Restricted 132-Dumont permutations

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Abstract

A permutation π is said to be a *Dumont permutation of the first kind* if each even integer in π must be followed by a smaller integer, and each odd integer is either followed by a larger integer or is the last element of π (see, for example, `www.theory.csc.uvic.ca/~cos/inf/perm/Genocchi Info.html`). In *Duke Math. J.* 41 (1974), 305–318, Dumont showed that certain classes of permutations on n letters are counted by the Genocchi numbers. In particular, Dumont showed that the $(n + 1)$st Genocchi number is the number of Dummont permutations of the first kind on $2n$ letters.

In this paper we study the number of Dumont permutations of the first kind on n letters avoiding the pattern 132 and avoiding (or containing exac tly once) an arbitrary pattern on k letters. In several interesting cases the generating function depends only on k.

1 Introduction

Classical patterns. Let $\alpha \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$ be two permutations. We say that α *contains* τ if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that α *avoids* τ, or is τ-*avoiding*, if such a subsequence does not exist. The set of all τ-avoiding permutations in $\mathfrak{S}_n$ is denoted $\mathfrak{S}_n(\tau)$. For an arbitrary finite collection of patterns T, we say that α avoids T if α avoids every $\tau \in T$; the corresponding subset of $\mathfrak{S}_n$ is denoted $\mathfrak{S}_n(T)$.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of

patterns τ_1, τ_2. This problem was solved completely for $\tau_1, \tau_2 \in \mathfrak{S}_3$ (see [19]), and for $\tau_1 \in \mathfrak{S}_3$ and $\tau_2 \in \mathfrak{S}_4$ (see [21]). Several recent papers [6, 9, 13, 14, 15, 16] deal with the case $\tau_1 \in \mathfrak{S}_3$, $\tau_2 \in \mathfrak{S}_k$ for various pairs τ_1, τ_2. Another natural question is to study permutations avoiding τ_1 and containing τ_2 exactly t times. Such a problem for certain $\tau_1, \tau_2 \in \mathfrak{S}_3$ and $t = 1$ was investigated in [17], and for certain $\tau_1 \in \mathfrak{S}_3$, $\tau_2 \in \mathfrak{S}_k$ in [1, 9, 13, 18]. The tools involved in these papers include generating trees, continued fractions, Chebyshev polynomials, and Dyck words. Also, the tools involved in these papers include many classical sequences, for example sequences of Catalan numbers, Fibonacci numbers, and Pell numbers.

We denote the nth *Catalan number* by $C_n = \frac{1}{n+1}\binom{2n}{n}$. The generating function for the Catalan numbers is denoted by $C(x)$, that is, $C(x) = \sum_{n \geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$.

Generalized patterns. In [2] generalized permutation patterns were introduced that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. We write a classical pattern with dashes between any two adjacent letters of the pattern, say 1342, as 1-3-4-2, and if we write, say 24-3-1, then we mean that if this pattern occurs in permutation $\pi \in \mathfrak{S}_n$, then the letters in the permutation π that correspond to 2 and 4 are adjacent (see [4]). For example, the permutation $\pi = 35421$ has only two occurrences of the pattern 23-1, namely the subsequences 352 and 351, whereas π has four occurrences of the pattern 2-3-1, namely the subsequences 352, 351, 342, and 341.

Claesson [4] presented a complete solution for the number of permutations avoiding any single generalized pattern of length three with exactly one adjacent pair of letters. Claesson and Mansour [5] presented a complete solution for the number of permutations avoiding any pair of generalized patterns of length three with exactly one adjacent pair of letters. Kitaev [8] investigated simultaneous avoidance of two or more 3-letter generalized patterns without internal dashes. Later, Mansour [10, 11] (for more details see [12]) presented a general approach to study the number of permutations avoiding 1-3-2 and avoiding (or containing exactly once) an arbitrary generalized pattern.

Dumont permutations. A permutation π is said to be a *Dumont permutation of the first kind* if each even integer in π must be followed by a smaller integer, and each odd integer is either followed by a larger integer or is the last element of π (see, for example, [22]). For example, 2143, 3421, and 4213 are all the Dumont permutations of the first kind of length 4.

A permutation π is said to be a *Dumont permutation of the second kind* if $\pi_i < i$ for any even position i, and $\pi_i \geq i$ for any odd position i. For example, 2143, 3142, and 4132 are all the Dumont permutations of the second kind of length 4.

Dumont [7] showed the number of Dumont permutations of the first (second) kind in $\mathfrak{S}_{2n}$ is given by the $(n+1)^{st}$ Genocchi number (see [20, Sequence A001469(M3041)]).

Remark 1.1 *Let $\pi \in \mathfrak{S}_n$ be any Dumont permutation of the second kind; since $\pi_2 < 2$ we get $\pi_2 = 1$. Hence, it is easy to see that there are no Dumont permutations of the second kind in $\mathfrak{S}_n(132)$ for all $n \geq 4$. So, in this paper we discuss only the case of Dumont permutations of the first kind and refer to them simply as Dumont*

permutations.

We define for all $r \geq 2$,

$$Q_r(x) = 1 + \frac{x^2 Q_{r-1}(x)}{1 - x^2 Q_{r-2}(x)}. \tag{1}$$

We denote the solution of Recurrence 1 with $Q_0(x) = 0$ and $Q_1(x) = 1$ by $F_r(x)$, and we denote the solution of Recurrence 1 with $Q_0(x) = Q_1(x) = 1$ by $G_r(x)$. For example, $F_2(x) = 1 + x^2$, $F_3(x) = \frac{1+x^4}{1-x^2}$, $G_2(x) = \frac{1}{1-x^2}$, and $G_3(x) = \frac{1-x^2+x^4}{(1-x^2)^2}$. Evidently, $F_r(x)$ and $G_r(x)$ are rational functions in x^2, and for all $r \geq 1$,

$$F_r(x) = 1 + \sum_{j=1}^{r-1} \frac{x^{2j}}{\prod_{m=r-1-j}^{r-2}(1 - x^2 F_m(x))}$$

$$\text{and } G_r(x) = 1 + \sum_{j=1}^{r-1} \frac{x^{2j}}{\prod_{m=r-1-j}^{r-2}(1 - x^2 G_m(x))}.$$

Example 1.2 *Using Recurrence 1 it is easy to see that*

$$F_4(\sqrt{x}) = \sum_{n \geq 0}(f_{n+2} + f_n - 2)x^n \text{ and } G_4(\sqrt{x}) = 1 + x + \sum_{n \geq 2}(3 \cdot 2^{n-2} - 1)x^n,$$

where f_n is the nth Fibonacci number.

Organization of the paper. In this paper we use generating function techniques to study those Dumont permutations in $\mathfrak{S}_n$ ($n \geq 0$) which avoid 132 and avoid (or contain exactly once) an arbitrary pattern on k letters. In several interesting cases the generating function depends only on k.

The paper is organized as follows. The case of Dumont permutations avoiding both 132 and τ is treated in Section 2. We present a simple structure for any Dumont permutation avoiding 132. This structure can be obtained explicitly for several interesting cases, including classical patterns and generalized patterns. This allows us to find explicitly some statistics on Dumont permutations which avoid 132. The case of avoiding 132 and containing another pattern τ exactly once is treated in Section 3. Again, we find explicitly the generating function for several interesting cases of τ, including classical patterns and generalized patterns.

Most of the explicit solutions obtained in Sections 2–4 involve the generating functions $F_k(x)$ and $G_k(x)$.

2 Dumont permutations which avoid 132 and another pattern

Let $\mathfrak{D}$ be the set of all Dumont permutations of all sizes including the empty permutation. Let $\mathfrak{D}_\tau(n)$ denote the number of Dumont permutations in $\mathfrak{S}_n(132, \tau)$, and

let $\mathfrak{D}_\tau(x) = \sum_{n\geq 0} \mathfrak{D}_\tau(n)x^n$ be the corresponding generating function. In this section we describe a method for enumerating Dumont permutations which avoid 132 and another pattern and we use our method to enumerate $\mathfrak{D}_\tau(n)$ for various τ. We begin with an observation concerning the structure of the Dumont permutations of the first kind avoiding 132 which holds immediately from definitions.

Proposition 2.1 *For any $\pi \in \mathfrak{D}_n(132)$ such that $\pi_j = n$, one of the following assertions holds:*

1. *if n is an odd number then $\pi = (\pi', n)$, where $\pi' \in \mathfrak{D}_{n-1}(132)$;*

2. *if n is an even number then $\pi = (\pi', n, \pi'')$ such that π' is a Dumont permutation on the numbers $n - j + 1, n - j + 2, \ldots, n - 1$, π'' is a nonempty Dumont permutation on the numbers $1, 2, \cdots, n - j$, and $j = 1, 2, 4, \ldots, n - 2$ (the minimal element of π' cannot be an even number).*

2.1 $\tau = \varnothing$

As a corollary of Proposition 2.1 we find an explicit formula for the number of 132-avoiding Dumont permutations in $\mathfrak{S}_n$.

Theorem 2.2 *The generating function for the number of 132-avoiding Dumont permutations in $\mathfrak{S}_n$ is given by $(1+x)C(x^2)$. In other words, the number of 132-avoiding Dumont permutations in $\mathfrak{S}_n$ is given by $C_{[n/2]}$, which is the $[n/2]$th Catalan number.*

Proof. By Proposition 2.1, we have two possibilities for block decomposition of an arbitrary $\pi \in \mathfrak{D}_n(132)$. Let us write an equation for $\mathfrak{D}_\varnothing(x)$. The contribution of the first decomposition above equals

$$\sum_{n\geq 0} \mathfrak{D}_\varnothing(2n+1)x^{2n+1} = x\sum_{n\geq 0} \mathfrak{D}_\varnothing(2n)x^{2n},$$

equivalently,

$$\mathfrak{D}_\varnothing(x) - \mathfrak{D}_\varnothing(-x) = x(\mathfrak{D}_\varnothing(x) + \mathfrak{D}_\varnothing(-x)). \tag{2}$$

The contribution of the second decomposition above equals

$$\sum_{n\geq 1} \mathfrak{D}_\varnothing(2n)x^{2n} = \sum_{n\geq 1} \mathfrak{D}_\varnothing(2n-1)x^{2n} + \sum_{n\geq 1}\sum_{j=0}^{n} \mathfrak{D}_\varnothing(2j+1)\mathfrak{D}_\varnothing(2n+2-2j)x^{2n},$$

equivalently,

$$\mathfrak{D}_\varnothing(x) + \mathfrak{D}_\varnothing(-x) - 2 =$$
$$= x(\mathfrak{D}_\varnothing(x) - \mathfrak{D}_\varnothing(-x)) + \tfrac{x}{2}(\mathfrak{D}_\varnothing(x) - \mathfrak{D}_\varnothing(-x))(\mathfrak{D}_\varnothing(x) + \mathfrak{D}_\varnothing(-x) - 2). \tag{3}$$

By putting $\mathfrak{D}_\varnothing(x) = (1 + x)A(x)$ in Equations 2 and 3 it is easy to see that $A(x) = C(x^2)$. $\qquad\square$

2.2 A classical pattern $\tau = 12 \ldots k$

Let us start with the following example.

Example 2.3 *By definition we have $\mathfrak{D}_1(x) = 1$ and $\mathfrak{D}_{12}(x) = 1 + x + x^2$.*

The case of varying k is more interesting. As an extension of Example 2.3, let us consider the case $\tau = 12 \ldots k$.

Theorem 2.4 *Let $A_k(x) = \frac{1}{2}(\mathfrak{D}_{12\ldots k}(x) + \mathfrak{D}_{12\ldots k}(-x))$ and $B_k(x) = \frac{1}{2}(\mathfrak{D}_{12\ldots k}(x) - \mathfrak{D}_{12\ldots k}(-x))$ for all $k \geq 0$. Then*

$$A_k(x) = F_k(x), \qquad B_k(x) = xF_{k-1}(x), \quad and \quad \mathfrak{D}_{12\ldots k}(x) = F_k(x) + xF_{k-1}(x).$$

Proof. Using the same arguments as in the proof of Theorem 2.2 we get

$$\mathfrak{D}_{12\ldots k}(x) - \mathfrak{D}_{12\ldots k}(-x) = x(\mathfrak{D}_{12\ldots(k-1)}(x) + \mathfrak{D}_{12\ldots(k-1)}(-x)),$$

and

$$\mathfrak{D}_{12\ldots k}(x) + \mathfrak{D}_{12\ldots k}(-x) - 2 = x(\mathfrak{D}_{12\ldots k}(x) - \mathfrak{D}_{12\ldots k}(-x)) +$$
$$+ \tfrac{x}{2}(\mathfrak{D}_{12\ldots(k-1)}(x) - \mathfrak{D}_{12\ldots(k-1)}(-x))(\mathfrak{D}_{12\ldots k}(x) + \mathfrak{D}_{12\ldots k}(-x) - 2).$$

The rest is easy to check by the definitions of A_k and B_k. $\qquad\square$

Example 2.5 *Theorem 2.4, for $k = 3$, yields $\mathfrak{D}_{123}(x) = \frac{1+x+x^4-x^5}{1-x^2}$. In other words, the number of 132-avoiding Dumont permutation in $\mathfrak{S}_n(123)$ is given by $1 + (-1)^n$ for all $n \geq 4$, and 1 for $n = 0, 1, 2, 3$. An another example, Theorem 2.4, for $k = 4$, yields $\mathfrak{D}_{1234}(x) = \frac{1+2x+x^2+2x^6+x^7+x^8}{(1+x)(1-x^2-x^4)}$. In other words, the number of 132-avoiding Dumont permutation in $\mathfrak{S}_n(1234)$ is $f_{n/2+2} + f_{n/2} - 2$ if n is even number, otherwise 2 for all $n \geq 2$, where f_n is the nth Fibonacci number.*

As an extension of Theorem 2.4, let us define

$$\mathfrak{A}(x_1, x_2, x_3, \ldots) = \sum_{\pi \in \mathfrak{D}} \prod_{j \geq 1} x_j^{12\ldots j(\pi)},$$

where $\tau(\pi)$ is the number of occurrences of τ in π. Let

$$A^{(1)}(x_1, x_2, x_3, \ldots) = \tfrac{1}{2}(\mathfrak{A}(x_1, x_2, x_3, \ldots) + \mathfrak{A}(-x_1, x_2, x_3, \ldots)),$$
$$B^{(1)}(x_1, x_2, x_3, \ldots) = \tfrac{1}{2}(\mathfrak{A}(x_1, x_2, x_3, \ldots) - \mathfrak{A}(-x_1, x_2, x_3, \ldots)).$$

Using the same arguments as in the proof of Theorem 2.4, we obtain the following.

Theorem 2.6 *We have*

$$A^{(1)}(x_1, x_2, x_3, \ldots) = 1 + \frac{x_1^2 A^{(1)}(x_1x_2, x_2x_3, x_3x_4, \ldots)}{1 - x_1^2 x_2 A^{(1)}(x_1x_2^2x_3, x_2x_3^2x_4, x_3x_4^2x_5, \ldots)},$$

and

$$B^{(1)}(x_1, x_2, x_3, \ldots) = x_1 A^{(1)}(x_1x_2, x_2x_3, x_3x_4, \ldots).$$

As an application of Theorem 2.6, for $x_1 = x$ and $x_j = 1$, $j \geq 2$, we get that

$$B^{(1)}(x, 1, 1, \dots) = xA^{(1)}(x, 1, 1, \dots),$$

and

$$A^{(1)}(x, 1, 1, \dots) = \cfrac{1}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{\ddots}}} = C(x^2).$$

Hence, we have $\mathfrak{D}_\varnothing(x) = (1 + x)C(x^2)$ (see Theorem 2.2).

Another application of Theorem 2.6 is to the number of right to left maxima. Let $\pi \in \mathfrak{S}_n$, π_i is a *right to left maxima* if $\pi_i > \pi_j$ for all $i < j$. We denote the number of right to left maxima of π by $rlm(\pi)$. Proposition 5 of [3] proved

$$lrm(\pi) = \sum_{j \geq 1} 12 \dots j(\pi)(-1)^{j-1}.$$

Therefore,

$$\sum_{\pi \in \mathfrak{D}} x^{|\pi|} y^{rlm(\pi)} = \mathfrak{A}(xy, y^{-1}, y, y^{-1}, \dots)$$

together with Theorem 2.6 and $A^{(1)}(x, 1, 1, \dots) = C(x^2)$ we get

$$\sum_{\pi \in \mathfrak{D}} x^{|\pi|} y^{rlm(\pi)} = 1 + xC(x^2)y + \sum_{n \geq 2} x^{2n-2} C^{n-1}(x^2) y^n.$$

Corollary 2.7 *The generating function for the number of Dumont permutations avoiding* 132 *and having exactly k right to left maxima is given by* $x^{2k-2} C^{k-1}(x^2)$ *for all $k \geq 2$, and $x^k C^k(x^2)$ for $k = 0, 1$.*

2.3   A classical pattern $\tau = 2134 \dots k$

Similarly as in Theorem 2.4, we obtain the case $\tau = 2134 \dots k$.

Theorem 2.8 *For all $k \geq 2$,*

$$\mathfrak{D}_{213\dots k}(x) = G_{k-1}(x) + xG_{k-2}.$$

Example 2.9 *Theorem 2.8 for $k = 3, 4$ yields* $\mathfrak{D}_{213}(x) = \dfrac{1 + x - x^3}{1 - x}$ *and* $\mathfrak{D}_{2134}(x) = \dfrac{1 + x - x^2 - x^3 + x^4}{(1 - x^2)^2}$.

2.4 A generalized pattern 12-3-⋯-k

In this subsection we use the notation of generalized patterns (see Section 1). For example, we write the classical pattern 132 as 1-3-2.

By definition, we get $\mathfrak{D}_{12}(x) = 1 + x + x^2$. So, by the same arguments as in the proof of Theorem 2.4, together with

$$\mathfrak{D}_{12}(x) = \mathfrak{D}_{1\text{-}2}(x) = 1 + x + x^2,$$

we obtain the following.

Theorem 2.10 *For all $k \geq 1$,*

$$\mathfrak{D}_{12\text{-}3\text{-}...\text{-}k}(x) = \mathfrak{D}_{1\text{-}2\text{-}3\text{-}...\text{-}k}(x) = F_k(x) + xF_{k-1}(x).$$

A comparison of Theorem 2.4 with Theorem 2.10 suggests that there should exist a bijection between the sets $\mathfrak{S}_n(1\text{-}3\text{-}2, 12\text{-}3\text{-}\cdots\text{-}k)$ and $\mathfrak{S}_n(1\text{-}3\text{-}2, 1\text{-}2\text{-}3\text{-}\cdots\text{-}k)$. However, we failed to produce such a bijection, and finding one remains a challenging open question.

Now, let us define

$$\mathfrak{B}(x_1, x_2, x_3, \dots) = \sum_{\pi \in \mathfrak{D}} x_1^{1(\pi)} \prod_{j \geq 2} x_1^{12\text{-}3\text{-}...\text{-}j(\pi)},$$

where $\tau(\pi)$ is the number of occurrences of τ in π. Let

$$A^{(2)}(x_1, x_2, x_3, \dots) = \tfrac{1}{2}(\mathfrak{B}(x_1, x_2, x_3, \dots) + \mathfrak{B}(-x_1, x_2, x_3, \dots)),$$
$$B^{(2)}(x_1, x_2, x_3, \dots) = \tfrac{1}{2}(\mathfrak{B}(x_1, x_2, x_3, \dots) - \mathfrak{B}(-x_1, x_2, x_3, \dots)).$$

Using the same arguments as those in the proof of Theorem 2.4, we get

Theorem 2.11

$$A^{(2)}(x_1, x_2, x_3, \dots) = 1 + \frac{x_1^2(1 - x_2 + x_2 A^{(2)}(x_1, x_2 x_3, x_3 x_4, \dots))}{1 - x_1^2 x_2(1 - x_2 x_3 + x_2 x_3 A^{(2)}(x_1, x_2 x_3^2 x_4, x_3 x_4^2 x_5, \dots))},$$

and

$$B^{(2)}(x_1, x_2, x_3, \dots) = x_1 - x_1 x_2 + x_1 x_2 A^{(2)}(x_1, x_2 x_3, x_3 x_4 \dots).$$

Let $\pi \in \mathfrak{S}_n$; we say π_j is a *rise* for π if $\pi_j < \pi_{j+1}$ for all $j = 1, 2, \dots, n - 1$. We denote the number of rises of π by $rises(\pi)$. By definition, we have

$$\sum_{\pi \in \mathfrak{D}} x^{|\pi|} y^{rises(\pi)} = x - xy + (1 + xy)A^{(2)}(x, y, 1, 1, \dots),$$

so as an application of Theorem 2.11 we get

Corollary 2.12 *The generating function $\sum_{\pi \in \mathfrak{D}} x^{|\pi|} y^{rises(\pi)}$ is given by*

$$\frac{1 + xy - 2x^2 y + 2x^2 y^2 - (1 + xy)\sqrt{1 - 4x^2 y}}{2x^2 y^2}.$$

In other words, the generating function for Dumont permutations avoiding 1-3-2 with exactly k rises is given by $C_k x^{2k+1} + C_{k+1} x^{2k+2}$ for all $k \geq 1$, and $1 + x + x^2$ for $k = 0$, where C_m is the mth Catalan number.

2.5   A generalized pattern $\tau = 21\text{-}3\text{-}\cdots\text{-}k$

In this subsection, we use the notation of generalized patterns (see Section 1). For example, we write the classical pattern 132 as 1-3-2.

By definition, we get $\mathfrak{D}_{21}(x) = 1 + x$. So, by the same arguments as in the proof of Theorem 2.4 together with

$$\mathfrak{D}_{21}(x) = \mathfrak{D}_{2\text{-}1}(x) = 1 + x,$$

we obtain the following.

Theorem 2.13 *For all $k \geq 2$,*

$$\mathfrak{D}_{21\text{-}3\text{-}\ldots\text{-}k}(x) = \mathfrak{D}_{2\text{-}1\text{-}3\text{-}\ldots\text{-}k}(x) = G_{k-1}(x) + xG_{k-2}(x).$$

A comparison of Theorem 2.8 with Theorem 2.13 suggests that there should exist a bijection between the sets $\mathfrak{S}_n(1\text{-}3\text{-}2, 21\text{-}3\text{-}\cdots\text{-}k)$ and $\mathfrak{S}_n(1\text{-}3\text{-}2, 2\text{-}1\text{-}3\text{-}\cdots\text{-}k)$. However, we failed to produce such a bijection, and finding one remains a challenging open question.

Now, let us define

$$\mathfrak{C}(x_1, x_2, x_3, \ldots) = \sum_{\pi \in \mathfrak{D}} x_1^{1(\pi)} \prod_{j \geq 2} x_1^{21\text{-}3\text{-}\ldots\text{-}j(\pi)},$$

where $\tau(\pi)$ is the number of occurrences of τ in π. Let

$$A^{(3)}(x_1, x_2, x_3, \ldots) = \tfrac{1}{2}(\mathfrak{C}(x_1, x_2, x_3, \ldots) + \mathfrak{C}(-x_1, x_2, x_3, \ldots)),$$
$$B^{(3)}(x_1, x_2, x_3, \ldots) = \tfrac{1}{2}(\mathfrak{C}(x_1, x_2, x_3, \ldots) - \mathfrak{C}(-x_1, x_2, x_3, \ldots)).$$

Using the same arguments as in the proof of Theorem 2.4, we get the following.

Theorem 2.14 *We have*

$$A^{(3)}(x_1, x_2, x_3, \ldots) = 1 + \frac{x_1^2 x_2 A^{(3)}(x_1, x_2 x_3, x_3 x_4, \ldots)}{1 - x_1^2 x_2 A^{(3)}(x_1, x_2 x_3^2 x_4, x_3 x_4^2 x_5, \ldots)},$$

and

$$B^{(3)}(x_1, x_2, x_3, \ldots) = x_1 A^{(3)}(x_1, x_2 x_3, x_3 x_4 \ldots).$$

Let $\pi \in \mathfrak{S}_n$; we say that π_j is a *descent* for π if $\pi_j > \pi_{j+1}$ for all $j = 1, 2, \ldots, n-1$. We denote the number of descents of π by $descents(\pi)$. By definitions, we have

$$\sum_{\pi \in \mathfrak{D}} x^{|\pi|} y^{descents(\pi)} = (1 + x) A^{(3)}(x, y, 1, 1, \ldots),$$

therefore as an application of Theorem 2.14 we get

Corollary 2.15   *The generating function $\sum_{\pi \in \mathfrak{D}} x^{|\pi|} y^{descents(\pi)}$ is given by $(1 + x) C(x^2 y)$. In other words, the generating function for Dumont permutations avoiding 1-3-2 with exactly k descents is given by $C_k x^{2k+1} + C_k x^{2k+2}$ for all $k \geq 0$, where C_m is the mth Catalan number.*

2.6   A classical pattern $\tau = 23\ldots k1$

Again, Proposition 2.1 gives a complete answer for $\tau = 23\ldots k1$.

Theorem 2.16 *For all $k \geq 3$,*

$$\mathfrak{D}_{23\ldots k1}(x) = 1 + x + \frac{x^2(1+x)}{1 - x^2 - x^2 F_{k-3}(x)}.$$

Proof.   Using the same arguments as in the proof of Theorem 2.2 we get

$$\mathfrak{D}_{23\ldots k1}(x) - \mathfrak{D}_{23\ldots k1}(-x) = x(\mathfrak{D}_{23\ldots k1}(x) + \mathfrak{D}_{23\ldots k1}(-x)),$$

and

$$\mathfrak{D}_{23\ldots k1}(x) + \mathfrak{D}_{23\ldots k1}(-x) - 2 = x(\mathfrak{D}_{23\ldots k1}(x) - \mathfrak{D}_{23\ldots k1}(-x)) +$$
$$+ \tfrac{x}{2}(\mathfrak{D}_{12\cdots(k-2)}(x) - \mathfrak{D}_{12\ldots(k-2)}(-x))(\mathfrak{D}_{23\ldots k1}(x) + \mathfrak{D}_{23\ldots k1}(-x) - 2).$$

The rest is easy to check by the definitions of $F_k(x)$ together with Theorem 2.4.   □

Example 2.17 *Theorem 2.16, for $k = 5$, yields $\mathfrak{D}_{23451}(x) = \frac{(1+x)(1-x^2-x^4)}{1-2x^2-x^4}$. In other words, the number of Dumont permutation in $\mathfrak{S}_n(132, 23451)$ is given by $P_{[n/2]}$, which is the $[n/2]$th Pell number for all $n \geq 2$.*

3   Dumont permutations which avoid 132 and contain another pattern exactly once

Let $\mathfrak{D}_{\tau;r}(n)$ denote the number of Dumont permutations in $\mathfrak{S}_n(132)$ containing τ exactly r times, and let $\mathfrak{D}_{\tau;r}(x) = \sum_{n \geq 0} \mathfrak{D}_{\tau;r}(n)x^n$ be the corresponding generating function.

3.1   A classical pattern $\tau = 12\ldots k$

Theorem 3.1 *Let*

$$A_k(x) = \frac{x^2}{1 - x^2 F_{k-2}(x)} A_{k-1}(x) + \frac{x^4 F_{k-1}(x)}{(1 - x^2 F_{k-2}(x))^2} A_{k-2}(x)$$

for all $k \geq 2$, where $A_1(x) = 0$ and $A_2(x) = x^4$. Then for all $k \geq 2$

$$\mathfrak{D}_{12\ldots k;1}(x) = A_k(x) + xA_{k-1}(x).$$

Proof.   By Proposition 2.1, we have two possibilities for the block decomposition of an arbitrary π in $\mathfrak{D}_n(132)$. Let us write an equation for $\mathfrak{D}_{12\ldots k;1}(x)$. The contribution of the first decomposition above is

$$\sum_{n \geq 0} \mathfrak{D}_{12\ldots k;1}(2n+1)x^{2n+1} = x \sum_{n \geq 0} \mathfrak{D}_{12\ldots(k-1);1}(2n)x^{2n},$$

equivalently

$$\mathfrak{D}_{12\ldots k;1}(x) - \mathfrak{D}_{12\ldots k;1}(-x) = x(\mathfrak{D}_{12\ldots(k-1);1}(x) + \mathfrak{D}_{12\ldots(k-1);1}(-x)). \qquad (4)$$

The contribution of the second decomposition above is

$$\sum_{n\geq1}\mathfrak{D}_{12\ldots k;1}(2n)x^{2n} = \sum_{n\geq1}\mathfrak{D}_{12\ldots k;1}(2n-1)x^{2n}+$$
$$+ \sum_{n\geq1}\sum_{j=0}^{n}\mathfrak{D}_{12\ldots(k-1);1}(2j+1)\mathfrak{D}_{12\ldots k;0}(2n+2-2j)x^{2n}+$$
$$+ \sum_{n\geq1}\sum_{j=0}^{n}\mathfrak{D}_{12\ldots(k-1);0}(2j+1)\mathfrak{D}_{12\ldots k;1}(2n+2-2j)x^{2n},$$

equivalently

$$\mathfrak{D}_{12\ldots k;1}(x) + \mathfrak{D}_{12\ldots k;1}(-x) = x(\mathfrak{D}_{12\ldots k;1}(x) - \mathfrak{D}_{12\ldots k;1}(-x))+$$
$$+\tfrac{x}{2}(\mathfrak{D}_{12\ldots(k-1);1}(x) - \mathfrak{D}_{12\ldots(k-1);1}(-x))(\mathfrak{D}_{12\ldots k;0}(x) + \mathfrak{D}_{12\ldots k;0}(-x) - 2)+$$
$$+\tfrac{x}{2}(\mathfrak{D}_{12\ldots(k-1);0}(x) - \mathfrak{D}_{12\ldots(k-1);0}(-x))(\mathfrak{D}_{12\ldots k;1}(x) + \mathfrak{D}_{12\ldots k;1}(-x)).$$
$$(5)$$

Using Theorem 2.4, Equation 4, Equation 5, and Definition 1, we get the desired result. $\square$

Example 3.2 *Theorem 3.1 for $k = 3$ we get*

$$\mathfrak{D}_{123;1}(x) = \frac{x^5(1+x-x^2)}{1-x^2},$$

and for $k = 4$ we get

$$\mathfrak{D}_{1234;1}(x) = \frac{x^7(1+x-3x^2+2x^3+3x^4+3x^5-x^6+x^7)}{(1-x^2)(1-x^2-x^4)^2}.$$

As an extension of Theorem 3.1, let us consider the case $r \geq 1$. Theorem 2.6, for given k and r, yields an explicit formula for $\mathfrak{D}_{12\ldots k;r}(x)$. For example, for $k = 3$ and $r = 0, 1, 2, 3, 4$, we have the following.

Theorem 3.3 *We have*

(i) $\mathfrak{D}_{123;0}(x) = \dfrac{1+x+x^4-x^5}{1-x^2}$;

(ii) $\mathfrak{D}_{123;1}(x) = \dfrac{x^5(1+x-x^2)}{1-x^2}$;

(iii) $\mathfrak{D}_{123;2}(x) = \dfrac{x^5(1+x^2)(1+2x-2x^2-x^3+x^4)}{(1-x^2)^2}$;

(iv) $\mathfrak{D}_{123;3}(x) = \dfrac{x^7(1+x-x^2+x^3-x^4-x^5+x^6)}{(1-x^2)^2}$;

(v) $\mathfrak{D}_{123;4}(x) = \dfrac{x^9(1+x^2)(-1-3x+3x^2+3x^3-3x^4-x^5+x^6)}{(1-x^2)^2}$.

3.2   A classical pattern $\tau = 2134\ldots k$

Similarly to Theorem 3.1, we have

Theorem 3.4 *Let*

$$A_k(x) = \frac{x^2}{1 - x^2 G_{k-2}(x)} A_{k-1}(x) + \frac{x^4 G_{k-1}(x)}{(1 - x^2 G_{k-2}(x))^2} A_{k-2}(x)$$

for all $k \geq 4$, where $A_1(x) = A_2(x) = x^2$ and $A_3(x) = \frac{x^4}{1-x^2}$. Then, for all $k \geq 2$,

$$\mathfrak{D}_{213\ldots k;1}(x) = A_k(x) + x A_{k-1}(x).$$

3.3   A generalized patterns $\tau = 12\text{-}3\text{-}\cdots\text{-}k$ and $\tau = 21\text{-}3\text{-}\cdots\text{-}k$

Similarly to Theorem 3.1, we get

Theorem 3.5 *Let*

$$A_k(x) = \frac{x^2}{1 - x^2 F_{k-2}(x)} A_{k-1}(x) + \frac{x^4 F_{k-1}(x)}{(1 - x^2 F_{k-2}(x))^2} A_{k-2}(x)$$

for all $k \geq 4$, where $A_1(x) = x^2$ and $A_2(x) = 2x^4$. Then, for all $k \geq 2$,

$$\mathfrak{D}_{12\text{-}3\text{-}\cdots\text{-}k;1}(x) = A_k(x) + x A_{k-1}(x).$$

As an extension of Theorem 3.5, let us consider the case $r \geq 1$. Theorem 2.11, for given k and r, yields an explicit formula for $\mathfrak{D}_{12\text{-}3\text{-}\cdots\text{-}k;r}(x)$. For example, for $k = 3$ and $r = 0, 1, 2, 3, 4$, we have the following.

Theorem 3.6 *We have*
(i) $\mathfrak{D}_{12\text{-}3;0}(x) = \dfrac{1 + x + x^4 - x^5}{1 - x^2}$;

(ii) $\mathfrak{D}_{12\text{-}3;1}(x) = \dfrac{x^5(2 + 3x - 4x^2 - x^3 + 2x^4)}{(1 - x^2)^2}$;

(iii) $\mathfrak{D}_{12\text{-}3;2}(x) = \dfrac{x^7(2 + 2x - 6x^2 - x^3 + 6x^4 + x^5 - 2x^6)}{(1 - x^2)^3}$;

(iv) $\mathfrak{D}_{12\text{-}3;3}(x) = \dfrac{x^7(3 + 5x - 10x^2 - 9x^3 + 10x^4 + 3x^5 + 4x^7 - 5x^8 - x^9 + 2x^{10})}{(1 - x^2)^4}$;

(v) $\mathfrak{D}_{12\text{-}3;4}(x) =$
$$\frac{x^9(5 + 5x - 23x^2 - 7x^3 + 40x^4 - x^5 - 30x^6 + 5x^7 + 5x^8 - x^9 + 5x^{10} + x^{11} - 2x^{12})}{(1 - x^2)^5}.$$

Similarly to Theorem 3.1, we have

Theorem 3.7 *Let*

$$A_k(x) = \frac{x^2}{1 - x^2 G_{k-2}(x)} A_{k-1}(x) + \frac{x^4 G_{k-1}(x)}{(1 - x^2 G_{k-2}(x))^2} A_{k-2}(x)$$

for all $k \geq 4$, where $A_1(x) = A_2(x) = x^2$, $A_3(x) = \frac{x^4}{1-x^2}$, and $A_4(x) = \frac{x^6(2-x^2)}{(1-x^2)^3}$. Then, for all $k \geq 2$,

$$\mathfrak{D}_{21\text{-}3\text{-}\cdots\text{-}k;1}(x) = A_k(x) + x A_{k-1}(x).$$

As an extension of Theorem 3.7, let us consider the case $r \geq 1$. Theorem 2.14, for given k and r, yields an explicit formula for $\mathfrak{D}_{21\text{-}3\text{-}\cdots\text{-}k;r}(x)$. For example, for $k = 3$ and $r = 0, 1, 2, 3, 4$, we have the following.

Theorem 3.8 *We have*

(i) $\mathfrak{D}_{21\text{-}3;0}(x) = \dfrac{1 + x + x^4 - x^5}{1 - x^2}$;

(ii) $\mathfrak{D}_{21\text{-}3;1}(x) = \dfrac{x^3(1 + x - x^2)}{1 - x^2}$;

(iii) $\mathfrak{D}_{21\text{-}3;2}(x) = \dfrac{x^5(1 + 2x - 2x^2 - x^3 + x^4)}{(1 - x^2)^2}$;

(iv) $\mathfrak{D}_{21\text{-}3;3}(x) = \dfrac{x^5(1 + x - x^2 + x^3 - x^4 - x^5 + x^6)}{(1 - x^2)^2}$;

(v) $\mathfrak{D}_{21\text{-}3;4}(x) = \dfrac{x^7(1 + 2x - 2x^2 - 2x^5 + 2x^6 + x^7 - x^8)}{(1 - x^2)^3}$.

4   Further results

Here we present three different directions to generalize the results of the previous sections. The first of these directions is to consider one occurrence of the classical pattern 132. For example, the following result is true.

Theorem 4.1 *There does not exist a Dumont permutation containing 132 (classical pattern) exactly once.*

Proof.   Let $\pi = (\pi', n, \pi'')$ be a Dumont permutation of length n, which contains the pattern 132 exactly once. It is easy to see that there does not exist a Dumont permutation where $n = 0, 1, 2, 3$. Suppose $n \geq 4$, and let us assume by induction on n that there does not exist a Dumont permutation of length $m \leq n - 1$ containing 132 exactly once. To prove this property for n, let us consider the following two cases together using Proposition 2.1: n is either an even number, or n is an odd number.

1. Let n be an odd number. Since π is a Dumont permutation, we get $\pi'' = \varnothing$, so π contains 132 exactly once if and only if π' contains 132 exactly once.

2. Let n be an even number. Since π is a Dumont permutation we have $\pi'' \neq \varnothing$. Now, let us consider two cases: either n does not appear in the occurrence of 132, or n does.

   (a) Let the occurrence of 132 not contain the element n. So, every element of π' is greater than every element of π''. Therefore, either π' is a Dumont permutation of length $m \leq n - 2$ that contains 132 exactly once, or π'' is a Dumont permutation of length $m \leq n - 1$ that contains 132 exactly once.

   (b) Let the occurrence of 132 contain the element n. So, $\pi = (\pi', a, n, \pi'', a + 1, \pi''')$ (see [16]) such that $\pi_p = n$ and $\pi_q = a + 1$, where every element of π' is greater than every element of π'' and every element of π'' is greater than every element of π'''. Since n is even number and maximal in π we have that a is an odd number, so $a + 1$ is an even number. Therefore, by using Proposition 2.1 we get that p, q are even numbers, (π', a) is of odd length, and π'' is of even length. On the other hand, $q = p + 1 + |\pi''|$, so q is an odd number, a contradiction.

Hence, by induction on n we get the desired result.   □

The second direction is to consider more than one additional restriction. For example, the following result is true.

Theorem 4.2 *Let $k \geq 2$. The generating function for the number of Dumont permutations in $\mathfrak{S}_n(1\text{-}3\text{-}2, 1\text{-}2\text{-}3 \cdots \text{-}k, 2\text{-}1\text{-}3 \cdots \text{-}k)$ is given by*

$$G_{k-1}(x) + xG_{k-2}(x).$$

A comparison of Theorem 4.2 with Theorem 2.8 suggests that there should exist a bijection between the sets

$$\mathfrak{S}_n(1\text{-}3\text{-}2, 2\text{-}1\text{-}3\text{-}\cdots\text{-}k) \quad \text{and} \quad \mathfrak{S}_n(1\text{-}3\text{-}2, 1\text{-}2\text{-}3\cdots\text{-}k, 2\text{-}1\text{-}3\text{-}\cdots\text{-}k).$$

However, we failed to produce such a bijection, and finding one remains an open question.

The third direction is to consider another 3-letter pattern instead of 1-3-2.

Theorem 4.3 *The number of Dumont permutation of the second kind in $\mathfrak{S}_n(3\text{-}2\text{-}1)$ is the same as the number of Dumont permutation in $\mathfrak{S}_n(2\text{-}3\text{-}1)$ (or in $\mathfrak{S}_n(3\text{-}1\text{-}2)$) which is equal to $C_{[n/2]}$.*

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