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Pascal Matrices and Stirling Numbers

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Abstract—This paper presents some relationships between Pascal matrices, Stirling numbers, and Bernouilli numbers.

Keywords—Combinatorial identities, Pascal matrices, Stirling numbers.

This paper begins with a well-known combinatorial expression for the the cube of a number

$$\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3} = n^3, \tag{1}$$

which can be extended to a solution for

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3,$$

or just

$$S_3 = \sum_{k=1}^{n} k^3. \tag{2}$$

Evaluation of (2) can be done simply by using Pascal's triangle identity

$$\binom{k}{l} = \binom{k-1}{l} + \binom{k-1}{l-1}$$

and (1), resulting in

$$\begin{aligned} S_3 &= \sum_{k=1}^{n} \left\{ \binom{k}{1} + 6\binom{k}{2} + 6\binom{k}{3} \right\} \\ &= \sum_{k=1}^{n} \left\{ \binom{k+1}{2} - \binom{k}{2} \right\} + 6\sum_{k=1}^{n} \left\{ \binom{k+1}{3} - \binom{k}{3} \right\} + 6\sum_{k=1}^{n} \left\{ \binom{k+1}{4} - \binom{k}{4} \right\} \\ &= \binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}. \end{aligned}$$

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Simplification yields the result

$$S_3 = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2, \tag{3}$$

which proves the well-known, but remarkable formula

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2. \tag{4}$$

As an aside, substituting $N = (n(n+1))/2 - 1$ in (3) gives

$$S_3 = N(N+1) + N + 1 = \sum_{i=0}^{N}(2i+1),$$

which proves the theorem of Nicomachus [1].

This suggests a means of finding a solution for the sums of the first n k^{th} powers

$$S_k(n) = \sum_{i=1}^{n} i^k$$

based on repeated use of the binomial expansion

$$\begin{aligned} a^k = \{(a-1)+1\}^k &= \sum_{i=0}^{k-1}\binom{k}{i}(a-1)^i + (a-1)^k \\ &= \sum_{i=0}^{k-1}\binom{k}{i}(a-1)^i + \sum_{i=0}^{k-1}\binom{k}{i}(a-2)^i + \cdots + \sum_{i=0}^{k-1}\binom{k}{i}(1)^i + 1 \\ &= \sum_{i=0}^{k-1}\binom{k}{i}\sum_{m=1}^{a-1}(m)^i + 1 \\ &= \sum_{i=0}^{k-1}\binom{k}{i}S_i(a-1) + 1. \end{aligned}$$

Substituting $a = n+1$ in the last expression for a^k gives

$$\sum_{i=0}^{k-1}\binom{k}{i}S_i(n) = (n+1)^k - 1.$$

This expression can be represented as a triangular system of equations

$$\hat{P}\,\mathbf{S} = P\,\mathbf{n}, \tag{5}$$

where for $k = 5$, the matrices are

$$\hat{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 \\ 1 & 5 & 10 & 10 & 5 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 \\ 4 & 6 & 4 & 1 & 0 \\ 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

and

$$\mathbf{S} = \begin{bmatrix} S_0(n) \\ S_1(n) \\ S_2(n) \\ S_3(n) \\ S_4(n) \end{bmatrix}, \qquad \mathbf{n} = \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \end{bmatrix}.$$

P is a Pascal matrix [2] with the first row and column deleted,

$$P_{ij} = \begin{cases} \binom{i}{j}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

$\hat{P}$ is the 'reverse' of P with

$$\hat{P}_{ij} = \begin{cases} \binom{i}{i-j+1}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

The inverse of P, $Q = P^{-1}$ is just the inverse of the corresponding Pascal matrix, with the first row and column deleted

$$Q_{ij} = \begin{cases} (-1)^{i-j}\binom{i}{j}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Q can easily be derived using the methods in [2,3]. Multiplying (4) by Q gives

$$\tilde{P}\,\mathbf{S} = \mathbf{n}, \tag{6}$$

where $\tilde{P} = Q\hat{P}$ is given by

$$\tilde{P}_{ij} = \begin{cases} (-1)^{i-j}\binom{i}{i-j+1}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

For $k = 5$, we have

$$\tilde{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 \\ 1 & -5 & 10 & -10 & 5 \end{bmatrix}.$$

The triangular system of equations (6) can easily be solved to obtain an infinite number of useful expressions

$$\begin{aligned} S_0(n) &= n \\ -S_0(n) + 2S_1(n) &= n^2 && \Rightarrow && S_1(n) = \frac{n^2+n}{2} \\ S_0(n) - 3S_1(n) + 3S_2(n) &= n^3 && \Rightarrow && S_2(n) = \frac{n(n+1)(2n+1)}{6} \\ -S_0(n) + 4S_1(n) - 6S_2(n) + 4S_3(n) &= n^3 && \Rightarrow && S_3(n) = \frac{n^4+2n^3+n^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2 \\ &\vdots && \vdots \end{aligned}$$

Perhaps more interesting is the fact that $\tilde{P}$ can be used to derive relationships between the binomial coefficients and Stirling numbers of the first and second kind, $s_1(i,j)$ and $s_2(i,j)$, respectively. Let N_1 denote the matrix with entries $s_1(i,j)$, and N_2, the matrix with entries $s_2(i,j)$. These matrices can be obtained using the recursive constructions for the Stirling numbers

$$s_1(i+1,j) = s_1(i,j-1) - is_1(i,j),$$

with $s_1(i,i) = 1$, $s_1(i,0) = 0$ for $i > 0$, and $s_1(i,j) = 0$ for $j < 0$ and $j > i$,

$$s_2(i+1,j) = s_2(i,j-1) + js_2(i,j),$$

with $s_2(i,i) = 1$, $s_2(i,0) = 0$ for $i > 0$, and $s_2(i,j) = 0$ for $j < 0$ and $j > i$. For $i,j \leq 5$, this gives

$$N_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 \\ 24 & -50 & 35 & -10 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix}$$

from which can be illustrated the interesting result for Stirling numbers [4]

$$N_1 N_2 = N_2 N_1 = I.$$

Now define a matrix Λ composed of the eigenvalues of $\tilde{P}$, which for the example is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Then

$$\tilde{P} N_2 = N_2 \Lambda \tag{7}$$

and

$$\tilde{P} = N_2 \Lambda N_1. \tag{8}$$

Equation (7) can be proven via a simple combinatorial interpretation. Equating the entries in row i, column j of the two matrices gives

$$\begin{aligned} &\binom{i}{1} s_2(i,j) - \binom{i}{2} s_2(i-1,j) + \binom{i}{3} s_2(i-2,j) - \cdots \\ &\qquad \cdots + (-1)^{i-j} \binom{i}{i-j+1} s_2(j,j) = j s_2(i,j). \end{aligned} \tag{9}$$

The right side of (9) can be interpreted as the number of ways to partition the set $\{1, 2, \ldots, i\}$ into j nonempty subsets, and then choose one of these j subsets to be distinguished from the others in some way. Now for $1 \leq k \leq i$, let A_k be the set of all such partitions in which k is an element of the distinguished subset. Then,

$$j s_2(i,j) = \left| \bigcup_{k=1}^{i} A_k \right|.$$

The cardinality of this union can also be computed by the inclusion-exclusion formula. First note that if $1 \leq k_1 < k_2 < \cdots < k_n \leq i$, then

$$\left| A_{k_1} \bigcap A_{k_2} \bigcap \cdots \bigcap A_{k_n} \right| = s_2(i-n+1, j),$$

since a partition in which $k_1, k_2, \ldots, k_n$ are all elements of the distinguished subset can be formed by first grouping $k_1, k_2, \ldots, k_n$ together and treating them as a single element, and then forming a partition of the resulting set of size $i-n+1$ into j nonempty subsets, taking the subset containing $k_1, k_2, \ldots, k_n$ to be the distinguished subset. There are $\binom{i}{n}$ such intersections of n of the A_k's, so the inclusion-exclusion formula gives the left side of (9). This proves (9) by showing that the two sides of the equation represent two ways of counting the same thing.

Generalizing (8), there exists an infinite number of such relations given by

$$\tilde{P}^m = N_2 \Lambda^m N_1,$$

which for $m = -1$ gives

$$\begin{aligned}\tilde{P}^{-1} &= N_2 \Lambda^{-1} N_1,\\ N_1 &= \Lambda N_2^{-1} \tilde{P}^{-1},\\ N_2 &= \tilde{P}^{-1} N_1^{-1} \Lambda.\end{aligned}$$

Using the previous matrices corresponding to $k = 5$ gives

$$\tilde{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \end{bmatrix}. \tag{10}$$

Returning to (5), multiplication by $\hat{P}^{-1} = \hat{Q}$ gives

$$\mathbf{S} = R\,\mathbf{n}, \tag{11}$$

which for $k = 5$ is equal to (10) (the proof that $R = \tilde{P}^{-1}$ is straightforward). One will note immediately that the first column of R contains the Bernoulli numbers B_i, as in Bernoulli's table [5]. Inverting $\hat{P}$ gives for $k = 5$

$$\hat{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} \end{bmatrix},$$

which is the same as R, except that now the diagonal below the main diagonal has value $-1/2$. Thus, the first column of $\hat{P}^{-1}$ contains the B_i as they are now defined [5,6].

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