Available online at www.sciencedirect.com
ScienceDirect
Discrete Applied Mathematics 156 (2008) 2620–2635
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Ultrametrics, Banach's fixed point theorem and the Riordan group

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Received 24 May 2006; received in revised form 1 October 2007; accepted 13 October 2007
Available online 20 February 2008

Abstract

We interpret the reciprocation process in $\mathbb{K}[[x]]$ as a fixed point problem related to contractive functions for certain adequate ultrametric spaces. This allows us to give a dynamical interpretation of certain arithmetical triangles introduced herein. Later we recognize, as a special case of our construction, the so-called Riordan group which is a device used in combinatorics. In this manner we give a new and alternative way to construct the proper Riordan arrays. Our point of view allows us to give a natural metric on the Riordan group turning this group into a topological group. This construction allows us to recognize a countable descending chain of normal subgroups.

Keywords: Banach's fixed point theorem; Pascal triangle; Ultrametrics; Riordan arrays; Riordan group; Arithmetical triangles

1. Introduction

It is a widely known fact that Banach's fixed point theorem, besides its simplicity, is one of the main tools for both the theoretical and the computational aspects in Mathematics:

A simple statement (with a simple proof in this case) with many applications.

This theorem is a theoretical framework of the successive approximation method used by Picard, even by Liouville. The well-known statement of this theorem is:

BANACH'S FIXED POINT THEOREM (BFPT)

Let (X, d) be a complete metric space and $f : X \to X$ contractive. Then f has a unique fixed point x_0 and $f^n(x) \to x_0$ for every $x \in X$.

In the above statement $f^n = f \circ f \circ \cdots \circ f$. Recall that a map is contractive, concretely c-contractive, if there is a real number $c \in [0, 1)$ such that $d(f(x), f(y)) \leq cd(x, y)$. We recommend, for example, [2] for the description of some of the applications of this result.

There is a mild generalization of BFPT that we will refer to as GBFPT (for Generalized Banach Fixed Point Theorem), which corresponds to a certain shadowing process in BFPT (for analogy to shadowing in discrete dynamical systems [3]). GBFPT is implicitly in the Fiber Contraction Principle given by Hirsch and Pugh in [5]. In [20], see Lemma 2 in page 212, GBFPT is explicitly stated. Sotomayor used GBFPT to get differentiability properties of vector fields associated to differential equations. For completeness we will recall this result as it appears in [20].

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0166-218X/$ - see front matter
doi:10.1016/j.dam.2007.10.026

Proposition 1 (*GBFPT*). *Let X be a complete metric space. Suppose $\{f_n\}_{n\in\mathbb{N}} : X \longrightarrow X$ is a sequence of contractive maps with the same contraction constant α and suppose that $\{f_n\} \longrightarrow f$ (point to point). Then f is α-contractive and for any point $z \in X$ the sequence $\{f_n \circ \cdots \circ f_1(z)\} \xrightarrow[n\to\infty]{} x_0$, where x_0 is the unique fixed point of f.*

The Riordan group, in the title, was so named for the first time by Shapiro et al. in [18]. In fact in [18] the authors named a subgroup of the group treated herein as the "Riordan group".

More general objects called Riordan arrays appear in the literature. A special kind of Riordan arrays, called renewal arrays, were introduced before by Rogers [17]. Any element in the Riordan group is a Riordan array. The literature on Riordan arrays originated mainly in the last decade of the last century, [8–10,18,21,22], and is still developing now, [4,6,11–13,19,23].

Researchers in enumerative combinatorics used Riordan arrays mainly to unify many themes in enumerations. For example Sprugnoli in [21,22] used that to find the generating function of many combinatorial sums.

The use of Riordan arrays was also related to inverse relations and to the so-called Schauder bases in [6], by using inverses in the Riordan group. We recommend the classical text of Riordan [15] for information on combinatorial topics and to make a comparison with the way it was treated before the Riordan arrays point of view appeared.

Although organized into six sections, including this introduction, this paper has three clearly different parts:

The first includes Sections 2–4 where we treat the elements of the Riordan group. The second coincides with Section 5 where the group operation and some basic algebraic properties are treated and in the third part, which is Section 6, we try to give some information about the global algebraic structure (the recognition of many normal subgroups) providing this group with an additional structure of non-Archimedean metrizable topological group for this aim. The guide line converting these three parts into a unit is the use of an adequate ultrametric framework.

In the first part, and after a motivation of our point of view given in Section 2, we convert, in Section 3, the problem of finding the quotient of two series into a fixed point problem associated to a contractive function defined in a suitable complete ultrametric space $(\mathbb{K}[[x]], d)$. Consequently Banach's fixed point Theorem gets an iterative algorithm to do that. In Section 4 we give a new algorithm to construct Riordan arrays depending on two given power series. The main difference with those known in the literature is that we do not have to use any extra object as the A-sequence or the Z-sequence, see for example Rogers [17], Sprugnoli [21] and Merlini et al. [8]. Anyway the A-sequence and the Z-sequence are very interesting objects to construct Riordan array as pointed out in the papers quoted above. Our algorithm covers, using only the initial data, the recurrence for any entry, even those entries in the first column, in the Riordan array. Our new point of view, based on Banach's fixed point Theorem, allows us to construct Riordan arrays using iteratively the classical algorithm to get the coefficients in the quotient of two given series. So we show that the structure of Riordan arrays and the reciprocation operation in the ring $\mathbb{K}[[x]]$ are intrinsically related where $\mathbb{K}$ is any field of characteristic zero.

In the second part, Section 5, we characterize the continuous endomorphisms in $(\mathbb{K}[[x]], d)$ and certain matrix representations. Analogously as in finite-dimensional Linear Algebra this representation gives us one of the main results about Riordan arrays, see Theorem 1.1 and the previous comments in Sprugnoli [21]. As a consequence we show that the Riordan group has a faithful representation as a group of $\mathbb{K}$-linear isometries in $(\mathbb{K}[[x]], d)$. Using this we get, in our own way, the known formula for the composition and the inverse in the group. We want to note that the group operation and the action on a power series can be given in terms of the so-called *Lagrangre group*, see Huang [6] and Sprugnoli [22]. Here we note that our parametrization of the elements in the Riordan group, in terms of a pair of power series, is different from the usual one. We end this section giving, in our own way, a result on the algebraic structure of the Riordan group in terms of a semidirect product of certain subgroups. There are some analogous results in the literature on this topic, [4,19].

In the third part, Section 6, and motivated by the last result in the previous section we find new normal subgroups of the Riordan group. This could help in obtaining new decomposition results. This is the reason why, motivated by the Banach space theory and by the classical Lie groups of finite matrices, we give an ultrametric in the set of continuous endomorphisms on $(\mathbb{K}[[x]], d)$. We prove later that this induces an invariant ultrametric in the group of isometries and eventually on the Riordan group. Consequently the identity has a neighborhood base formed by open and closed normal subgroups. We describe these groups in terms of the involved power series. Even so we think that our results on describing new normal subgroups are still modest but we also think that the ultrametric defined on the Riordan group could help with further developments on these and other topics.

To finish this introduction we have to say that, as we will explain in the next section, all the work in this paper was motivated by three simple observations:

First, the relation between Banach's Fixed Point Theorem and the way to sum the geometric progression $\sum_{n\geq 0} x^n$.

Second, the relation between the Generalized Banach Fixed Point Theorem and the way to sum the arithmetic–geometric progression $\sum_{n\geq 1} nx^n$.

Third, the coefficients of the above series are respectively in the first and the second columns in the Pascal Triangle.

2. Motivation: GBFPT and Pascal triangle

The usual proof of Banach's fixed Point Theorem, see [2] for example, relies strongly on the fact that $\lim_{n\to\infty} x^n = 0$ when $x \in \mathbb{R}$ is such that $|x| < 1$. Also the partial sums of the geometric series are involved in the proof.

We can reverse this chain of relations. In fact, if our starting point is the Banach's fixed point theorem, we can obtain that $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$ iterating the function $f(t) = xt$ starting at $t = 1$. On the other hand, let us consider the real, or complex, function $f(t) = xt+1$ with $|x| < 1$. It is obvious that $f^n(0) = \sum_{k=0}^{n-1} x^k$. Consequently $f^n(0) \longrightarrow \frac{1}{1-x}$ which is the unique fixed point of f. Here again f^n represents the n-iteration of f.

The next classical and easier series to sum is the arithmetic–geometric series $\sum_{k=1}^{\infty} kx^k$. It easy to see that there are no one-degree polynomial $f(t) = g(x)t + h(x)$ and any point x_0 such that the partial sum $\sum_{k=1}^{n} kx^k = f^n(x_0)$. But using the Generalized Fixed Point Theorem we can solve our problem by means of crossed iterations of an equicontractive sequence of one-degree polynomials where the geometric series is involved. Consider the sequence of one-degree polynomials

$$h_m(t) = xt + x\sum_{k=0}^{m-1} x^k \quad (m = 0, 1, 2, \ldots)$$

with the agreement $\sum_{k=0}^{-1} = 0$.

Note that $h_m(t) = xt + xT_{m-1,1}(x)$ where $T_{m-1,1}$ is the $(m-1)$-Taylor polynomial of the geometric series, it is the first column in the Pascal triangle (see below). If $|x| < 1$ then $\{h_m\}_{m\in\mathbb{N}\cup\{0\}}$ is a sequence of $|x|$-contractive functions and

$$\{h_m\} \longrightarrow h(t) = xt + \frac{x}{1-x}.$$

Using GBFPT we obtain that

$$(h_m \circ \cdots \circ h_0)(0) \longrightarrow \frac{x}{(1-x)^2}$$

which is the unique fixed point of the limit function $h(t) = xt + \frac{x}{1-x}$. Now $h_0(0) = 0$, $(h_1 \circ h_0)(0) = x$, $h_2((h_1 \circ h_0))(0) = x+2x^2$. By induction $(h_m \circ \cdots h_0)(0) = \sum_{k=0}^{m} kx^k$. Consequently we get that $\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$.

One of the usual ways to describe the Pascal Triangle is by means of an infinite triangular matrix whose rows are the coefficients of the polynomial $P_n(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. That is

$$\begin{array}{ccccccccccccc}
1 & & & & & & & & & & \to & (1+x)^0 \\
1 & 1 & & & & & & & & & \to & (1+x)^1 \\
1 & 2 & 1 & & & & & & & & \to & (1+x)^2 \\
1 & 3 & 3 & 1 & & & & & & & \to & (1+x)^3 \\
1 & 4 & 6 & 4 & 1 & & & & & & \to & (1+x)^4 \\
1 & 5 & 10 & 10 & 5 & 1 & & & & & \to & (1+x)^5 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & & & & \to & (1+x)^6 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & & & \to & (1+x)^7 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} & \binom{n}{5} & \binom{n}{6} & \binom{n}{7} & \cdots & \binom{n}{n} & \to & (1+x)^n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\frac{1}{1-x} & \frac{x}{(1-x)^2} & \frac{x^2}{(1-x)^3} & \frac{x^3}{(1-x)^4} & \frac{x^4}{(1-x)^5} & \frac{x^5}{(1-x)^6} & \frac{x^6}{(1-x)^7} & \frac{x^7}{(1-x)^8} & \cdots & \frac{x^{n-1}}{(1-x)^n} & \to & \ddots
\end{array}$$

We can interpret the Pascal Triangle, by columns, as a countable set of powers series. In the first column the coefficients of $\sum_{k=0}^{\infty} x^k$ appear, in the second those of $\sum_{k=0}^{\infty}(k+1)x^k$, in the third the coefficients of $\sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} x^k$ appear, and so on. But each column is shifted one step below the previous one. This means that j-column represents the series $\frac{x^{j-1}}{(1-x)^j}$. In fact this is what really happens if we write in the triangle not only the coefficients but the powers of x (in increasing order) in the development of $(1+x)^n$ as it is by rows. As we saw before, the first and the second columns in the Pascal triangle are just the coefficients of the geometric and the arithmetic geometric series respectively.

Let us try only the next column: consider now the sequence $h_m(t) = xt + x\sum_{k=0}^{m-1} kx^k$. Note that, as before, the independent term of the polynomial h_m is just $xT_{m-1,2}(x)$ where $T_{m-1,2}$ is the $(m-1)$-Taylor polynomial of the second column. In this case the limit function is $h(t) = xt + x\frac{x}{(1-x)^2}$ whose unique fixed point is $t = \frac{x^2}{(1-x)^3}$ and $h_0(0) = h_1(h_0(0)) = 0, h_2(h_1(h_0(0))) = x^2$ and by induction $(h_m \cdots h_0)(0) = \sum_{k=0}^{m} \frac{(k-1)k}{2} x^k \longrightarrow \frac{x^2}{(1-x)^3}$.

So we can conclude without a real proof yet:

Proposition 2. *For $n \geq 2$, the n-column in the Pascal triangle is obtained from the $(n-1)$-column applying the crossed iterations in **GBFPT** to the sequence $\{h_{k,n}\}_{k\in\mathbb{N}\cup\{0\}}$ where $h_{k,n}(t) = xt + xT_{k-1,n-1}(x)$, $|x| < 1$ being $T_{k-1,n-1}$ the $(k-1)$-Taylor polynomial of the $(n-1)$-column.*

3. Reciprocation in $\mathbb{K}[[x]]$ as a fixed point problem

Let $\mathbb{K}$ be a field (of characteristic zero). Consider the ring of power series $\mathbb{K}[[x]]$ with coefficients in $\mathbb{K}$. Let $f \in \mathbb{K}[[x]]$ given by $f = \sum_{n\geq 0} a_n x^n$ and denote by $\omega(f)$ the order of f. Recall that $\omega(f)$ is the smallest non-negative integer number p such that $a_p \neq 0$ if any exists. Otherwise, that is if $f = 0$, we write $\omega(f) = \infty$.

Given a non-negative integer k and the series f as above we denote by $T_k(f)$ the corresponding Taylor polynomial of order k.

It is well known that $(\mathbb{K}[[x]], d)$ is a complete ultrametric space where $d(f, g) = \frac{1}{2^{\omega(f-g)}}$, $f, g \in \mathbb{K}[[x]]$. In the previous formula we understand that $\frac{1}{2^\infty} = 0$. In order to refer to this and some other related fact we put this in the following ($\mathbb{R}_+$ represents the non-negative real numbers).

Proposition 3. *The map $d : \mathbb{K}[[x]] \times \mathbb{K}[[x]] \to \mathbb{R}_+$ defined by $d(f, g) = \frac{1}{2^{\omega(f-g)}}$ is a complete ultrametric on $\mathbb{K}[[x]]$. Moreover $d(f, g) \leq \frac{1}{2^{k+1}}$ if and only if $T_k(f) = T_k(g)$. Finally the sum and product of the series are continuous if we consider the corresponding product topology in $\mathbb{K}[[x]] \times \mathbb{K}[[x]]$.*

Remark 4. The proofs of all the facts above are easy consequences of the properties of the order of a series.

Note that if $f \in \mathbb{K}[[x]]$ then $\lim_{k\to\infty} T_k(f) = f$ in $(\mathbb{K}[[x]], d)$ and then the set of polynomials $\mathbb{K}[x]$ is, topologically, dense in the space of the series. Moreover the relative topology induced on $\mathbb{K}[x]$ is discrete. The induced metric d on $\mathbb{K}_l[x]$ is uniformly discrete, where the subscript l means "degree less than or equal to l".

It is obviously known that g is a unit in the ring $\mathbb{K}[[x]]$ if and only if $g(0) \neq 0$. We want to show here that the Banach's fixed point Theorem allows us to give a different proof of this fact which, in addition, gives us an expression for the reciprocal. In the next proposition f^n represents the n-power of the series f.

Proposition 5. *Let $f, h \in \mathbb{K}[[x]]$ with $f(0) = 0$ then the first-degree polynomial map $P : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ defined by $P(S) = fS + h$ is $\frac{1}{2}$-contractive independently on f and h. In fact $d(P(S_1), P(S_2)) = \frac{1}{2^{\omega(f)}} d(S_1, S_2)$. Moreover the unique fixed point of P is just $\frac{h}{1-f}$ and consequently*

$$\frac{h}{1-f} = \left(\sum_{n\geq 0} f^n\right) h.$$

Proof. Let $S_1, S_2 \in \mathbb{K}[[x]]$, then $d(P(S_1), P(S_2)) = \frac{1}{2^{\omega(P(S_1)-P(S_2))}}$ but $\omega(P(S_1) - P(S_2)) = \omega(f(S_1 - S_2)) = \omega(f) + \omega(S_1 - S_2)$ so $d(P(S_1), P(S_2)) = \frac{1}{2^{\omega(P(S_1)-P(S_2))}} = \frac{1}{2^{\omega(f)}} d(S_1, S_2)$. Since $f(0) = 0$, then $\omega(f) \geq 1$ and so

$d(P(S_1), P(S_2)) \leq \frac{1}{2}d(S_1, S_2)$. Using now the Banach's fixed point theorem we obtain that the unique fixed point u of P is just $u = \lim_{n\to\infty} P^n(0)$, where $P^n(0)$ is the n-iteration at 0. But $P^n(0) = (\sum_{k=0}^{n-1} f^k)h$. So $u = (\sum_{k=0}^{\infty} f^k)h$ and $u = \frac{h}{1-f}$ because it is the unique solution of $fu + h = u$. □

Corollary 6. *If $g \in \mathbb{K}[[x]]$ and $g(0) \neq 0$, then g is a unit in $\mathbb{K}[[x]]$ and*

$$\frac{1}{g} = \frac{1}{g(0)} \sum_{n\geq 0} \left(\frac{g(0) - g}{g(0)}\right)^n.$$

Proof. Consider the series $\frac{g(0)-g}{g(0)}$, then $\omega(\frac{g(0)-g}{g(0)}) \geq 1$. Take, as in the above proposition, $P(S) = (\frac{g(0)-g}{g(0)})S + \frac{1}{g(0)}$. So P is contractive and the unique fixed point u satisfies $(\frac{g(0)-g}{g(0)})u + \frac{1}{g(0)} = u$. Using the algebraic operations in $\mathbb{K}[[x]]$ one obtains that $u \cdot g = 1$. Consequently g is a unit and $\frac{1}{g} = \frac{1}{g(0)} \sum_{n\geq 0}(\frac{g(0)-g}{g(0)})^n$. □

Remark 7. Note that in Proposition 5 there appears a convergent series of the series $\sum_{n\geq 0} f^n$. This also can be explained by means of the ultrametric d in the following way:

In an ultrametric space, see [16], a sequence $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if $d(a_n, a_{n+1}) \longrightarrow 0$. Take the sequence $S_n = \sum_{k=0}^{n} f^k$. Since $\omega(f) \geq 1$ then $\omega(f^n) \geq n$. Consequently $d(S_{n+1}, S_n) = \frac{1}{2^{\omega(S_{n+1}-S_n)}} \leq \frac{1}{2^{n+1}}$. But S_n is just the corresponding partial sum of $\sum_{k\geq 0} f^k$.

The following proposition will be important for the rest of the paper. In fact it is a refined version, in our context, of GBFPT. It gives not only convergence but controls the remainders.

Proposition 8. *Let $f, g \in \mathbb{K}[[x]]$ with $g(0) \neq 0$. Fix the one-degree polynomial function $P : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ defined by $P(S) = (\frac{g(0)-g}{g(0)})S + \frac{f}{g(0)}$. Consider the sequence of one-degree polynomial functions: $\{P_m\}_{m\in\mathbb{N}} : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ defined by $P_m(S) = T_m(\frac{g(0)-g}{g(0)})S + T_m(\frac{f}{g(0)})$. Then $\{P_m\}_{m\in\mathbb{N}} \longrightarrow P$ uniformly in $(\mathbb{K}[[x]], d)$. P_m is $\frac{1}{2}$-contractive for every $m \in \mathbb{N}$. Moreover $d(P_m \circ P_{m-1} \circ \cdots \circ P_0(0), f/g) \leq \frac{1}{2^{m+1}}$ and consequently $T_m(P_m \circ P_{m-1} \circ \cdots \circ P_0(0)) = T_m(f/g)$.*

Proof. First of all note that $\omega(T_m(S) - S) \geq m + 1$ for any $S \in \mathbb{K}[[x]]$. Then $d(T_m(S), S) \leq \frac{1}{2^{m+1}}$. So $d(P_m(S), P(S)) = \frac{1}{2^{\omega(P_m(S)-P(S))}}$, but $P_m(S) - P(S) = (T_m(\frac{g(0)-g}{g(0)}) - \frac{g(0)-g}{g(0)})S + (T_m(\frac{f}{g(0)}) - \frac{f}{g(0)})$. Consequently $\omega(P_m(S) - P(S)) \geq \min\{\omega(T_m(\frac{g(0)-g}{g(0)}) - \frac{g(0)-g}{g(0)}) + \omega(S), \omega(T_m(\frac{f}{g(0)}) - \frac{f}{g(0)})\} \geq m + 1$ for any $S \in \mathbb{K}[[x]]$. Hence $d(P_m(S), P(S)) \leq \frac{1}{2^{m+1}}$ independently on S. Now $d(P_m \circ P_{m-1} \circ \cdots \circ P_0(0), f/g) \leq \max\{d(P_m \circ P_{m-1} \circ \cdots \circ P_0(0)), P_m(f/g), d(P_m(f/g), P(f/g))\}$ by the strong triangle inequality and the fact that $P(f/g) = f/g$. So $d(P_m(f/g), P(f/g)) \leq \frac{1}{2^{m+1}}$. We only have to control the number $\delta_m = d(P_m \circ P_{m-1} \circ \cdots \circ P_0(0), P_m(f/g))$. Let us prove by induction that $\delta_m \leq \frac{1}{2^{m+1}}$. If $m = 0$ then $\delta_0 = 0$, because $P_0(S) = f(0)/g(0)$ for any $S \in \mathbb{K}[[x]]$. Since P_1 is $\frac{1}{2}$-contractive we obtain that $d(P_1 \circ P_0(0), P_1(f/g)) \leq \frac{1}{2}d(P_0(0), f/g) \leq \frac{1}{2^2}$. Suppose that $\delta_k \leq \frac{1}{2^{k+1}}$. Now $d(P_{k+1} \circ P_k \circ \cdots \circ P_0(0), P_{k+1}(f/g)) \leq \max\{d(P_{k+1} \circ P_k \circ \cdots \circ P_0(0), P_{k+1}(P_k(f/g))), d(P_{k+1}(P_k(f/g)), P_{k+1}(f/g))\}$, but $d(P_{k+1} \circ P_k \circ \cdots \circ P_0(0), P_{k+1}(P_k(f/g))) \leq \frac{1}{2}d(P_k \circ \cdots \circ P_0(0), P_k(f/g)) \leq \frac{1}{2^{k+2}}$ and $d(P_{k+1}(P_k(f/g)), P_{k+1}(f/g)) \leq \frac{1}{2}d(P_k(f/g), P(f/g)) \leq \frac{1}{2^{k+2}}$. Putting all together and using induction we have proved that $\delta_m \leq \frac{1}{2^{m+1}}$. Consequently $d(P_m \circ P_{m-1} \circ \cdots \circ P_0(0), f/g) \leq \frac{1}{2^{m+1}}$. So $T_m(P_{m-1} \circ \cdots \circ P_0(0)) = T_m(f/g)$. □

In order to avoid useless operations in the procedure described in Proposition 8, we can refine the obtained recurrence process as follows:

Corollary 9. *Let $f, g \in \mathbb{K}[[x]]$ with $g(0) \neq 0$. Then for every $m \in \mathbb{N}$ we have:*

$$T_m(f/g) = T_m(P_m(T_{m-1}(P_{m-1} \cdots (T_1(P_1(T_0(P_0(0))))) \cdots))).$$

Proof. Using the same notation as in the last proposition we have

$$d(P_m(T_{m-1}(f/g)), f/g) \leq \max\{d(P_m(T_{m-1}(f/g)), P_m(f/g)), d(P_m(f/g), P(f/g))\} \leq \frac{1}{2^{m+1}}.$$

Hence $T_m(P_m(T_{m-1}(f/g))) = T_m(f/g)$. Consequently one can avoid all operations related to the remainder $R_m(x) = P_m \circ P_{m-1} \circ \cdots \circ P_0(0) - T_m(f/g)$. □

Corollary 9 provides an algorithm that can be summarized as follows:

$$T_k(f/g) = T_k\left(\left(\frac{g(0) - T_k(g)}{g(0)}\right) T_{k-1}(f/g) + \frac{1}{g(0)} T_k(f)\right).$$

If we write the result above in, we will say, “coordinates” we have

Corollary 10. *Let $f, g \in \mathbb{K}[[x]]$ with $g(0) \neq 0$. If $f = \sum_{n\geq 0} a_n x^n$ and $g = \sum_{n\geq 0} b_n x^n$ and $f/g = \sum_{n\geq 0} d_n x^n$, then $d_n = -\frac{b_1}{b_0} d_{n-1} - \frac{b_2}{b_0} d_{n-2} \cdots - \frac{b_n}{b_0} d_0 + \frac{a_n}{b_0}$, for $n \geq 1$, $d_0 = \frac{a_0}{b_0}$.*

In the above result there are hidden known recurrences:

Bernoulli numbers

Recall that Bernoulli numbers $\{B_k\}_{k\in\mathbb{N}}$ are defined by means of their exponential generating function: $\frac{x}{e^x - e^0} = \sum_{k\geq 0} \frac{B_k}{k!} x^k$.

Suppose that $f \equiv 1$, and take $g(x) = \frac{e^x - e^0}{x} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} x^k$ but $\frac{x}{e^x - e^0} = \frac{1}{g}$. Using the recurrence above we have $d_0 = 1$, $d_n = \frac{B_n}{n!} = \sum_{\mu=0}^{n-1} -\frac{1}{(n-\mu+1)!} \frac{B_\mu}{\mu!}$ or $B_n = \sum_{\mu=0}^{n-1} -\frac{n!}{(n-\mu+1)!\mu!} B_\mu$. Multiplying both sides by $n+1$ we obtain $(n+1)B_n + \sum_{\mu=0}^{n-1} \frac{n!}{(n-\mu+1)!\mu!} B_\mu = 0$ or $\sum_{\mu=0}^{n-1} \binom{n+1}{\mu} B_\mu = 0$, $B_0 = 1$ which is the usual recurrence for Bernoulli numbers.

Generalized Fibonacci and Lucas numbers

Suppose now that $g(x) = a + bx + cx^2$ with $a \neq 0$. $\frac{1}{g} = \frac{1}{a+bx+cx^2}$. In this case $\frac{1}{g(0)} = \frac{1}{a}$, $b_1 = b$, $b_2 = c$, $b_k = 0$ if $k > 2$. Suppose that $\frac{1}{g} = \sum_{n\in\mathbb{N}} d_n x^n$. So using Corollary 10 one obtains $d_n = -\frac{b}{a} d_{n-1} + (-\frac{c}{a}) d_{n-2}$ if $n \geq 2$, $d_0 = \frac{1}{a}$ and $d_1 = -\frac{b}{a^2}$. So one obtains the usual recurrence for generalized Fibonacci numbers.

As in the above example suppose now that $g(x) = a + bx + cx^2$ $a \neq 0$ and $f = 2a + bx$. Suppose now that $\frac{2a+bx}{a+bx+cx^2} = \sum_{n\geq 0} c_n x^n$. In this case $a_0 = 2a$, $a_1 = b$, $a_k = 0$ for $k > 1$, $b_0 = a$, $b_1 = b$, $b_2 = c$, $b_k = 0$, $k > 2$. Hence, by the above recurrence, we obtain $c_n = -\frac{b}{a} c_{n-1} + (-\frac{c}{a}) c_{n-2}$, $c_0 = 2$, $c_1 = -\frac{b}{a}$ which is the usual recurrence of the so-called in the literature, see [7] for example, Lucas sequence $\{c_k\}$ associated to the generalized Fibonacci sequence $\{d_k\}$ in the above example.

4. Arithmetical triangles arising from Banach’s fixed point theorem

With the next construction we are going to capture and extend, using GBFPT, the pattern of formation of the Pascal Triangle in Section 2.

Given are $f = \sum_{n\geq 0} a_n x^n$ and $g = \sum_{n\geq 0} b_n x^n$ with $g(0) = b_0 \neq 0$. We are going to construct a lower triangular matrix that we call *the arithmetical triangle of the power series f with rate g*. We denote it by $T(f \mid g)$. We construct $T(f \mid g)$ by columns. In the first column are the coefficients of the series $\frac{f}{g}$, in the second those of $\frac{xf}{g^2}$, so in the j-column appear the coefficients of $\frac{x^{j-1}f}{g^j}$. As one can see the j-column is the j-th term of a geometric progression (in $\mathbb{K}[[x]]$), whose rate is $\frac{x}{g}$ and first term $\frac{f}{g}$. This corresponds to the following crossed iteration. To construct the j-column, $j \geq 2$, we consider $\frac{1}{2}$-equicontractive sequences given by

$$h_m(S) = T_m\left(\frac{g(0) - g}{g(0)}\right) S + x T_{m-1,j-1}\left(\frac{x^{j-2} f}{g(0) g^{j-1}}\right) \tag{1}$$

$\{h_m\} \to h$ (in particular point to point) where $h(S) = (\frac{g(0)-g}{g(0)}) S + x(\frac{x^{j-2} f}{g(0) g^{j-1}})$. Note that h is also $\frac{1}{2}$-contractive and its fixed point is $\frac{x^{j-1} f}{g^j}$ which is the j-th column.

Note also that $(\frac{x^{j-2}f}{g^{j-1}})$ is just the $j-1$-column. This implies that (1) gives us a recursive algorithm to construct $T(f \mid g)$ using only the coefficients of f and g. To find this algorithm of construction we need an *auxiliary column*, the 0-column, formed by the coefficients $(a_n)_{n\in\mathbb{N}}$ of f and to follow the rule described in Corollary 10 to calculate the coefficients of $\frac{f}{g}$ which is the first column in $T(f \mid g)$. To get the second we use just the same arguments but resting on this time in the first column, not in the 0-column. So we construct the j-th column using the algorithm in Corollary 10 and resting on the $(j-1)$-th column.

We then obtain

$$\begin{array}{c|llll}
a_0 & & & & \\
a_1 & a_0/b_0 & & & \\
a_2 & -\frac{b_1a_0}{b_0^2}+\frac{a_1}{b_0} & a_0/b_0^2 & & \\
a_3 & \frac{b_1^2a_0}{b_0^3}-\frac{b_1a_1}{b_0^2}-\frac{b_2a_0}{b_0^2}+\frac{a_2}{b_0} & -\frac{2b_1a_0}{b_0^3}+\frac{a_1}{b_0^2} & a_0/b_0^3 & \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
f & \frac{f}{g} & \frac{xf}{g^2} & \frac{x^2f}{g^3} & \cdots
\end{array}$$

Suppose that $T(f \mid g)=\{c_{ij}\}_{\substack{i\in\mathbb{N}\\ j\in\mathbb{N}}}$, so the element c_{ij} depends only the elements above in its column, they are $c_{i-1,j}\cdots c_{1,j}$ and the element $c_{i-1,j-1}$ just to its left in the row above. Moreover $c_{ij}=0$ if $j>i$. Collecting all that above we obtain:

Theorem 11. *Let $f=\sum_{n\geq 0}a_nx^n$, $g=\sum_{n\geq 0}b_nx^n$ with $b_0\neq 0$ then the matrix $T(f \mid g)=\{c_{ij}\}_{\substack{i\in\mathbb{N}\\ j\in\mathbb{N}}}$, $i,j\geq 1$, is defined by $\frac{x^{j-1}f}{g^j}=\sum_{i=1}^{\infty}c_{ij}x^{i-1}$. Consequently $T(f \mid g)$ is a Riordan array.*

The general construction is very easy to understand. We have the following:

Algorithm for $T(f \mid g)$

$f=\sum_{n\geq 0}a_nx^n$, $g=\sum_{n\geq 0}b_nx^n$ with $b_0\neq 0$, $T(f \mid g)=\{c_{ij}\}_{\substack{i\in\mathbb{N}\\ j\in\mathbb{N}}}$, $i,j\geq 1$.

$$\begin{array}{c|cccccc}
a_0 & & & & & & \\
a_1 & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & \cdots \\
a_2 & c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & \cdots \\
a_3 & c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
a_n & c_{n1} & c_{n2} & c_{n3} & c_{n4} & c_{n5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

with $c_{ij}=0$ if $j>i$ and the following rules for $i\geq j$:

If $j>1$

$$c_{i,j}=-\frac{b_1}{b_0}c_{i-1,j}-\frac{b_2}{b_0}c_{i-2,j}\cdots-\frac{b_{i-1}}{b_0}c_{1,j}+\frac{c_{i-1,j-1}}{b_0}=\frac{1}{b_0}\left(c_{i-1,j-1}-\sum_{k=1}^{i-1}b_kc_{i-k,j}\right)$$

and if $j=1$

$$c_{i,1}=-\frac{b_1}{b_0}c_{i-1,1}-\frac{b_2}{b_0}c_{i-2,1}\cdots-\frac{b_{i-1}}{b_0}c_{1,1}+\frac{a_{i-1}}{b_0}=\frac{1}{b_0}\left(a_{i-1}-\sum_{k=1}^{i-1}b_kc_{i-k,1}\right)$$

with the agreement $\sum_{k=1}^{0}=0$. Note that $c_{11}=a_0/b_0$.

Example 12. $T(1 \mid 1-x)\equiv$ Pascal triangle.

$a_0 = 1, a_n = 0,\ n \geq 1, b_0 = 1,\ b_1 = -1,\ b_n = 0$, for $n \geq 2$. In our notation $c_{i,j} = \binom{i-1}{j-1}$. The recurrence in the algorithm is $c_{i,j} = c_{i-1,j} + c_{i-1,j-1},\ j > 1, c_{i,1} = 1$, which is a new proof of the known recurrence of the binomial numbers.

Example 13. Fibonacci numbers. Consider the arithmetical triangle $T(1 \mid 1 - x - x^2)$. In this case $a_0 = 1, a_n = 0,\ n \geq 1, b_0 = 1,\ b_1 = b_2 = -1,\ b_n = 0,\ n \geq 3$. The corresponding recurrence is $c_{i,j} = c_{i-1,j} + c_{i-2,j} + c_{i-1,j-1}$, if $j > 1$. Also $c_{i,1} = c_{i-1,1} + c_{i-2,1}$, for $i \geq 3$, and $c_{11} = 1$, $c_{21} = 1$. Note that this last one is the recurrence for Fibonacci numbers.

Sprugnoli [21], pages 269–270, identified many generating functions associated to a Riordan array. We are going to choose two of them, the so-called bivariate generating function and the sum by shallow diagonals, as examples of how this can be interpreted in our context. Anyway it is a curious thing that Banach's fixed point Theorem can explain the fact that Fibonacci numbers are obtained from the Pascal triangle if one considers the shallow diagonals. Now it is convenient to rename the triangles as follows. Consider the arithmetical triangle

$$T(f \mid g) = \begin{pmatrix} c_{00} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ c_{10} & c_{11} & 0 & \cdots & 0 & 0 & \cdots \\ c_{20} & c_{21} & c_{22} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & \cdots \\ c_{n0} & c_{n1} & c_{n2} & \cdots & c_{nn} & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Bivariate generating function

Consider the bivariate generating function of $T(f \mid g)$ as defined by Sprugnoli in [21] page 269,

$$h(x,t) = \sum_{n,k \geq 0} c_{nk} x^n t^k.$$

So any column is a series in $\mathbb{K}[[x,t]]$. The first column is $\frac{f}{g}$ (independent on t), the second is $\frac{xtf}{g^2}$. The n-column is $\frac{(xt)^{n-1} f}{g^n}$. Consider the contractive map $P : ((\mathbb{K}[[x]]), d) \longrightarrow ((\mathbb{K}[[x]]), d)$ given by $P(S) = \frac{x}{g} S + \frac{f}{g}$. As we saw in Section 3, P is contractive. If we iterate P at the point $S = 0$ we have $P^m(0) = \sum_{k=0}^{m} \frac{f}{g^{m+1}} x^m$ which is just the sum of the first m-columns. So $\lim_{m\to\infty} P^m(0) = \frac{f}{g-x}$ which is the unique solution of $\frac{x}{g} S + \frac{f}{g} = S$. Now, as in the columns of $T(f \mid g)$, change the series x by the series xt. Then one obtains that the sum, by columns, in $T(f \mid g)$ is $\frac{f}{g-xt}$. But, obviously, if we sum by rows we have $h(x,t)$. Consequently $h(x,t) = \frac{f}{g-xt}$, with $g(0) \neq 0$. If we describe, as we will do, $T(f \mid g)$ in the notation (d,h) of [21], we obtain (1.2) in page 269 in [21].

Shallow diagonals

If we sum along shallow diagonals in $T(f \mid g)$, we obtain the following sequence of numbers: $F_0(f \mid g) = c_{00}$, $F_1(f \mid g) = c_{10}$, $F_2(f \mid g) = c_{20} + c_{11}$, $F_3(f \mid g) = c_{30} + c_{21}$ and so on. Consider the series $h = \sum_{k\geq 0} F_k(f \mid g) x^k$. Since each column in $T(f \mid g)$ represents the series $\frac{x^{n-1} f}{g^n}$, one can easily see that to sum along the shallow diagonals corresponds to summing by column the following: the first $+x$(the second)$+x^2$(the third)$+x^3$(the fourth)$+\cdots$ and so on. So we are summing $\frac{f}{g} + \frac{x^2 f}{g^2} + \frac{x^4 f}{g^3} + \frac{x^6 f}{g^4} + \cdots + \frac{x^{2n} f}{g^{n+1}} + \cdots$ whose partial sums in $\mathbb{K}[[x]]$ correspond to $H^m(0)$, where $H : \mathbb{K}[[x]] \longrightarrow \mathbb{K}[[x]]$ is given by $H(S) = \frac{x^2}{g} S + \frac{f}{g}$. H is obviously contractive, so $\sum_{k\geq 0} F_k(f \mid g) x^k = \frac{f}{g-x^2}$ because $\frac{f}{g-x^2}$ is the unique fixed point of H. Note that the Pascal triangle is just $T(1 \mid 1-x)$, consequently $\sum_{k\geq 0} F_k(1 \mid 1-x) x^k = \frac{1}{1-x-x^2}$. Hence $\{F_k(1 \mid 1-x)\}_{k\in\mathbb{N}}$ is the Fibonacci sequence $1, 1, 2, 3, \ldots$.

5. Arithmetical triangles as $\mathbb{K}$-linear continuous functions

We are going to study these triangles $T(f \mid g)$, $f \in \mathbb{K}[[x]]$, $g \in \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]$ considered as the matrix representations of continuous endomorphism in $(\mathbb{K}[[x]], d)$. We recall here, for notational facts, that $0 \in \mathbb{N}$. First of all we have:

Proposition 14. *Consider $\mathbb{K}[[x]]$ as a $\mathbb{K}$ vector space. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathbb{K}[[x]]$. Then there is a linear continuous function $\Phi : (\mathbb{K}[[x]], d) \to (\mathbb{K}[[x]], d)$ such that $\Phi(x^n) = f_n$, $n \in \mathbb{N}$ if and only if $\{f_n\}_{n\in\mathbb{N}} \to 0$ in $(\mathbb{K}[[x]], d)$. Moreover Φ is unique with the above properties and $\Phi(g) = \sum_{n\geq 0} a_n f_n$ where $g = \sum_{n\geq 0} a_n x^n$.*

Proof. Suppose first that there is a linear continuous function Φ, such that $\Phi(x^n) = f_n$. Then by continuity at 0, $f_n \to 0$ in $(\mathbb{K}[[x]], d)$.

For the opposite direction suppose now that $\{f_n\} \to 0$. Let $g = \sum_{n\geq 0} a_n x^n$ be any series. Consider the series of the series $\Phi(g) = \sum_{n\geq 0} a_n f_n$. We have to prove that $\sum_{n\geq 0} a_n f_n$ converges in $(\mathbb{K}[[x]], d)$. So take $S_m = \sum_{k=0}^{m} a_k f_k$, consequently $S_{m+1} - S_m = a_{m+1} f_{m+1}$. Hence $\lim_{m\to\infty}(S_{m+1} - S_m) = 0$. Since d is an ultrametric it implies, [16] page 73, that S_m is a Cauchy sequence and then convergent. By this way we define $\Phi(g)$ for every $g \in \mathbb{K}[[x]]$ and obviously Φ is linear. Moreover $\Phi(x^n) = f_n$. Take now $\{g_n\} \to g$ and $\epsilon > 0$, then there is a $m_0 \in \mathbb{N}$ such that $\frac{1}{2^{m_0}} < \epsilon$ and $d(f_p, 0) < \epsilon$ for $p \geq m_0$. Consider now $m_1 \geq m_0$ such that $d(g_n, g) < \frac{1}{2^{m_0}}$ for $n \geq m_1$. This means that $g_n - g = \sum_{k\geq m_0} a_{kn} x^k$. So we obtain $\Phi(g_n - g) = \sum_{k\geq m_0} a_{kn} f_k$. Hence $d(\Phi(g_n), \Phi(g)) \leq \max_{k\geq m_0}\{d(f_k, 0)\} < \epsilon$ for $n \geq m_1$. The uniqueness of Φ is clear. □

The following is now obvious.

Proposition 15. *Let $\Phi : (\mathbb{K}[[x]], d) \to (\mathbb{K}[[x]], d)$ be a linear continuous function and suppose that $f_n = \Phi(x^n) = \sum_{k=0}^{\infty} a_{kn} x^k$. Let $g = \sum_{k\geq 0} \alpha_k x^k$ be any series and suppose that $\Phi(g) = \sum_{k\geq 0} \beta_k x^k$. Then*

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{00} & \cdots & a_{0j} & \cdots \\ a_{10} & \cdots & a_{1j} & \cdots \\ \vdots & \cdots & \vdots & \cdots \\ a_{n0} & \cdots & a_{nj} & \cdots \\ \vdots & \cdots & \vdots & \cdots \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \\ \vdots \end{pmatrix}$$

it is $\beta_n = \sum_{k=0}^{\infty} a_{nk}\alpha_k$, where each of these sums is in fact finite.

Definition 16. We call the matrix defined above by means of Φ

$$M(\Phi) = \{a_{ij}\}_{\substack{i\in\mathbb{N} \\ j\in\mathbb{N}}}$$

the matrix associated to Φ.

Now we can rewrite Proposition 14 in the following way.

Corollary 17. *Let $M = \{a_{ij}\}_{\substack{i\in\mathbb{N} \\ j\in\mathbb{N}}}$ be an infinite matrix with entries in $\mathbb{K}$, then M represents a continuous linear mapping $\Phi_M : (\mathbb{K}[[x]], d) \to (\mathbb{K}[[x]], d)$ (i.e. $M = M(\Phi_M)$, Φ_M continuous) if and only if for every $n \in \mathbb{N}$ there is a $m \in \mathbb{N}$ such that $a_{0,p} = a_{1,p} = \cdots = a_{n,p} = 0$ for every $p \geq m$.*

Corollary 18. *Let $M = \{a_{i,j}\}_{\substack{i\in\mathbb{N} \\ j\in\mathbb{N}}}$, $a_{i,j} \in \mathbb{K}$ be a matrix satisfying conditions as in Corollary 17. Then*

(a) *$\Phi_M : (\mathbb{K}[[x]], d) \to (\mathbb{K}[[x]], d)$ is an onto isometry if and only if M is lower triangular and $a_{i,i} \neq 0$ for every $i \in \mathbb{N}$.*
(b) *$\Phi_M : (\mathbb{K}[[x]], d) \to (\mathbb{K}[[x]], d)$ is contractive if and only if M is lower triangular and $a_{i,i} = 0$ for every $i \in \mathbb{N}$.*

Proof. (a) If Φ_M is an onto isometry, we have in particular that $d(x^n, 0) = d(\Phi_M(x^n), 0)$ since $\Phi_M(x^n) = \sum_{i=0}^{\infty} a_{i,n} x^i$ we have that $a_{0,n} = a_{1,n} = \cdots = a_{n-1,n} = 0$ but $a_{n,n} \neq 0$ because $\omega(x^n) = \omega(\Phi_M(x^n))$. On the contrary suppose that M is lower triangular and $a_{i,i} \neq 0$ for $i \in \mathbb{N}$, it implies that $d(x^n, 0) = d(\Phi_M(x^n), 0)$. Take $f = \sum_{k\geq 0} \alpha_k x^k$ and $g = \sum_{k\geq 0} \beta_k x^k$. Then $d(\Phi_M(f), \Phi_M(g)) = \frac{1}{2^{\omega(\Phi_M(f-g))}}$, $\Phi_M(f - g) = \Phi_M(\sum_{k\geq 0}(\alpha_k - \beta_k)x^k) = \sum_{k\geq 0}(\alpha_k - \beta_k)\Phi_M(x^k)$. If $\alpha_0 \neq \beta_0$ then $d(f, g) = 1$ and since $d(\Phi_M(1), 0) = 1$ and $d(\Phi_M(x^n), 0) \leq 1/2$ for $n \geq 1$ we obtain that $d(\Phi(f), \Phi(g)) = 1$. Suppose on the contrary that $\alpha_0 = \beta_0$ and let $p \geq 1$ be such that

$p = \omega(f - g)$. So $d(f, g) = \frac{1}{2^p}$. Consequently $\Phi_M(f - g) = (\alpha_p - \beta_p) + \sum_{k\geq p+1}(\alpha_k - \beta_k)\Phi_M(x^k)$ with $d((\alpha_p - \beta_p)\Phi_M(x^p), 0) = \frac{1}{2^p}$ and $d(\sum_{k\geq p+1}(\alpha_k - \beta_k)\Phi_M(x^k), 0) \leq \frac{1}{2^{p+1}}$. Since d is an ultrametric we obtain that $d(\Phi_M(f - g), 0) = \frac{1}{2^p} = d(f, g)$. Consequently $d(\Phi_M(f), \Phi_M(g)) = d(f, g)$. In order to prove that Φ_M is an onto map we only have to show that $x^n \in \operatorname{Im}\Phi_M$ for every $n \in \mathbb{N}\cup\{0\}$. Using the above results we only have to prove that for every $n \in \mathbb{N}$ there is a $f_n \in \mathbb{K}[[x]]$, $f_n = \sum_{k\geq 0}\alpha_{n,k}x^k$ such that

$$\begin{pmatrix} a_{00} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ a_{10} & a_{11} & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \cdots & 0 & 0 & \cdots \\ a_{n0} & a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \alpha_{n,0} \\ \alpha_{n,1} \\ \vdots \\ \alpha_{n,n} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$$

but since $\mathbb{K}$ is a field and $a_{l,l} \neq 0$, one can easily deduce the existence of f_n.

By analogous arguments we can get (b). $\square$

Let us go back to our *arithmetical triangles*. In the next proposition we describe the action of our $T(f \mid g)$ as a linear map, on a power series. This result was an assumption in the original definition of the Riordan group in Shapiro [18] and it was stated as one of the main results on Riordan arrays in Sprugnoli [21], Sprugnoli [22], Merlini et al. [8].

Proposition 19. *Let $f, g \in \mathbb{K}[[x]]$ with $g(0) \neq 0$ then the arithmetical triangle $T(f \mid g)$ of f with rate g induces a linear continuous function, that we denote with the same symbol, $T(f \mid g) : (\mathbb{K}[[x]], d) \to (\mathbb{K}[[x]], d)$ defined by*

$$T(f \mid g)(h) = \frac{f}{g}h\left(\frac{x}{g}\right).$$

Moreover $T(f \mid g)$ is an onto isometry if and only if $f(0) \neq 0$ and $T(f \mid g)$ is contractive if and only if $f(0) = 0$.

Proof. First of all let us say a few words on the expression $h(\frac{x}{g})$. It is no more than the composition of the series h and $\frac{x}{g}$ i.e. $h(\frac{x}{g}) = h \circ \frac{x}{g}$, which is defined in the following way. Let $h(x) = \sum_{k\geq 0}\alpha_k x^k$. Since $(\frac{x}{g})(0) = 0$ we obtain that the series of the series $\sum_{k\geq 0}\alpha_k(\frac{x}{g})^k$ converges in $(\mathbb{K}[[x]], d)$ because $\omega((\frac{x}{g})^k) = k\omega(\frac{x}{g})$ and $\omega(\frac{x}{g}) = 1$ so $\lim_{k\to\infty}(\frac{x}{g})^k = 0$. Recall that, Theorem 11, $T(f \mid g) = \{c_{ij}\}_{\substack{i \in \mathbb{N} \\ j \in \mathbb{N}}}$ with $\frac{x^{j-1}f}{g^j} = \sum_{i=1}^{\infty} c_{ij}x^{i-1}$. Suppose that $f = \sum_{n\geq 0}a_n x^n$ and $g = \sum_{n\geq 0}b_n x^n$ with $b_0 \neq 0$. Recall now the rule of construction of $T(f \mid g)$

$$\begin{matrix} a_0/b_0 & & & \\ -\frac{b_1a_0}{b_0^2} + \frac{a_1}{b_0} & a_0/b_0^2 & & \\ \frac{b_1^2a_0}{b_0^3} - \frac{b_1a_1}{b_0^2} - \frac{b_2a_0}{b_0^2} + \frac{a_2}{b_0} & -\frac{2b_1a_0}{b_0^3} + \frac{a_1}{b_0^2} & a_0/b_0^3 & \\ \vdots & \vdots & \vdots & \ddots \\ \frac{f}{g} & \frac{xf}{g^2} & \frac{x^2f}{g^3} & \cdots \end{matrix}$$

So it is a lower triangular matrix and $c_{n,n} \neq 0 \forall\, n \in \mathbb{N}$ if and only if $f(0) \neq 0$. Now $T(f \mid g)(h) = \sum_{k\geq 0}\alpha_k\frac{x^kf}{g^{k+1}} = \frac{f}{g}(\sum_{k\geq 0}\alpha_k(\frac{x}{g})^k) = \frac{f}{g}h(\frac{x}{g})$ and the proof is finished. $\square$

Using the classical definition of composition of maps and the behavior of the associated matrix, we can easily find the formula for the product and the inverse, when it exists, of Riordan arrays. These expressions can be found in the quoted literature.

Proposition 20. (a) *The product of two arithmetical triangles is again an arithmetical triangle. In fact $T(f_1 \mid g_1)T(f_2 \mid g_2) = T(f_1f_2(\frac{x}{g_1}) \mid g_1g_2(\frac{x}{g_1}))$ for $f_1, f_2 \in \mathbb{K}[[x]]$, $g_1, g_2 \in \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]$.*

(b) *If $A(\mathbb{K}[[x]]) = \{T(f \mid g),\ f, g \in \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]\}$ then $(A(\mathbb{K}[[x]]), \cdot)$ ($\cdot$ being the usual product of matrices) is a group.*

Proof. It is obvious that the matrix assignment, as the finite-dimensional vector spaces case, satisfies that if $T, S : (\mathbb{K}[[x]], d) \to (\mathbb{K}[[x]], d)$ are linear continuous functions, then $M(S \circ T) = M(S)M(T)$. The product $M(S)M(T)$ of these infinite matrices makes sense because, for continuity, all sums are in fact finite sums.

Consider the continuous linear functions $T(f_1 \mid g_1)$ and $T(f_2 \mid g_2)$, $g_1(0) \neq 0$, $g_2(0) \neq 0$. Then $(T(f_1 \mid g_1)T(f_2 \mid g_2))(h) = T(f_1 \mid g_1)(\frac{f_2}{g_2}h(\frac{x}{g_2})) = \frac{f_1}{g_1}(\frac{f_2(\frac{x}{g_1})}{g_2(\frac{x}{g_1})}h(\frac{x}{g_1 g_2(\frac{x}{g_1})}))$. Consequently $(T(f_1 \mid g_1) \circ T(f_2 \mid g_2)) = T(f_1 f_2(\frac{x}{g_1}) \mid g_1 g_2(\frac{x}{g_1}))$ and the proof of (a) is finished.

(b) Suppose now that $f, g \in \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]$. Then $T(f \mid g)$ is a linear onto isometry (then invertible). Let us calculate $(T(f \mid g))^{-1}$. First of all recall that $T(f \mid g)(h) = \frac{f}{g}h(\frac{x}{g})$. The series $k = \frac{x}{g}$ satisfies that $k(0) = 0$ and $k'(0) = D(k)(0) = \frac{1}{g(0)} \neq 0$, ($D$ denotes the usual derivative). So it is invertible for composition. This means that there is a series k^{-1} such that $\frac{x}{g} \circ k^{-1} = k^{-1} \circ \frac{x}{g} = x$. Consider now $s = \frac{1}{f \circ k^{-1}}$ and $t = \frac{1}{g \circ k^{-1}}$ then $(T(f \mid g)T(s \mid t))(h) = T(f \mid g)(\frac{s}{t}h(\frac{x}{t})) = T(f \mid g)(\frac{g \circ k^{-1}}{f \circ k^{-1}}h(xg \circ k^{-1})) = \frac{f}{g}(\frac{g \circ k^{-1} \circ \frac{x}{g}}{f \circ k^{-1} \circ \frac{x}{g}}h(\frac{x}{g}(g \circ k^{-1} \circ \frac{x}{g}))) = h$. The same arguments prove that $T(s \mid t) \circ T(f \mid g) \equiv I$ but the identity $I = T(1 \mid 1)$.

Using (a) we have proved that $f_1, g_1, f_2, g_2 \in \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]$ then $T(f_1 \mid g_1)(T(f_2 \mid g_2))^{-1} \in A(\mathbb{K}[[x]])$. Consequently $A(\mathbb{K}[[x]])$ is a subgroup of the group of isometries of $(\mathbb{K}[[x]], d)$. $\square$

In order to study some algebraic properties of the group $(A(\mathbb{K}[[x]]), \cdot)$ we are going to describe some special subsets of the set of arithmetical triangles.

First of all note that the set of arithmetical triangles contains a natural algebraic copy of $\mathbb{K}[[x]]$. In fact given $f = \sum_{k \geq 0} a_k x^k$ we have that

$$T(f \mid 1) = \begin{pmatrix} a_0 & & & & & & \\ a_1 & a_0 & & & & & \\ a_2 & a_1 & a_0 & & & & \\ a_3 & a_2 & a_1 & a_0 & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is the matrix representation of multiplying by the series f. It is obvious that $T(\alpha f + \beta g \mid 1) = \alpha T(f \mid 1) + \beta T(g \mid 1)$ for $\alpha\ \beta \in \mathbb{K}[[x]]$, where the sum in the right part of the equality is the usual sum of matrices (also the usual product by scalars). Moreover $T(f \cdot g \mid 1) = T(f \mid 1)T(g \mid 1) = T(g \mid 1)T(f \mid 1)$.

Related to the algebraic structure of the ring $\mathbb{K}[[x]]$, we can consider it as a module over the ring $\mathbb{K}[[x]]$. For this module structure there is a related concept of linear map, we will call it $\mathbb{K}[[x]]$-linear map. Of course any $\mathbb{K}[[x]]$-linear map is a $\mathbb{K}$-linear. In fact more can be said.

Proposition 21. *For any $\mathbb{K}[[x]]$-linear map $\Phi : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ there is a $f \in \mathbb{K}[[x]]$ such that $\Phi(h) = fh$. Consequently the $\mathbb{K}[[x]]$-linear maps are continuous in $(\mathbb{K}[[x]], d)$ and their matricial representations are just the arithmetical triangles of the form $T(f \mid 1)$.*

We also have the following rules of products (or compositions). Let $f \in \mathbb{K}[[x]]$ and $g \in \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]$, $T(f(\frac{x}{g}) \mid g) = T(1 \mid g)T(f \mid 1)$. If in addition $f \in \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]$, then $T(1 \mid gf(\frac{x}{g})) = T(1 \mid g)T(1 \mid f)$. Let $U(\mathbb{K}[[x]])$ denote the multiplicative group of unities of the ring $\mathbb{K}[[x]]$. We have:

Proposition 22. *Consider the group of arithmetical triangles $(A(\mathbb{K}[[x]]), \cdot)$ and let $N = \{T(f \mid g) \in A(\mathbb{K}[[x]]) \ / \ g \equiv 1\}$, $M = \{T(f \mid g) \in A(\mathbb{K}[[x]]) \ / \ f = 1\}$. Then: M is a subgroup and N is a normal subgroup of $A(\mathbb{K}[[x]])$, $N \cdot M = A(\mathbb{K}[[x]])$ and $N \cap M = T(1 \mid 1)$ the neutral element. Consequently $A(\mathbb{K}[[x]])$ is isomorphic to the semidirect product $N \times_{\varphi} M$. $\varphi : M \to Aut(N)$ is the homomorphism defined by $\varphi(T(1 \mid g))(T(f \mid 1)) = T(1 \mid g)T(f \mid 1)T(1 \mid g)^{-1}$ and $Aut(N)$ is the group of automorphism of the group N. Moreover N is isomorphic to $U(\mathbb{K}[[x]])$.*

Proof. It is obvious that N is a subgroup of $A(\mathbb{K}[[x]])$. Take now $T(f \mid 1) \in N$ and $T(s \mid t) \in A(\mathbb{K}[[x]])$. Then $(T(s \mid t)T(f \mid 1)(T(s \mid t))^{-1})(h) = T(s \mid t)T(f \mid 1)(\frac{t \circ k^{-1}}{s \circ k^{-1}})h(x(t \circ k^{-1}))$ (*) where, recalling Proposition 20, k^{-1} is the compositional inverse of $k = \frac{x}{t}$ so $(*) = T(s \mid t)(\frac{f(t \circ k^{-1})}{s \circ k^{-1}})h(x(t \circ k^{-1})) = \frac{s}{t}\frac{f(\frac{x}{t})(t \circ k^{-1} \circ k)}{s \circ k^{-1} \circ k}h(\frac{x}{t}t \circ k^{-1} \circ k) = f(\frac{x}{t})h = T(f(\frac{x}{t}) \mid 1) \in N$. Consequently N is a normal subgroup. Now, in order to prove that M is a subgroup of $A(\mathbb{K}[[x]])$, let $T(1 \mid f), T(1 \mid g) \in M$. First of all recall that, see the proof of Proposition 20, $(T(1 \mid f))^{-1} = T(1 \mid \frac{1}{f \circ k^{-1}})$ where, in this case, k^{-1} is the compositional inverse of $k = \frac{x}{f}$ which exists because $k(0) = 0$ and $D(k)(0) \neq 0$. So we obtain that $T(1 \mid g)(T(1 \mid f))^{-1} = T(1 \mid g)T(1 \mid \frac{1}{f \circ k^{-1}}) = T(1 \mid \frac{g}{f \circ k^{-1}(\frac{x}{g})}) \in M$. Consequently M is a subgroup of $A(\mathbb{K}[[x]])$. It is obvious that $N \cap M = T(1 \mid 1)$ and it is a standard fact in group theory, see for example [1] page 133, that in the above conditions $A(\mathbb{K}[[x]]) \simeq N \times_{\varphi} M$ for such a φ. Note also that this is not a direct product. In particular M is not a normal subgroup of $A(\mathbb{K}[[x]])$. □

In the literature there is no unified way to describe the elements in the Riordan group. It is even called Riordan group to denote different but related things. [6,18,19,21]. In order to end this section we are going to point out that our group $A(\mathbb{K}[[x]])$ of arithmetical triangles $T(f \mid g)$, f, $g \in \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]$ is in fact the Riordan group but parametrized by $(\mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]) \times (\mathbb{K}[[x]] \setminus x\mathbb{K}[[x]])$ in a different form. To do this we have chosen a concrete description of the Riordan group. In fact we are going to choose that in [8] or [6].

Note that in [6] an element of the Riordan group is denoted by a pair of series (u, v) where $u, v, \in \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]$. With this notation we have:

Corollary 23. *For any $f, g, u, v \in \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]$,*

$$T(f \mid g) = \left(\frac{f}{g}, \frac{1}{g}\right) \quad or \quad (u, v) = T\left(\frac{u}{v} \mid \frac{1}{v}\right).$$

Consequently our group $A(\mathbb{K}[[x]])$ is no more than the Riordan group.

6. Ultrametrics in spaces of linear functions: The Riordan group as a non-Archimedean topological group

It is widely known that normal subgroups are very important to clarify the algebraic structure in any group and then for the classification problem of groups. The constructions in the following, we think that it could be of independent interest, will allow us to recognize many normal subgroups of the Riordan group. Maybe it is still a modest contribution but we think that it could help in further developments. This time we were inspired by the theory of Banach spaces and the classical Lie groups of finite real or complex square matrices.

Consider the ultrametric space $(\mathbb{K}[[x]], d)$. Denote by $End_d(\mathbb{K}[[x]])$ the set of all continuous endomorphisms in $(\mathbb{K}[[x]], d)$ considered as a $\mathbb{K}$-vector space. As in the case of classical Banach spaces we can define what we will call the norm associated to d. We will denote it by $\| \ \|_d$.

Definition 24. Let $T : (\mathbb{K}[[x]], d) \to (\mathbb{K}[[x]], d)$ be a continuous endomorphism. We define the norm of T as the number

$$\|T\|_d = l.u.b._{f \in \mathbb{K}[[x]]}\{d(T(f), 0)\}$$

where $l.u.b.$ means the least upper bound.

Note that $\|T\|_d$ exists for any $T \in End(\mathbb{K}[[x]])$ because d is bounded above by 1.

Since the unique accumulation point of the values of the metric d is zero, it follows that for any T there is a series $f_T \in \mathbb{K}[[x]]$ such that $\|T\|_d = d(T(f_T), 0)$. Some elementary properties of this norm are:

Proposition 25. 1. $0 \leq \|T\|_d \leq 1$ *for any $T \in End_d(\mathbb{K}[[x]])$.*
2. $0 = \|T\|_d$ *if and only if $T \equiv 0$.*
3. $\|\lambda T\|_d = \|T\|_d$ *for any $\lambda \in \mathbb{K}$, with $\lambda \neq 0$.*
4. $\|T_1 + T_2\|_d \leq \max\{\|T_1\|_d, \|T_2\|_d\}$.

Proof. Only a proof of (4) is needed. Suppose $T_1, T_2 \in End_d(\mathbb{K}[[x]])$. Choose $f \in \mathbb{K}[[x]]$ such that $\|T_1 + T_2\|_d = d((T_1 + T_2)(f), 0)$. Since $d((T_1 + T_2)(f), 0) = \frac{1}{2^{\omega((T_1+T_2)(f))}}$ and $\omega(T_1(f) + T_2(f)) \geq \min\{\omega(T_1(f)), \omega(T_2(f))\}$, we obtain that $\|T_1 + T_2\|_d \leq \max\{\|T_1\|_d, \|T_2\|_d\}$. □

So this norm satisfies also a strong version of the triangular inequality. Using the above proposition we obtain

Corollary 26. *The assignment* $d^* : End_d(\mathbb{K}[[x]]) \times End_d(\mathbb{K}[[x]]) \to \mathbb{R}_+$ *given by* $d^*(T_1, T_2) = \|T_1 - T_2\|_d$ *defines an ultrametric in* $End_d(\mathbb{K}[[x]])$.

Remark 27. More can be said about this ultrametric, in particular about its completeness and about the property of approximate any continuous endomorphism by a sequence of them with finite-dimensional range. We are not going to do this at this moment because we are interested in the group of isometries (with composition as operation) and eventually in the Riordan group.

The metric d^* defined above can be visualized when we know the matricial representations, that given in Definition 16 in Section 5, of two continuous endomorphisms.

Suppose that $A = \{a_{ij}\}_{\substack{i \in \mathbb{N} \\ j \in \mathbb{N}}}$ is an infinite matrix. Remember that, for us, $0 \in \mathbb{N}$. Then the first row is just the 0-row $\{a_{0,j}\}_{j\in\mathbb{N}}$. The first column is the 0-column $\{a_{i,0}\}_{i\in\mathbb{N}}$. Let us define the following:

Definition 28. Let $\mathbb{K}$ be a field (of characteristic zero) and $A = \{a_{ij}\}_{\substack{i \in \mathbb{N} \\ j \in \mathbb{N}}}$, $A \in M_{\mathbb{N}\times\mathbb{N}}(\mathbb{K})$. We define the order of A (and we denote it again by $\omega(A)$) as $\omega(A) = \infty$ if $A = 0$. Otherwise $\omega(A) = k$, $(k \in \mathbb{N})$ if k is the unique natural number with the following properties: $a_{l,m} = 0$ for every $m \in \mathbb{N}$ and $0 \le l \le k-1$ and there is an $m_0 \in \mathbb{N}$ with $a_{k,m_0} \neq 0$.

Note that $\omega(A) = 0$ means that there is a non-zero entry in the 0-row. On the other hand $\omega(A) = k \ge 1$, $(k \in \mathbb{N})$ if the submatrix $\{a_{ij}\}_{\substack{i = 0,\ldots,k-1 \\ j \in \mathbb{N}}}$ is the null one and the row $\{a_{k,m}\}_{m\in\mathbb{N}}$ is non-null.

Proposition 29. *Let* $T, S \in End_d(\mathbb{K}[[x]])$. *Suppose that* $A = \{a_{ij}\}_{\substack{i \in \mathbb{N} \\ j \in \mathbb{N}}} = M(T)$ *and* $B = \{b_{ij}\}_{\substack{i \in \mathbb{N} \\ j \in \mathbb{N}}} = M(S)$ *are the corresponding associated matrices as in Section 5. Then*

$$d^*(T, S) = \frac{1}{2^{\omega(A-B)}}.$$

Proof. If $T = S$ then the equality is obvious if we interpret $\frac{1}{2^\infty} = 0$. So, we can suppose that $d^*(T, S) = \frac{1}{2^{k_0}}$ for a $k_0 \in \mathbb{N}$. In particular we have that $\omega((T - S)(x^l)) \ge k_0$ for every $l \in \mathbb{N}$. This means that for every $l \in \mathbb{N}$, $a_{m,l} - b_{m,l} = 0$ for $0 \le m \le k_0 - 1$ if $k_0 \ge 1$. Note also that the equality is clear if $k_0 = 0$. Consequently $\omega(A - B) \ge k_0$. Hence $\frac{1}{2^{\omega(A-B)}} \le d^*(T, S)$. On the other hand suppose that $f = \sum_k \alpha_k x^k \in \mathbb{K}[[x]]$. Since $T - S$ is obviously continuous, $M(T - S) = A - B$. If $(T - S)(f) = \sum_k \beta_k x^k$, then

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{00} - b_{00} & a_{01} - b_{01} & a_{02} - b_{02} & \cdots & a_{0n} - b_{0n} & \cdots \\ a_{10} - b_{10} & a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} & \cdots \\ a_{20} - b_{20} & a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ a_{n0} - b_{n0} & a_{n1} - b_{n1} & a_{n2} - b_{n2} & \cdots & a_{nn} - b_{nn} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ \vdots \end{pmatrix}.$$

So if $\omega(A - B) = k_1$ then $\beta_0 = \beta_1 = \cdots = \beta_{k-1} = 0$. It implies that $\omega((T - S)(f)) \ge 1$ for every $f \in \mathbb{K}[[x]]$. Consequently $d^*(T, S) \le \frac{1}{2^{\omega(A-B)}}$ and the proof is finished. $\square$

Note that the above proposition points out that $d^*(T, S)$ can be computed using only the set of series $\{1, x, x^2, \ldots, x^n, \ldots\}$.

Let us denote by $Isom_d(\mathbb{K}[[x]])$ the group, with the composition of maps as operation, of linear isometries in $(\mathbb{K}[[x]], d)$. We recommend the paper [14] for the definitions and results that we will use from now on.

Proposition 30. *The metric d^*, when restricted to the group $Isom_d(\mathbb{K}[[x]])$ (with composition as operation), gives rise to an invariant complete ultrametric. Moreover $(Isom_d(\mathbb{K}[[x]]), d^*)$ is a non-Archimedean metrizable topological group (in the sense of* [14]*).*

Proof. We are going to prove first that d^* is left and right invariant. That is if $T_1, T_2, S \in Isom_d(\mathbb{K}[[x]])$ then $d^*(T_1 \circ S, T_2 \circ S) = d^*(T_1, T_2) = d^*(S \circ T_1, S \circ T_2)$. Let $f \in \mathbb{K}[[x]]$ be such that $d^*(T_1, T_2) = d((T_1 - T_2)(f), 0)$. Since S is onto we have that $f = S(g)$ for some $g \in \mathbb{K}[[x]]$. So $d^*(T_1, T_2) = d((T_1 - T_2)(S(g)), 0) \le \sup_{h\in\mathbb{K}[[x]]}\{d((T_1 - T_2)(S(h)), 0)\} = \|(T_1 - T_2) \circ S\|_d = d^*(T_1 \circ S, T_2 \circ S)$.

Suppose now that $h \in \mathbb{K}[[x]]$ in such that $\|T_1 \circ S - T_2 \circ S\|_d = d((T_1 \circ S - T_2 \circ S)(h), 0) = d((T_1 - T_2)(S(h)), 0) \le \|T_1 - T_2\|_d = d^*(T_1, T_2)$. So we have the right invariance of d^*.

Take again the series $f \in \mathbb{K}[[x]]$ satisfying $d^*(T_1, T_2) = \|T_1 - T_2\|_d = d((T_1 - T_2)(f), 0)$. Since S is a linear isometry we have $d(S(T_1 - T_2)(f), 0) = d^*(T_1, T_2)$. By definition $\|S \circ (T_1 - T_2)\|_d = d^*(S \circ T_1, S \circ T_2) \ge d(S \circ (T_1 - T_2), 0) = \|T_1 - T_2\|_d = d^*(T_1, T_2)$.

Take now $m \in \mathbb{K}[[x]]$ such that $d^*(S \circ T_1, S \circ T_2) = d(S \circ (T_1 - T_2)(m), 0)$ because S is an isometry but $d((T_1 - T_2)(m), 0) \le d^*(T_1, T_2)$ by definition. So we have proved that d^* is invariant.

To prove that $Isom_d(\mathbb{K}[[x]])$ is a topological group with the topology induced by d^* we have:

Suppose that $\{T_n, S_n\}_{n\in\mathbb{N}} \in Isom_d(\mathbb{K}[[x]]) \times Isom_d(\mathbb{K}[[x]])$ with $T_n \to T$ and $S_n \to S$ in $(Isom_d(\mathbb{K}[[x]]), d^*)$. Using the strong triangle inequality and the invariance we get $d^*(T_n \circ S_n, T \circ S) \le \max\{d^*(T_n \circ S_n, T_n \circ S), d^*(T_n \circ S, T \circ S)\} = \max\{d^*(S_n, S), d^*(T_n, T)\}$. Consequently the composition is continuous. Suppose now that $\{T_n\}_{n\in\mathbb{N}} \to T$ in $(Isom_d(\mathbb{K}[[x]]), d^*)$. $d^*(T_n^{-1}, T^{-1})$ is also continuous (in fact an isometry in $(Isom_d(\mathbb{K}[[x]]), d^*)$). Consequently $(Isom_d(\mathbb{K}[[x]]), d^*)$ is a non-Archimedean (or ultrametric) metrizable topological group in the sense of [14]. Moreover d^* is invariant.

In order to prove the completeness, consider a Cauchy sequence $\{T_n\}_{n\in\mathbb{N}} \subset (Isom_d(\mathbb{K}[[x]]), d^*)$. Let $f \in \mathbb{K}[[x]]$, then $\{T_n(f)\}_{n\in\mathbb{N}} \subset (\mathbb{K}[[x]], d)$ is a Cauchy sequence and then it converges to a series that we denote by $T(f)$. So we have defined a function $T : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$. The linearity of T is obvious because $\lim_{n\to\infty}(T_n(\alpha f + \beta g)) = \lim_{n\to\infty}(\alpha T_n(f) + \beta T_n(g))$ for $f, g \in \mathbb{K}[[x]]$, $\alpha, \beta \in \mathbb{K}$. Moreover $d(T(f), T(g)) = \lim_{n\to\infty} d(T_n(f), T_n(g)) = d(f, g)$. Let us prove now that $\{T_n\}_{n\in\mathbb{N}} \to T$ in $(End_d(\mathbb{K}[[x]]), d^*)$. Since $\{T_n\}_{n\in\mathbb{N}}$ is d^*-Cauchy, then for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $d^*(T_n, T_m) < \epsilon$ for every $n, m \ge n_0$. This means that $d(T_n(f), T_m(f)) < \epsilon$ for $n, m \ge n_0$ and for every $f \in \mathbb{K}[[x]]$. Given a particular $f \in \mathbb{K}[[x]]$ there is a number $m_0(f) \ge n_0$ such that $d(T_{m_0(f)}(f), T(f)) < \epsilon$. Consequently, for every $f \in \mathbb{K}[[x]]$ and $n \ge n_0$ we have $d(T_n(f), T(f)) \le \max\{d(T_n(f), T_{m_0(f)}(f)), d(T_{m_0(f)}(f), T(f))\} < \epsilon$. So, $d^*(T_n, T) < \epsilon$ for $n \ge n_0$ in $(End_d(\mathbb{K}[[x]]), d^*)$.

It remains to prove only that T is a surjective isometry. First of all note that $\{T_n^{-1}\}_{n\in\mathbb{N}}$ is also a d^*-Cauchy sequence. Using the same arguments as before we have a linear into isometry $S : (\mathbb{K}[[x]], d) \to (\mathbb{K}[[x]], d)$ with $\lim_{n\to\infty} T_n^{-1} = S$ in $(End_d(\mathbb{K}[[x]]), d^*)$. It is now clear that $T \circ S = S \circ T = I$ and T is an onto isometry. □

Corollary 31. *For every $k \in \mathbb{N}$, $G_k = \{T \in Isom_d(\mathbb{K}[[x]]) \ / \ d^*(T, I) \le \frac{1}{2^k}\}$, where I is the identity, is a nested sequence of normal subgroups which are open and closed for the topology induced by d^*. Moreover $\bigcap_{k\in\mathbb{N}} G_k = \{I\}$.*

Proof. Using [14] we have that G_k is a normal subgroup for any $k \in \mathbb{N}$, because d^* is left and right invariant. Moreover $G_k = B_c\left(I, \frac{1}{2^k}\right)$ is just the closed ball, for the metric d^*, of center the identity and radius $\frac{1}{2^k}$. Of course it is closed in the metric space $(Isom_d(\mathbb{K}[[x]]), d^*)$. They are also open. In fact if $k = 0$ then $G_0 = Isom_d(\mathbb{K}[[x]])$ the whole space. Suppose that $k \ge 1$. Take ϵ such that $\frac{1}{2^k} < \epsilon < \frac{1}{2^{k-1}}$. Obviously G_k is the open ball of center I and radius ϵ and then open for the metric d^*. Finally $\{G_k\}_{k\in\mathbb{N}}$ is a base of open neighborhoods for the identity in $(Isom_d(\mathbb{K}[[x]]), d^*)$. By the Hausdorff separation axiom we have $\bigcap_{k\in\mathbb{N}} G_k = \{I\}$. □

We are going to restrict the constructions made before to the Riordan group. Remember that for us the Riordan group $A(\mathbb{K}[[x]])$ is just $A(\mathbb{K}[[x]]) = \{T(f \mid g) \ / \ f, g \in \mathbb{K}[[x]]$ with $f(0), g(0) \ne 0\}$. Considering as a subgroup of $Isom_d(\mathbb{K}[[x]])$ we have the obvious observation:

Proposition 32. *$(A(\mathbb{K}[[x]]), d^*)$ is a non-Archimedean metrizable topological group and d^* is an invariant metric. Moreover $A_k(\mathbb{K}[[x]]) = A(\mathbb{K}[[x]]) \bigcap G_k$ is a normal subgroup for any $k \ge 1$.*

We are going now to recognize, in terms of f and g, when $T(f \mid g) \in A_k(\mathbb{K}[[x]])$.

Proposition 33. *Let* $f = \sum_{k \geq 0} a_k x^k$ *and* $g = \sum_{k \geq 0} b_k x^k$ *with* $f(0) \neq 0$, $g(0) \neq 0$. *Then*

(1) $T(f \mid g) \in A_1(\mathbb{K}[[x]])$ *if and only if* $a_0 = b_0$.
(2) $T(f \mid g) \in A_2(\mathbb{K}[[x]])$ *if and only if* $a_0 = b_0 = 1$ *and* $a_1 = b_1$.
(3) *If* $k \geq 3$, $T(f \mid g) \in A_k(\mathbb{K}[[x]])$ *if and only if* $a_0 = b_0 = 1$, $a_j = b_j = 0$ *for* $1 \leq j \leq k-2$ *and* $a_{k-1} = b_{k-1}$.

The proof of this proposition follows easily from the algorithm for $T(f \mid g)$ at the end of Section 4.

Note also that the above proposition implies the following symmetry relation $T(f \mid g) \in A_k(\mathbb{K}[[x]])$ if and only if $T(g \mid f) \in A_k(\mathbb{K}[[x]])$.

To finish we add the following comparison table between the standard and the $T(f \mid g)$ notations:

Name	$(d(t), h(t))$	$T(f \mid g)$
Pascal	$\left(\frac{1}{1-t}, \frac{1}{1-t}\right)$	$T(1 \mid 1-t)$
Catalan	$\left(\frac{1-\sqrt{1-4t}}{2t}, \frac{1-\sqrt{1-4t}}{2t}\right)$	$T\left(1 \mid \frac{2t}{1-\sqrt{1-4t}}\right)$
Stirling first kind	$\left(1, \frac{1}{t}\ln\frac{1}{1-t}\right)$	$T\left(\frac{-t}{\ln(1-t)} \mid \frac{-t}{\ln(1-t)}\right)$
Stirling second kind	$\left(1, \frac{e^t-1}{t}\right)$	$T\left(\frac{t}{e^t-1} \mid \frac{t}{e^t-1}\right)$
$d(t) = h(t) = \frac{1-t-\sqrt{1-6t+t^2}}{2t}$	$\left(\frac{1-t-\sqrt{1-6t+t^2}}{2t}, \frac{1-t-\sqrt{1-6t+t^2}}{2t}\right)$	$T\left(1 \mid \frac{2t}{1-t-\sqrt{1-6t+t^2}}\right)$
$d(t) = \frac{1}{1-t-t^2}$, $h(t) = \frac{(1+t)}{(1-t)(1-t-t^2)}$	$\left(\frac{1}{1-t-t^2}, \frac{(1+t)}{(1-t)(1-t-t^2)}\right)$	$T\left(\frac{1-t}{1+t} \mid \frac{(1-t)(1-t-t^2)}{1+t}\right)$
Appel subgroup element	$(d(t), 1)$	$T(d \mid 1)$
Associated subgroup element	$(1, h(t))$	$T\left(\frac{1}{h} \mid \frac{1}{h}\right)$
Bell subgroup element	$(d(t), d(t))$	$T\left(1 \mid \frac{1}{d}\right)$

Acknowledgments

The authors thank the referee(s) for his/her suggestions and comments which improved an earlier version of this paper. They also thank J. Dydak for his helpful comments. The first author was partially supported by DGES grant MAT 2005-05730-C02-02. The second author was partially supported by DGES grant MTM-2006-0825.

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