationships involving the (generalized) Bernoulli and (generalized) Euler polynomials. Other relationships associated with the Stirling numbers of the second kind are also considered. © 2006 Elsevier Ltd. All rights reserved.

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## 1. INTRODUCTION

The *generalized Bernoulli polynomials*  $B_n^{(\alpha)}(x)$  of order  $\alpha$  and the *generalized Euler polynomials*  $E_n^{(\alpha)}(x)$  of order  $\alpha$ , each of degree  $n$  in  $x$  as well as in  $\alpha$ , are defined by means of the following generating functions (see, for details, [3, p. 253 *et seq.*; 4, Section 2.8; 5, Section 1.6]):

$$\left(\frac{t}{e^t - 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!} \quad (|t| < 2\pi; 1^\alpha := 1) \quad (1)$$

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and

$$\left(\frac{2}{e^t + 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^\infty E_n^{(\alpha)}(x) \frac{t^n}{n!} \quad (|t| < \pi; 1^\alpha := 1), \tag{2}$$

respectively.

Clearly, the corresponding *generalized Bernoulli numbers*  $B_n^{(\alpha)}$  of order  $\alpha$  and the *generalized Euler numbers*  $E_n^{(\alpha)}$  of order  $\alpha$  are given by

$$B_n^{(\alpha)} := B_n^{(\alpha)}(0) \quad \text{and} \quad E_n^{(\alpha)} := 2^n E_n^{(\alpha)}\left(\frac{\alpha}{2}\right) \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{3}$$

Also, the classical *Bernoulli polynomials*  $B_n(x)$  and the classical *Euler polynomials*  $E_n(x)$  are given by

$$B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x) \quad (n \in \mathbb{N}_0), \tag{4}$$

respectively,  $\mathbb{N}$  being (as usual) the set of *positive* integers.

Moreover, the classical *Bernoulli numbers*  $B_n$  and the classical *Euler numbers*  $E_n$  are given by

$$B_n := B_n^{(1)} = B_n(0) = (-1)^n B_n(1) = \frac{1}{2^{1-n} - 1} B_n\left(\frac{1}{2}\right) \quad (n \in \mathbb{N}_0) \tag{5}$$

and

$$E_n := E_n^{(1)} = 2^n E_n\left(\frac{1}{2}\right) \quad (n \in \mathbb{N}_0), \tag{6}$$

respectively. From the generating functions (1) and (2), it is easily seen that

$$B_n^{(0)}(x) = E_n^{(0)}(x) = x^n \quad (n \in \mathbb{N}_0). \tag{7}$$

Numerous interesting (and useful) properties and relationships involving each of these polynomials and numbers can be found in many books and tables on this subject (for example, see [5–10]).

Recently, by making use of some fairly standard techniques based upon series rearrangement, Srivastava and Pintér [1] derived each of the following elegant theorems (cf. [1, Theorem 1, p. 379; Theorem 2, p. 380]).

**THEOREM A.** (See, [1, Theorem 1, p. 379].) *The following relationship:*

$$B_n^{(\alpha)}(x + y) = \sum_{k=0}^n \binom{n}{k} \left[ B_k^{(\alpha)}(y) + \frac{k}{2} B_{k-1}^{(\alpha-1)}(y) \right] E_{n-k}(x) \quad (\alpha \in \mathbb{C}; n \in \mathbb{N}_0) \tag{8}$$

*holds true between the generalized Bernoulli polynomials and the classical Euler polynomials.*

**THEOREM B.** (See [1, Theorem 2, p. 380].) *The following relationship:*

$$E_n^{(\alpha)}(x + y) = \sum_{k=0}^n \frac{2}{k+1} \binom{n}{k} \left[ E_{k+1}^{(\alpha-1)}(y) - E_{k+1}^{(\alpha)}(y) \right] B_{n-k}(x) \quad (\alpha \in \mathbb{C}; n \in \mathbb{N}_0) \tag{9}$$

*holds true between the generalized Euler polynomials and the classical Bernoulli polynomials.*

Upon setting  $\alpha = 1$  in assertion (8) of Theorem A, if we let  $y \rightarrow 0$  and make use of (7), we can deduce the aforementioned *main* relationship in Cheon’s work (cf., [2, p. 368, Theorem 3]),

$$B_n(x) = \sum_{\substack{k=0 \\ (k \neq 1)}}^n \binom{n}{k} B_k E_{n-k}(x) \quad (n \in \mathbb{N}_0), \tag{10}$$

just as it was accomplished by Srivastava and Pintér [1, p. 379].

In Section 2 of the present paper, we propose to prove some properties and relationships involving the generalized Apostol-Bernoulli polynomials (see, for details, [11]) and the generalized Apostol-Euler polynomials (see, for details, [12]). In Section 3, we shall consider some interesting generalizations and analogues of the Srivastava-Pintér addition theorems (Theorem A and Theorem B above). Finally, in Section 4, we derive some *explicit* representations of these general families of Apostol-Bernoulli and Apostol-Euler polynomials in terms of the Stirling numbers of the second kind.

## 2. PROPERTIES AND RELATIONSHIPS INVOLVING THE GENERALIZED APOSTOL-BERNOULLI POLYNOMIALS AND THE GENERALIZED APOSTOL-EULER POLYNOMIALS

For arbitrary real or complex parameters  $\alpha$  and  $\lambda$ , the *generalized Apostol-Bernoulli polynomials*  $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$  (see, for details, [11]) and the *generalized Apostol-Euler polynomials*  $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$  (see, for details, [12]) are defined by means of the following generating functions:

$$\left(\frac{t}{\lambda e^t - 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!} \quad (|t + \log \lambda| < 2\pi; 1^\alpha := 1) \tag{11}$$

and

$$\left(\frac{2}{\lambda e^t + 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!} \quad (|2t + \log \lambda| < \pi; 1^\alpha := 1), \tag{12}$$

respectively.

Clearly, the corresponding *generalized Apostol-Bernoulli numbers*  $\mathfrak{B}_n^{(\alpha)}(\lambda)$  and the *generalized Apostol-Euler numbers*  $\mathfrak{E}_n^{(\alpha)}(\lambda)$  are given by

$$\mathfrak{B}_n^{(\alpha)}(\lambda) := \mathfrak{B}_n^{(\alpha)}(0; \lambda) \quad \text{and} \quad \mathfrak{E}_n^{(\alpha)}(\lambda) := 2^n \mathfrak{E}_n^{(\alpha)}\left(\frac{\alpha}{2}; \lambda\right) \quad (n \in \mathbb{N}_0), \tag{13}$$

respectively. The so-called *Apostol-Bernoulli polynomials*  $\mathfrak{B}_n(x; \lambda)$  (see, for details, [13, pp. 161–167] and [5, pp. 126–127]) and the so-called *Apostol-Euler polynomials*  $\mathfrak{E}_n(x; \lambda)$  are given by

$$\mathfrak{B}_n(x; \lambda) := \mathfrak{B}_n^{(1)}(x; \lambda) \quad \text{and} \quad \mathfrak{E}_n(x; \lambda) := \mathfrak{E}_n^{(1)}(x; \lambda) \quad (n \in \mathbb{N}_0; \lambda \in \mathbb{C}), \tag{14}$$

respectively. Furthermore, the corresponding *Apostol-Bernoulli numbers*  $\mathfrak{B}_n(\lambda)$  and the *Apostol-Euler numbers*  $\mathfrak{E}_n(\lambda)$  are given by

$$\begin{aligned} \mathfrak{B}_n(\lambda) &:= \mathfrak{B}_n(0; \lambda) = \frac{(-1)^n}{\lambda} \mathfrak{B}_n\left(1; \frac{1}{\lambda}\right) \\ &= 2^n \left[ \frac{1}{2} \mathfrak{B}_n(\lambda^2) + \frac{\lambda}{2} \mathfrak{B}_n\left(\frac{1}{2}; \lambda^2\right) \right] \quad (n \in \mathbb{N}_0) \end{aligned} \tag{15}$$

and

$$\mathfrak{E}_n(\lambda) := 2^n \mathfrak{E}_n\left(\frac{1}{2}; \lambda\right) \quad (n \in \mathbb{N}_0), \tag{16}$$

respectively.

Obviously, when  $\lambda = 1$  in (11) to (16), we readily arrive at the corresponding *well-known* forms given by (1) to (6).

Moreover, it can be deduced from the generating functions (11) and (12) that (see also [11,12])

$$\mathfrak{B}_n^{(\alpha)}(x; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k^{(\alpha)}(\lambda) x^{n-k}, \tag{17}$$

$$\mathfrak{E}_n^{(\alpha)}(x; \lambda) = \sum_{k=0}^n \binom{n}{k} \frac{\mathfrak{E}_k^{(\alpha)}(\lambda)}{2^k} \left(x - \frac{\alpha}{2}\right)^{n-k}, \tag{18}$$

$$\mathfrak{B}_n^{(\alpha)}(\alpha - x; \lambda) = \frac{(-1)^n}{\lambda^\alpha} \mathfrak{B}_n^{(\alpha)}(x; \lambda^{-1}), \tag{19}$$

$$\mathfrak{E}_n^{(\alpha)}(\alpha - x; \lambda) = \frac{(-1)^n}{\lambda^\alpha} \mathfrak{E}_n^{(\alpha)}(x; \lambda^{-1}), \tag{20}$$

$$\mathfrak{B}_n^{(\alpha+\beta)}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k^{(\alpha)}(x; \lambda) \mathfrak{B}_{n-k}^{(\beta)}(y; \lambda), \tag{21}$$

and

$$\mathfrak{E}_n^{(\alpha+\beta)}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{(\alpha)}(x; \lambda) \mathfrak{E}_{n-k}^{(\beta)}(y; \lambda). \tag{22}$$

From the generating functions (11) and (12), it follows also that (see [11,12]):

$$\lambda \mathfrak{B}_n^{(\alpha)}(x + 1; \lambda) - \mathfrak{B}_n^{(\alpha)}(x; \lambda) = n \mathfrak{B}_{n-1}^{(\alpha-1)}(x; \lambda) \tag{23}$$

and

$$\lambda \mathfrak{E}_n^{(\alpha)}(x + 1; \lambda) + \mathfrak{E}_n^{(\alpha)}(x; \lambda) = 2 \mathfrak{E}_n^{(\alpha-1)}(x; \lambda), \tag{24}$$

respectively. Now, since

$$\mathfrak{B}_n^{(0)}(x; \lambda) = \mathfrak{E}_n^{(0)}(x; \lambda) = x^n \quad (n \in \mathbb{N}_0), \tag{25}$$

upon setting  $\beta = 0$  in addition theorems (21) and (22), if we interchange  $x$  and  $y$ , we obtain

$$\mathfrak{B}_n^{(\alpha)}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k^{(\alpha)}(y; \lambda) x^{n-k} \tag{26}$$

and

$$\mathfrak{E}_n^{(\alpha)}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{(\alpha)}(y; \lambda) x^{n-k}, \tag{27}$$

respectively.

Next, by combining (23) and (26) (with  $x = 1$  and  $y \mapsto x$ ), we find that

$$\mathfrak{B}_n^{(\alpha-1)}(x; \lambda) = \frac{1}{n+1} \left[ \lambda \sum_{k=0}^{n+1} \binom{n+1}{k} \mathfrak{B}_k^{(\alpha)}(x; \lambda) - \mathfrak{B}_{n+1}^{(\alpha)}(x; \lambda) \right] \quad (n \in \mathbb{N}_0), \tag{28}$$

which, in the special case when  $\alpha = 1$ , yields the following expansion:

$$x^n = \frac{1}{n+1} \left[ \lambda \sum_{k=0}^{n+1} \binom{n+1}{k} \mathfrak{B}_k(x; \lambda) - \mathfrak{B}_{n+1}(x; \lambda) \right] \quad (n \in \mathbb{N}_0) \tag{29}$$

in series of the Apostol-Bernoulli polynomials  $\{\mathfrak{B}_n(x; \lambda)\}_{n=0}^\infty$ .

In the special case of (29) when  $\lambda = 1$ , we obtain the following familiar expansion (cf., e.g., [10, p. 26]):

$$x^n = \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} B_k(x) \quad (n \in \mathbb{N}_0) \tag{30}$$

in series of the classical Bernoulli polynomials  $\{B_n(x)\}_{n=0}^\infty$ .

In precisely the same manner, the addition theorem (27) in conjunction with (24) would lead us to the following companions of (28) and (29):

$$\mathfrak{E}_n^{(\alpha-1)}(x; \lambda) = \frac{1}{2} \left[ \lambda \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{(\alpha)}(x; \lambda) + \mathfrak{E}_n^{(\alpha)}(x; \lambda) \right] \quad (n \in \mathbb{N}_0) \tag{31}$$

and

$$x^n = \frac{1}{2} \left[ \lambda \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k(x; \lambda) + \mathfrak{E}_n(x; \lambda) \right] \quad (n \in \mathbb{N}_0). \tag{32}$$

In view of (25), this last expansion (32) in series of the Apostol-Euler polynomials  $\{\mathfrak{E}_n(x; \lambda)\}_{n=0}^\infty$  is indeed an immediate consequence of (31) when  $\alpha = 1$ .

By using (11) (with  $\alpha = 1$ ) and (12) (with  $\alpha = 1$ ), we have

$$\begin{aligned} \sum_{n=0}^\infty \mathfrak{B}_n(x; \lambda^2) \frac{t^n}{n!} &= \frac{te^{xt}}{\lambda^2 e^t - 1} = \frac{t/2}{\lambda e^{t/2} - 1} \cdot \frac{2e^{xt}}{\lambda e^{t/2} + 1} \\ &= \sum_{n=0}^\infty 2^{-n} \mathfrak{B}_n(\lambda) \frac{t^n}{n!} \cdot \sum_{n=0}^\infty 2^{-n} \mathfrak{E}_n(2x; \lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^\infty \left[ 2^{-n} \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_{n-k}(\lambda) \mathfrak{E}_k(2x; \lambda) \right] \frac{t^n}{n!}, \end{aligned}$$

which yields the following relationship between the Apostol-Bernoulli and Apostol-Euler polynomials:

$$\mathfrak{B}_n(x; \lambda^2) = 2^{-n} \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_{n-k}(\lambda) \mathfrak{E}_k(2x; \lambda) \tag{33}$$

or, equivalently,

$$2^n \mathfrak{B}_n\left(\frac{x}{2}; \lambda^2\right) = \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k(\lambda) \mathfrak{E}_{n-k}(x; \lambda). \tag{34}$$

If we set  $\lambda = 1$  in (33) and (34), we find the corresponding familiar relationship between the classical Bernoulli and the classical Euler polynomials as follows [1, p. 376, equations (10),(11)] (see also [6, p. 806, Entry (23.1.29); 5, p. 66, equation (63)]:

$$B_n(x) = 2^{-n} \sum_{k=0}^n \binom{n}{k} B_{n-k} E_k(2x) \tag{35}$$

or, equivalently,

$$2^n B_n\left(\frac{x}{2}\right) = \sum_{k=0}^n \binom{n}{k} B_k E_{n-k}(x). \tag{36}$$

By applying similar arguments, it is not difficult to get the following explicit representation for the Apostol-Euler polynomials  $\mathfrak{E}_n(x; \lambda)$  in terms of the Apostol-Bernoulli polynomials  $\mathfrak{B}_n(x; \lambda)$ :

$$\mathfrak{E}_{n-1}(x; \lambda) = \frac{2^n}{n} \left[ \lambda \mathfrak{B}_n\left(\frac{x+1}{2}; \lambda^2\right) - \mathfrak{B}_n\left(\frac{x}{2}; \lambda^2\right) \right] \tag{37}$$

or, equivalently,

$$\mathfrak{E}_{n-1}(x; \lambda) = \frac{2}{n} \left[ \mathfrak{B}_n(x; \lambda) - 2^n \mathfrak{B}_n\left(\frac{x}{2}; \lambda^2\right) \right]. \tag{38}$$

By setting  $\lambda = 1$  in (37) and (38), we deduce the corresponding well-known relationship between the classical Bernoulli and the classical Euler polynomials as given below [1, p. 377, equation (14)] (see also [6, p. 806, Entry (23.1.27)]; 5, p. 65, equation 1.6 (60]):

$$E_{n-1}(x) = \frac{2^n}{n} \left[ B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right) \right] \tag{39}$$

or, equivalently,

$$E_{n-1}(x) = \frac{2}{n} \left[ B_n(x) - 2^n B_n\left(\frac{x}{2}\right) \right]. \tag{40}$$

In addition, from the relationships (37) (with  $x = 0$ ) and (38) (with  $x = 0$ ), we find that

$$\lambda \mathfrak{B}_n\left(\frac{1}{2}; \lambda^2\right) = 2^{-n} \mathfrak{B}_n(\lambda) + n \cdot 2^{-n-1} \mathfrak{E}_{n-1}(0; \lambda). \tag{41}$$

Thus, by substituting for  $\mathfrak{E}_{n-1}(0; \lambda)$  from (38) (with  $x = 0$ ) into (41), we obtain the above-asserted relationship (15), that is,

$$\mathfrak{B}_n(\lambda) = 2^{n-1} \left[ \mathfrak{B}_n(\lambda^2) + \lambda \mathfrak{B}_n\left(\frac{1}{2}; \lambda^2\right) \right] \quad (n \in \mathbb{N}_0). \tag{42}$$

For numerous other properties and relationships involving the (ordinary as well as generalized) Apostol-Bernoulli and the (ordinary as well as generalized) Apostol-Euler polynomials, see the recent works [5,11-14].

### 3. GENERALIZATIONS AND ANALOGUES OF THE SRIVASTAVA-PINTÉR ADDITION THEOREMS

Making use of some known formulas and identities given in Section 2, we now prove an interesting generalization of the Srivastava-Pintér addition theorem (8), which is given by Theorem 1 below.

**THEOREM 1.** *The following relationship:*

$$\mathfrak{B}_n^{(\alpha)}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \left[ \mathfrak{B}_k^{(\alpha)}(y; \lambda) + \frac{k}{2} \mathfrak{B}_{k-1}^{(\alpha-1)}(y; \lambda) \right] \mathfrak{E}_{n-k}(x; \lambda) \tag{43}$$

$(\alpha, \lambda \in \mathbb{C}; n \in \mathbb{N}_0)$

*holds true between the generalized Apostol-Bernoulli polynomials and the Apostol-Euler polynomials.*

**PROOF.** First of all, if we substitute from (32) into the right-hand side of (26), we get

$$\begin{aligned} \mathfrak{B}_n^{(\alpha)}(x + y; \lambda) &= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k^{(\alpha)}(y; \lambda) \left[ \mathfrak{E}_{n-k}(x; \lambda) + \lambda \sum_{j=0}^{n-k} \binom{n-k}{j} \mathfrak{E}_j(x; \lambda) \right] \\ &= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k^{(\alpha)}(y; \lambda) \mathfrak{E}_{n-k}(x; \lambda) \\ &\quad + \frac{\lambda}{2} \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k^{(\alpha)}(y; \lambda) \sum_{j=0}^{n-k} \binom{n-k}{j} \mathfrak{E}_j(x; \lambda), \end{aligned} \tag{44}$$

which, upon inverting the order of summation and using the following elementary combinatorial identity:

$$\binom{m}{l} \binom{l}{n} = \binom{m}{n} \binom{m-n}{m-l} \quad (m \geq l \geq n; l, m, n \in \mathbb{N}_0), \tag{45}$$

yields

$$\begin{aligned} \mathfrak{B}_n^{(\alpha)}(x+y; \lambda) &= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k^{(\alpha)}(y; \lambda) \mathfrak{E}_{n-k}(x; \lambda) \\ &+ \frac{\lambda}{2} \sum_{j=0}^n \binom{n}{j} \mathfrak{E}_j(x; \lambda) \sum_{k=0}^{n-j} \binom{n-j}{k} \mathfrak{B}_k^{(\alpha)}(y; \lambda). \end{aligned} \tag{46}$$

The innermost sum in (46) can be evaluated by means of (26) with, of course,

$$x = 1 \quad \text{and} \quad n \mapsto n - j \quad (0 \leq j \leq n; n, j \in \mathbb{N}_0).$$

Thus, we find from (46) that

$$\begin{aligned} \mathfrak{B}_n^{(\alpha)}(x+y; \lambda) &= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k^{(\alpha)}(y; \lambda) \mathfrak{E}_{n-k}(x; \lambda) \\ &+ \frac{\lambda}{2} \sum_{j=0}^n \binom{n}{j} \mathfrak{B}_{n-j}^{(\alpha)}(y+1; \lambda) \mathfrak{E}_j(x; \lambda) \end{aligned} \tag{47}$$

or, equivalently, that

$$\mathfrak{B}_n^{(\alpha)}(x+y; \lambda) = \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \left[ \mathfrak{B}_k^{(\alpha)}(y; \lambda) + \lambda \mathfrak{B}_k^{(\alpha)}(y+1; \lambda) \right] \mathfrak{E}_{n-k}(x; \lambda), \tag{48}$$

which, in light of the recurrence relation (23), leads us at once to the relationship (43) asserted by Theorem 1.

REMARK 1. By setting  $\lambda = 1$  in Theorem 1, we readily obtain the Srivastava-Pintér addition Theorem 8 as asserted by Theorem A.

REMARK 2. In terms of the generalized Apostol-Bernoulli numbers  $\{\mathfrak{B}_n^{(\alpha)}(\lambda)\}_{n=0}^\infty$ , by setting  $y = 0$  in Theorem 1, we obtain the following special case:

$$\mathfrak{B}_n^{(\alpha)}(x; \lambda) = \sum_{k=0}^n \binom{n}{k} \left( \mathfrak{B}_k^{(\alpha)}(\lambda) + \frac{k}{2} \mathfrak{B}_{k-1}^{(\alpha-1)}(\lambda) \right) \mathfrak{E}_{n-k}(x; \lambda) \quad (\alpha \in \mathbb{C}; n \in \mathbb{N}_0). \tag{49}$$

Since, by the definition (11),

$$B_1 := \mathfrak{B}_1(1) = -\frac{1}{2} \quad \text{and} \quad \mathfrak{B}_n^{(0)}(\lambda) = \delta_{n,0} \quad (n \in \mathbb{N}_0), \tag{50}$$

$\delta_{m,n}$  being the Kronecker symbol, a further special case of (49), when  $\alpha = 1$ , would yield the following new relationship between the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials:

$$\mathfrak{B}_n(x; \lambda) = \sum_{\substack{k=0 \\ (k \neq 1)}}^n \binom{n}{k} \mathfrak{B}_k(\lambda) \mathfrak{E}_{n-k}(x; \lambda) + n \left[ \mathfrak{B}_1(\lambda) + \frac{1}{2} \right] \mathfrak{E}_{n-1}(x; \lambda) \quad (\lambda \in \mathbb{C}; n \in \mathbb{N}_0). \tag{51}$$

In view of (50), Cheon’s *main* result (10) would follow also from (51) for  $\lambda = 1$ . When  $\lambda \neq 1$ , by using the following special values of the Apostol-Bernoulli numbers  $\mathfrak{B}_n(\lambda)$  (see [5, p. 126, equation 2.5 (46)]):

$$\mathfrak{B}_0(\lambda) = 0 \quad \text{and} \quad \mathfrak{B}_1(\lambda) = \frac{1}{\lambda - 1} \quad (\lambda \in \mathbb{C} \setminus \{1\}), \tag{52}$$

we find from (51) that

$$\mathfrak{B}_n(x; \lambda) = \sum_{k=1}^n \binom{n}{k} \mathfrak{B}_k(\lambda) \mathfrak{E}_{n-k}(x; \lambda) + \frac{n}{2} \mathfrak{E}_{n-1}(x; \lambda) \quad (\lambda \in \mathbb{C} \setminus \{1\}; n \in \mathbb{N}_0). \tag{53}$$

REMARK 3. Alternatively, in view of (25), the assertion (43) of Theorem 1 gives us the following relationship between the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials when  $\alpha = 1$ :

$$\mathfrak{B}_n(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \left[ \mathfrak{B}_k(y; \lambda) + \frac{k}{2} y^{k-1} \right] \mathfrak{E}_{n-k}(x; \lambda), \tag{54}$$

which, upon letting  $\lambda = 1$  and  $y \rightarrow 0$ , yields Cheon’s *main* result (10) once again.

Next, by appealing instead to (27) and (29), our demonstration of Theorem 1 can be applied *mutatis mutandis* in order to derive an interesting analogue of the Srivastava-Pintér addition formula (9), which is given by Theorem 2 below.

THEOREM 2. *The following relationship:*

$$\mathfrak{E}_n^{(\alpha)}(x + y; \lambda) = \sum_{k=0}^n \frac{2}{k + 1} \binom{n}{k} \left[ \mathfrak{E}_{k+1}^{(\alpha-1)}(y; \lambda) - \mathfrak{E}_{k+1}^{(\alpha)}(y; \lambda) \right] \mathfrak{B}_{n-k}(x; \lambda) \tag{55}$$

$(\alpha, \lambda \in \mathbb{C}; n \in \mathbb{N}_0)$

*holds true between the generalized Apostol-Euler polynomials and the Apostol-Bernoulli polynomials.*

REMARK 4. By setting  $\lambda = 1$  in Theorem 2, we get the Srivastava-Pintér addition formula (9) as asserted by Theorem B.

REMARK 5. In light of (25), a special case of assertion (25) of Theorem 2 when  $\alpha = 1$  gives us the following relationship:

$$\mathfrak{E}_n(x + y; \lambda) = \sum_{k=0}^n \frac{2}{k + 1} \binom{n}{k} [y^{k+1} - \mathfrak{E}_{k+1}(y; \lambda)] \mathfrak{B}_{n-k}(x; \lambda) \quad (n \in \mathbb{N}_0), \tag{56}$$

which, for  $y = 0$ , yields

$$\mathfrak{E}_n(x; \lambda) = - \sum_{k=0}^n \frac{2}{k + 1} \binom{n}{k} \mathfrak{E}_{k+1}(0; \lambda) \mathfrak{B}_{n-k}(x; \lambda) \quad (n \in \mathbb{N}_0). \tag{57}$$

Moreover, by setting  $\lambda = 1$  in (57), we obtain the Srivastava-Pintér formula given below (see [1, p. 380, Equation (40)]):

$$E_n(x) = - \sum_{k=0}^n \frac{2}{k + 1} \binom{n}{k} E_{k+1}(0) B_{n-k}(x) \quad (n \in \mathbb{N}_0). \tag{58}$$

Relationship (57) between the Apostol-Euler polynomials and the Apostol-Bernoulli polynomials is evidently analogous to the equivalent relationships (33) and (51). From (20) (with  $\alpha = 1$



and  $x = 1$ ) and (38) (with  $x = 0$  and  $n \mapsto n + 1$ ), we obtain

$$\mathfrak{E}_n(0; \lambda) = \frac{(-1)^n}{\lambda} \mathfrak{E}_n\left(1; \frac{1}{\lambda}\right) = \frac{2}{n+1} [\mathfrak{B}_{n+1}(\lambda) - 2^{n+1}\mathfrak{B}_{n+1}(\lambda^2)] \quad (59)$$

$(n \in \mathbb{N}).$

Thus, in view of (59), the relationship (57) can be rewritten in the following *equivalent* form:

$$\mathfrak{E}_{n-2}(x; \lambda) = 2 \binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} [2^{n-k}\mathfrak{B}_{n-k}(\lambda^2) - \mathfrak{B}_{n-k}(\lambda)] \mathfrak{B}_k(x; \lambda) \quad (60)$$

$(n \in \mathbb{N} \setminus \{1\}).$

By putting  $\lambda = 1$  in (60), it is easy to derive

$$E_{n-2}(x) = 2 \binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} (2^{n-k} - 1) B_{n-k} B_k(x) \quad (61)$$

$(n \in \mathbb{N} \setminus \{1\}),$

which incidentally is a known result recorded by (for example) Srivastava and Pintér [1, p. 380, Equation (42); 6, p. 806, Entry (23.1.28)] (see also [10, p. 29]).

REMARK 6. By setting  $y = 0$  and  $\lambda = 1$  in Theorem 2, we get the following relationship between the generalized Euler polynomials and the classical Bernoulli polynomials:

$$E_n^{(\alpha)}(x) = \sum_{k=0}^n \frac{2}{k+1} \binom{n}{k} [E_{k+1}^{(\alpha-1)}(0) - E_{k+1}^{(\alpha)}(0)] B_{n-k}(x) \quad (62)$$

$(\alpha \in \mathbb{C}; n \in \mathbb{N}_0),$

which, in the special case when  $\alpha = 1$ , yields the Srivastava-Pintér formula (58) once again.

REMARK 7. By letting  $x = 0$  and  $y \mapsto x$  in Theorem 2, appealing to the first formula in (13), we find the following relationship between the generalized Apostol-Euler polynomials and the Apostol-Bernoulli numbers:

$$\mathfrak{E}_n^{(\alpha)}(x; \lambda) = \sum_{k=0}^n \frac{2}{k+1} \binom{n}{k} [\mathfrak{E}_{k+1}^{(\alpha-1)}(x; \lambda) - \mathfrak{E}_{k+1}^{(\alpha)}(x; \lambda)] \mathfrak{B}_{n-k}(\lambda) \quad (63)$$

$(\alpha, \lambda \in \mathbb{C}; n \in \mathbb{N}_0),$

which, for  $x = \frac{\alpha}{2}$  and in conjunction with second formula in (13), yields the following relationship between the generalized Apostol-Euler numbers and the Apostol-Bernoulli numbers:

$$\mathfrak{E}_n^{(\alpha)}(\lambda) = \sum_{k=0}^n \frac{2^{n-k}}{k+1} \binom{n}{k} [2^{k+1}\mathfrak{E}_{k+1}^{(\alpha-1)}\left(\frac{\alpha}{2}; \lambda\right) - \mathfrak{E}_{k+1}^{(\alpha)}(\lambda)] \mathfrak{B}_{n-k}(\lambda) \quad (64)$$

$(\alpha, \lambda \in \mathbb{C}; n \in \mathbb{N}_0).$

Furthermore, by setting  $\alpha = 1$  and  $\lambda = 1$  in (64), we get the following relationship between the classical Euler numbers and the classical Bernoulli numbers:

$$E_n = \sum_{k=0}^n \frac{2^{n-k}}{k+1} \binom{n}{k} (1 - E_{k+1}) B_{n-k} \quad (n \in \mathbb{N}_0), \quad (65)$$

which is, in fact, the same as a known result given by (cf. [5, p. 61, Equation (18)])

$$\sum_{k=0}^n \frac{4^{n-k}}{2k+1} \binom{2n}{2k} B_{2n-2k} = 1 \quad (n \in \mathbb{N}_0) \tag{66}$$

or, equivalently, by

$$\sum_{k=0}^n \frac{2^{2k} B_{2k}}{(2k)!(2n-2k+1)!} = \frac{1}{(2n)!} \quad (n \in \mathbb{N}_0). \tag{67}$$

### 4. FORMULAS INVOLVING THE STIRLING NUMBERS OF THE SECOND KIND

Apostol [13] not only gave elementary properties of the so-called Apostol-Bernoulli polynomials  $\mathfrak{B}_n(x, \lambda)$ , but also obtained the following recursion formulas for the so-called Apostol-Bernoulli numbers  $\mathfrak{B}_n(\lambda)$  (see [13, p. 166, Equation (3.7)]):

$$\mathfrak{B}_n(\lambda) = n \sum_{k=0}^{n-1} \frac{k!(-\lambda)^k}{(\lambda-1)^{k+1}} S(n-1, k) \quad (\lambda \in \mathbb{C} \setminus \{1\}; n \in \mathbb{N}_0). \tag{68}$$

Luo and Srivastava [11] established substantially more general recursion formulas for the generalized Apostol-Bernoulli polynomials  $\mathfrak{B}_n^{(l)}(x; \lambda)$  ( $l \in \mathbb{N}_0$ ) and the generalized Apostol-Bernoulli numbers  $\mathfrak{B}_n^{(l)}(\lambda)$  ( $l \in \mathbb{N}_0$ ) as follows (see [11, Equations (27),(30)]):

$$\begin{aligned} \mathfrak{B}_n^{(l)}(x; \lambda) &= l! \sum_{k=l}^n \binom{n}{k} \binom{k}{l} x^{n-k} \sum_{j=0}^{k-l} \binom{l+j-1}{j} \\ &\cdot \frac{j!(-\lambda)^j}{(\lambda-1)^{j+l}} S(k-l, j) \quad (\lambda \in \mathbb{C} \setminus \{1\}; n, l \in \mathbb{N}_0) \end{aligned} \tag{69}$$

and

$$\mathfrak{B}_n^{(l)}(\lambda) = l! \binom{n}{l} \sum_{k=0}^{n-l} \binom{l+k-1}{k} \frac{k!(-\lambda)^k}{(\lambda-1)^{k+l}} S(n-l, k) \quad (\lambda \in \mathbb{C} \setminus \{1\}; n, l \in \mathbb{N}_0). \tag{70}$$

Luo [12], on the other hand, obtained the following general recursion formulas for the generalized Apostol-Euler polynomials  $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$  and the generalized Apostol-Euler numbers  $\mathfrak{E}_n^{(\alpha)}(\lambda)$  (see [12, equations (20),(29)]):

$$\mathfrak{E}_n^{(\alpha)}(x; \lambda) = 2^\alpha \sum_{k=0}^n \binom{n}{k} x^{n-k} \sum_{j=0}^k \binom{\alpha+j-1}{j} \frac{j!(-\lambda)^j}{(\lambda+1)^{j+\alpha}} S(k, j) \quad (\alpha, \lambda \in \mathbb{C}; n \in \mathbb{N}_0) \tag{71}$$

and

$$\begin{aligned} \mathfrak{E}_n^{(\alpha)}(\lambda) &= (-1)^n \sum_{k=0}^n \binom{n}{k} 2^{k+\alpha} \alpha^{n-k} \sum_{j=0}^k \binom{\alpha+j-1}{j} \frac{j!(-\lambda)^j}{(\lambda+1)^{j+\alpha}} S(k, j) \\ &(\alpha, \lambda \in \mathbb{C}; n \in \mathbb{N}_0). \end{aligned} \tag{72}$$

Here, and in what follows,  $S(n, k)$  denotes the Stirling numbers of the second kind, which are defined by (see [7, p. 207, Theorem B])

$$x^n = \sum_{k=0}^n \binom{x}{k} k! S(n, k), \tag{73}$$

so that

$$S(n, 0) = \delta_{n,0}, \quad S(n, 1) = S(n, n) = 1, \quad \text{and} \quad S(n, n - 1) = \binom{n}{2}, \tag{74}$$

$\delta_{m,n}$  being the Kronecker symbol (see also [5, p. 58, equation (25) *et seq.*]).

Finally, we give an addition formula for each of the generalized Apostol-Bernoulli and the generalized Apostol-Euler polynomials. Indeed, from the addition theorems (26) and (27) in conjunction with (73), we can deduce the addition formulas asserted by Theorem 3 below (see also [15]).

**THEOREM 3.** *The following relationships:*

$$\mathfrak{B}_n^{(\alpha)}(x + y; \lambda) = \sum_{k=0}^n \binom{x}{k} k! \sum_{j=0}^{n-k} \binom{n}{j} \mathfrak{B}_j^{(\alpha)}(y; \lambda) S(n - j, k) \tag{75}$$

and

$$\mathfrak{E}_n^{(\alpha)}(x + y; \lambda) = \sum_{k=0}^n \binom{x}{k} k! \sum_{j=0}^{n-k} \binom{n}{j} \mathfrak{E}_j^{(\alpha)}(y; \lambda) S(n - j, k) \quad (\alpha, \lambda \in \mathbb{C}; n \in \mathbb{N}_0) \tag{76}$$

hold true between the generalized Apostol-Bernoulli polynomials, the generalized Apostol-Euler polynomials, and the Stirling numbers of the second kind.

**REMARK 8.** By setting  $\lambda = 1$  in (75) and (76), it is easy to deduce the following interesting consequences of Theorem 3:

$$B_n^{(\alpha)}(x + y) = \sum_{k=0}^n \binom{x}{k} k! \sum_{j=0}^{n-k} \binom{n}{j} B_j^{(\alpha)}(y) S(n - j, k) \quad (\alpha \in \mathbb{C}; n \in \mathbb{N}_0) \tag{77}$$

and

$$E_n^{(\alpha)}(x + y) = \sum_{k=0}^n \binom{x}{k} k! \sum_{j=0}^{n-k} \binom{n}{j} E_j^{(\alpha)}(y) S(n - j, k) \quad (\alpha \in \mathbb{C}; n \in \mathbb{N}_0) \tag{78}$$

for the generalized Bernoulli polynomials and the generalized Euler polynomials of order  $\alpha$ .

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