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The generalized Touchard polynomials revisited

Toufik Mansour [a,*], Matthias Schork [b]

[a] Department of Mathematics, University of Haifa, 31905 Haifa, Israel
[b] Camillo-Sitte-Weg 25, 60488 Frankfurt, Germany

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ABSTRACT

We discuss the generalized Touchard polynomials introduced recently by Dattoli et al. as well as their extension to negative order introduced by the authors with operational methods. The connection to generalized Stirling and Bell numbers is elucidated and analogs to Burchnall's identity are derived. A recursion relation for the generalized Touchard polynomials is established and it is shown that one can interpret some of the resulting formulas as binomial theorems for particular noncommuting variables. We suggest to generalize the generalized Touchard polynomials still further and introduce so called Comtet–Touchard functions which are associated to the powers of an arbitrary derivation.

1. Introduction

The Touchard polynomials (also called *exponential polynomials*) may be defined (see, e.g., [1,2,6,16,17]) for $n \in \mathbb{N}$ by

$$T_n(x) := e^{-x}\left(x\frac{d}{dx}\right)^n e^x. \tag{1.1}$$

In the following we also use the notation $D = \partial_x = \frac{d}{dx}$. If we further denote the operator of multiplication with x by X, i.e., $(Xf)(x) = xf(x)$ for all functions f considered, we can make contact with operational formulas [16,17]. For example, using the fact that $(XD)^n = \sum_{k=0}^{n} S(n,k)X^kD^k$, where $S(n,k)$ denotes the Stirling numbers of the second kind, one obtains directly from the definition of the Touchard polynomials the relation

$$T_n(x) = \sum_{k=0}^{n} S(n,k)x^k = B_n(x), \tag{1.2}$$

where the second equality corresponds to the definition of the conventional Bell polynomials. Dattoli et al. introduced in [6, Eq. (36)] *Touchard polynomials of higher order*. They are defined for $m \in \mathbb{N}$ (and $n \in \mathbb{N}$) by

$$T_n^{(m)}(x) := e^{-x}(x^m\partial_x)^n e^x, \tag{1.3}$$

and reduce for $m = 1$ to the conventional Touchard polynomials from above. Many of their properties are discussed in [6]. In particular, noting that the normal ordering of $(x^m\partial_x)^n$ leads to the generalized Stirling numbers $S_{m,1}(n,k)$ considered, e.g., by Lang [12], one has a close connection between the Touchard polynomials of order m and the Stirling numbers $S_{m,1}(n,k)$. It was mentioned in [6, Eq. (37)] that the higher order Touchard polynomials satisfy the recursion relation

$$(x^m + x^m\partial_x)T_n^{(m)}(x) = T_{n+1}^{(m)}(x). \tag{1.4}$$

* Corresponding author.
E-mail addresses: toufik@math.haifa.ac.il, tmansour@univ.haifa.ac.il (T. Mansour), mschork@member.ams.org (M. Schork).

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In [15] the Touchard polynomials of negative integer order $-m$ (with $m \in \mathbb{N}$) are defined for all $n \in \mathbb{N}$ by

$$T_n^{(-m)}(x) := e^{-x}(x^{-m}\partial_x)^n e^x, \tag{1.5}$$

and several of their properties are discussed (in close analogy to the Touchard polynomials of higher order considered in [6]). For example, from the definition above, it is easy to see that the analog of (1.4) holds true, i.e.,

$$(x^{-m} + x^{-m}\partial_x)T_n^{(-m)}(x) = T_{n+1}^{(-m)}(x). \tag{1.6}$$

Note that one can also define the Touchard polynomials of order zero, but due to $T_n^{(0)}(x) = e^{-x}\partial_x^n e^x = 1$ for all $n \in \mathbb{N}$, no interesting polynomials result. As a particular example let us mention that one has a very close connection (mentioned in the main text below) between $T_n^{(-1)}(x)$ and Bessel polynomials. In the general case it was shown in [15] that the Touchard polynomials (of arbitrary order $m \in \mathbb{Z}$) have a nice expression in terms of the generalized Stirling and Bell numbers discussed by the authors in [13,14]. The aim of the present paper is to continue the study of the Touchard polynomials of negative order. Several results will be derived in analogy to the case of Touchard polynomials of higher order (treated in [6]) and new results will be given for arbitrary order, e.g., a new recursion relation. Furthermore, several connections of the generalized Touchard polynomials to other combinatorial objects will be drawn and a further generalization is suggested.

Let us outline the structure of the paper in more detail. In Section 2 we consider the Touchard polynomials of order -1 in detail since it is possible to derive all results completely explicit. In particular, using these results it is possible to find an explicit binomial formula for variables U, V satisfying $UV = VU - V^3$ involving Bessel polynomials. In Section 3 the Touchard polynomials of arbitrary negative integer order are discussed along the same lines. A recursion relation for the generalized Touchard polynomials of arbitrary integer order is derived in Section 4. In Section 5 it is suggested to consider generalized Touchard functions of arbitrary real order. As a particular example the case of order 1/2 is discussed and it is shown that the associated Touchard functions are given by Hermite polynomials. In Section 6 we motivate a further generalization and introduce so called Comtet–Touchard functions associated to powers of an arbitrary derivation. Finally, in Section 7 some conclusions are presented and some possible avenues for future research are outlined.

2. The Touchard polynomials of order -1

It was observed in [15] that for the Touchard polynomials of negative order the first nontrivial case $m = 1$ seems to be much nicer than the general case. This is the reason why we treat in the present section this case explicitly before turning to the general case in the next section. From (1.6) we obtain in the case $m = 1$ that

$$(x^{-1} + x^{-1}\partial_x)T_n^{(-1)}(x) = T_{n+1}^{(-1)}(x).$$

Introducing the operator $M_1 := (x^{-1} + x^{-1}\partial_x)$ we can denote this as $M_1^k T_n^{(-1)}(x) = T_{n+k}^{(-1)}(x)$, yielding

$$\sum_{k\geqslant 0} \frac{t^k}{k!} T_{n+k}^{(-1)}(x) = e^{tM_1} T_n^{(-1)}(x). \tag{2.1}$$

To understand the right-hand side, we have to study the operator

$$e^{tM_1} = e^{t(x^{-1}+x^{-1}\partial_x)}, \tag{2.2}$$

closer, i.e., we need a *disentanglement identity* which would allow us to write this as a product of two operators where each factor depends on only one of the operators. Let us recall from [8, Eq. (I.2.34)] the following crucial result: Given two operators A, B satisfying $[A, B] = mA^n$, one has the following disentanglement identity

$$e^{A+B} = \exp\left[\frac{1}{m(n-2)A^{n-2}}\left\{(1 + m(n-1)A^{n-1})^{\frac{n-2}{n-1}} - 1\right\}\right]e^B. \tag{2.3}$$

For the case we are interested in we identify $A \equiv tx^{-1}$ and $B \equiv tx^{-1}\partial_x$ with commutation relation

$$[A, B] = t^{-1}A^3.$$

Thus, we can use (2.3) with $m = t^{-1}$ and $n = 3$ to find after some simplifications

$$e^{A+B} = \exp\left[tA^{-1}\left\{(1 + 2t^{-1}A^2)^{\frac{1}{2}} - 1\right\}\right]e^B,$$

which can be expressed in terms of the original operators as

$$e^{t(x^{-1}+x^{-1}\partial_x)} = e^{\sqrt{x^2+2t}-x}e^{tx^{-1}\partial_x}. \tag{2.4}$$

This is the sought-for disentanglement identity. Inserting this into (2.1), we obtain

$$\sum_{k\geqslant 0}\frac{t^k}{k!}T^{(-1)}_{n+k}(x)=e^{\sqrt{x^2+2t}-x}e^{tx^{-1}\partial_x}T^{(-1)}_n(x).$$

Recalling from [15, Corollary 6.15] that

$$e^{\lambda x^{-1}\partial_x}f(x)=f\left(\sqrt{x^2+2\lambda}\right), \tag{2.5}$$

we have shown the following proposition as analog to [6, Eq. (38)].

Proposition 2.1. *The Touchard polynomials of order* -1 *satisfy the relation*

$$\sum_{k\geqslant 0}\frac{t^k}{k!}T^{(-1)}_{n+k}(x)=e^{\sqrt{x^2+2t}-x}T^{(-1)}_n\left(\sqrt{x^2+2t}\right). \tag{2.6}$$

According to [15, Corollary 6.15], $e^{\sqrt{x^2+2t}-x}$ is the exponential generating function of the $T^{(-1)}_n(x)$. It follows that

$$\sum_{k\geqslant 0}(x^{-1}+x^{-1}\partial_x)^k\frac{t^k}{k!}=e^{t(x^{-1}+x^{-1}\partial_x)}=e^{\sqrt{x^2+2t}-x}e^{tx^{-1}\partial_x}=\left\{\sum_{m\geqslant 0}\frac{t^m}{m!}T^{(-1)}_m(x)\right\}\left\{\sum_{l\geqslant 0}\frac{t^l}{l!}(x^{-1}\partial_x)^l\right\}$$
$$=\sum_{s\geqslant 0}\frac{t^s}{s!}\left\{\sum_{r=0}^{s}\binom{s}{r}T^{(-1)}_{s-r}(x)(x^{-1}\partial_x)^r\right\}.$$

Comparing coefficients yields the following corollary as analog to [6, Eq. (39)].

Corollary 2.2. *For* $k\in\mathbb{N}_0$ *one has the operational relation*

$$(x^{-1}+x^{-1}\partial_x)^k=\sum_{r=0}^{k}\binom{k}{r}T^{(-1)}_{k-r}(x)(x^{-1}\partial_x)^r. \tag{2.7}$$

In the particular case we are considering one has a beautiful expression for $T^{(-1)}_\ell(x)$ in terms of Bessel polynomials. The *n-th Bessel polynomial* (with $n\in\mathbb{N}_0$) is defined by

$$y_n(x):=\sum_{k=0}^{n}\frac{(n+k)!}{2^k k!(n-k)!}x^k, \tag{2.8}$$

see, e.g., [20] where also some properties of these polynomials as well as those of the corresponding Bessel numbers can be found. The first few Bessel polynomials are given by

$$y_0(x)=1,\quad y_1(x)=1+x,\quad y_2(x)=1+3x+3x^2. \tag{2.9}$$

According to [15, Theorem 6.10], one has the relation

$$T^{(-1)}_\ell(x)=x^{-\ell}y_{\ell-1}\left(-\frac{1}{x}\right).$$

This is valid for $\ell>0$, but if we set for convenience $y_{-1}(x)=1$ it also holds true for $\ell=0$. Using this relation, (2.7) can be written alternatively as

$$(x^{-1}+x^{-1}\partial_x)^k=\sum_{r=0}^{k}\binom{k}{r}x^{-(k-r)}y_{k-r-1}(-x^{-1})(x^{-1}\partial_x)^r. \tag{2.10}$$

This identity can now be interpreted in a straightforward fashion as a particular example of a *noncommutative binomial theorem*. In [13] variables U,V were considered which satisfy the commutation relation

$$UV=VU+hV^s, \tag{2.11}$$

for arbitrary $s\in\mathbb{R}$ (and $h\neq 0$). In the particular case $s=0$ and $h=1$ this reduces to the commutation relation of the Weyl algebra generated by $\{X,\partial_x\}$ satisfying $\partial_x\circ X=X\circ\partial_x+1$ (thus, $U\mapsto\partial_x$ and $V\mapsto X$). For variables U,V satisfying (2.11) expressions for $(U+V)^k$ were discussed in [13] (see also the many references to the literature for particular choices of parameters s,h given therein). In the case at hand we identify $X^{-1}\mapsto V$ and $X^{-1}\partial_x\mapsto U$ and find that $X^{-1}\partial_x\circ X^{-1}=X^{-1}\circ X^{-1}\partial_x-(X^{-1})^3$, or

$$UV=VU-V^3. \tag{2.12}$$

Thus, the algebra generated by $\{X^{-1},X^{-1}\partial_x\}$ (with the above commutation relation) yields an example for (2.11) with $s=3$ and $h=-1$. The identity (2.10) can then be interpreted as follows.

Theorem 2.3. *Let U, V be variables satisfying $UV = VU - V^3$. Then the following binomial formula holds true for any $k \in \mathbb{N}_0$*

$$(V+U)^k = \sum_{r=0}^{k} \binom{k}{r} V^{k-r} y_{k-r-1}(-V) U^r, \tag{2.13}$$

where y_ℓ are the Bessel polynomials introduced in (2.8).

In the beautiful formula (2.13) the factor $y_{k-r-1}(-V)$ arises due to the noncommutative nature of the variables and completely describes it (for commuting variables U, V this factor is absent in the binomial formula).

Example 2.4. Let us consider the first few instances of (2.13). For $k = 0$ the identity (2.13) reduces to the trivial statement $1 = 1$ (note that we have here used the convention $y_{-1}(x) = 1$). For $k = 1$ the left-hand side yields $V + U$, whereas the right-hand side yields explicitly $\binom{1}{0} V y_0(-V) U^0 + \binom{1}{1} V^0 y_{-1}(-V) U^1 = V + U$, where we have used the convention $y_{-1}(-V) = 1$ as well as the explicit expression $y_0(-V) = 1$, see (2.9). For $k = 2$ the right-hand side of (2.13) gives the explicit expression

$$\binom{2}{0} V^2 y_1(-V) U^0 + \binom{2}{1} V y_0(-V) U + \binom{2}{2} V^0 y_{-1}(-V) U^2 = V^2 + 2VU - V^3 + U^2,$$

where we have used the convention $y_{-1}(-V) = 1$ as well as the explicit expressions $y_0(-V) = 1$ and $y_1(-V) = 1 - V$, see (2.9). On the other hand, computing $(V+U)^2 = (V+U)(V+U) = V^2 + VU + UV + U^2$ and using the commutation relation $UV = VU - V^3$ leads to the same result, as it should. As a final example, we consider $k = 3$ where the right-hand side of (2.13) gives the explicit expression

$$\binom{3}{0} V^3 y_2(-V) U^0 + \binom{3}{1} V^2 y_1(-V) U + \binom{3}{2} V y_0(-V) U^2 + \binom{3}{3} V^0 y_{-1}(-V) U^3.$$

Using the convention $y_{-1}(-V) = 1$ as well as the explicit expressions given in (2.9), this equals

$$V^3(1 - 3V + 3V^2) + 3V^2(1 - V)U + 3VU^2 + U^3.$$

On the other hand, one may compute directly

$$(V+U)^3 = (V+U)(V+U)^2 = (V+U)(V^2 - V^3 + 2VU + U^2),$$

which gives – after using several times the commutation relation – the same result. Note that the highest power of V which appears is V^5. From (2.13) one reads off that in general the highest power of V which appears in the explicit expression of $(V+U)^k$ is V^{2k-1}.

Remark 2.5. It is interesting to compare the above result to the ones derived in [13]. For example, it was shown in [13, Proposition 5.2] that one has for variables U, V satisfying (2.11) that (we have switched the notation to the one used here)

$$(V+U)^k = \sum_{r=0}^{k} \left(\sum_{i=0}^{k-r-1} h^i d_k^{(s)}(r,i) V^{k-r+i(s-2)} \right) U^r, \tag{2.14}$$

where the coefficients $d_k^{(s)}(j,i)$ satisfy a particular recursion relation (and their generating function is stated in [13, Theorem 5.3]). This expression reduces for $s = 3$ and $h = -1$ to

$$(V+U)^k = \sum_{r=0}^{k} V^{k-r} \left(\sum_{i=0}^{k-r-1} (-V)^i d_k^{(3)}(r,i) \right) U^r,$$

which, upon comparison with (2.13), allows us to conclude that

$$\binom{k}{r} y_{k-r-1}(-V) = \sum_{i=0}^{k-r-1} (-V)^i d_k^{(3)}(r,i).$$

Using the explicit expression for $y_n(x)$ given in (2.8), one finds

$$d_k^{(3)}(r,i) = \binom{k}{r} \frac{(k-r+i-1)!}{2^i i!(k-r-i-1)!}.$$

Before we close this section, we would like to point out how the above methods can be used to derive a summation formula for Bessel functions in a slightly alternative fashion to [7]. Let us denote for $l \in \mathbb{N}$ by $J_l(x)$ the *cylindrical Bessel function* which satisfy the identity $\left(\frac{1}{x}\frac{d}{dx}\right)^k \{x^{-l} J_l(x)\} = \frac{(-1)^k}{x^{k+l}} J_{l+k}(x)$, see [19, Section 17–211]. If we consider $l = 0$ as well as $k = n$, then multiplication with $\frac{\tau^n}{n!}$ on both sides and summing over n yields

$$\sum_{n \geqslant 0} \frac{\tau^n}{n!} \left(\frac{1}{x} \frac{d}{dx} \right)^n J_0(x) = \sum_{n \geqslant 0} \frac{(-1)^n}{x^n} \frac{\tau^n}{n!} J_n(x),$$

which is Eq. (8) of [7]. Letting $\tau = -tx$, one obtains

$$e^{\tau\left(\frac{1}{x}\frac{d}{dx}\right)}J_0(x) = \sum_{n\geqslant 0}\frac{t^n}{n!}J_n(x).$$

Using (2.5), the left-hand side equals $J_0(\sqrt{x^2+\tau})$, giving finally the identity [7, Eq. (16)].

$$J_0\left(\sqrt{x^2-2xt}\right) = \sum_{n\geqslant 0}\frac{t^n}{n!}J_n(x).$$

3. The Touchard polynomials of arbitrary negative integer order

In the previous section we considered the Touchard polynomials of order -1. In this section we transfer the above treatment to the Touchard polynomials of arbitrary negative (integer) order $-m$. Introducing the operator $M_m := (x^{-m} + x^{-m}\partial_x)$, we can write as above

$$\sum_{k\geqslant 0}\frac{t^k}{k!}T_{n+k}^{(-m)}(x) = e^{tM_m}T_n^{(-m)}(x) \tag{3.1}$$

and have to study the operator

$$e^{tM_m} = e^{t(x^{-m}+x^{-m}\partial_x)}. \tag{3.2}$$

Identifying $A \equiv tx^{-m}$ and $B \equiv tx^{-m}\partial_x$, one obtains the commutation relation

$$[A,B] = mt^{-\frac{1}{m}}A^{\frac{2m+1}{m}}. \tag{3.3}$$

Thus, we can use the disentanglement identity (2.3) (with parameters $mt^{-\frac{1}{m}}$ and $\frac{2m+1}{m}$) to find after some simplifications

$$e^{A+B} = \exp\left[(tA^{-1})^{\frac{1}{m}}\left\{\left(1+(m+1)t^{-\frac{1}{m}}A^{\frac{m+1}{m}}\right)^{\frac{1}{m+1}} - 1\right\}\right]e^B.$$

This can be expressed in terms of the original operators as

$$e^{t(x^{-m}+x^{-m}\partial_x)} = e^{\sqrt[m+1]{x^{m+1}+(m+1)t}-x}e^{tx^{-m}\partial_x}. \tag{3.4}$$

Inserting this into (3.1), we obtain

$$\sum_{k\geqslant 0}\frac{t^k}{k!}T_{n+k}^{(-m)}(x) = e^{\sqrt[m+1]{x^{m+1}+(m+1)t}-x}e^{tx^{-m}\partial_x}T_n^{(-m)}(x).$$

Recalling from [15, Eq. (6.27)] that

$$e^{\lambda x^{-m}\partial_x}f(x) = f\left(\sqrt[m+1]{x^{m+1}+(m+1)\lambda}\right),$$

we have shown the following generalization of Proposition 2.1 to arbitrary order $-m$.

Proposition 3.1. *The Touchard polynomials of order $-m$ with $m \in \mathbb{N}$ satisfy the relation*

$$\sum_{k\geqslant 0}\frac{t^k}{k!}T_{n+k}^{(-m)}(x) = e^{\sqrt[m+1]{x^{m+1}+(m+1)t}-x}T_n^{(-m)}\left(\sqrt[m+1]{x^{m+1}+(m+1)t}\right). \tag{3.5}$$

According to [15, Theorem 6.14], $e^{\sqrt[m+1]{x^{m+1}+(m+1)t}-x}$ is the exponential generating function of the $T_n^{(-m)}(x)$. As in the case $m = 1$ one can use this to derive from the disentanglement identity (3.4) the following generalization of Corollary 2.2 to arbitrary order $-m$.

Corollary 3.2. *One has for $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$ the operational relation*

$$(x^{-m}+x^{-m}\partial_x)^k = \sum_{r=0}^{k}\binom{k}{r}T_{k-r}^{(-m)}(x)(x^{-m}\partial_x)^r. \tag{3.6}$$

Similar to the case $m = 1$, we can interpret this identity in terms of noncommuting variables U, V. In the case at hand we identify $X^{-m} \mapsto V$ as well as $X^{-m}\partial_x \mapsto U$ and obtain from (3.3) the commutation relation

$$UV = VU - mV^{\frac{2m+1}{m}}, \tag{3.7}$$

which reduces to (2.12) for $m=1$. Thus, the case of order $-m$ corresponds to the choice of parameters $h=-m$ and $s=\frac{2m+1}{m}$ in the relation (2.11). The identity of Corollary 3.2 can be written in terms of the variables U,V as

$$(V+U)^k=\sum_{r=0}^{k}\binom{k}{r}T_{k-r}^{(-m)}(V^{-\frac{1}{m}})U^r. \tag{3.8}$$

In the case $m=1$ we could use an expression of the Touchard polynomials of order -1 in terms of Bessel polynomials to derive a nice binomial formula. In the case of arbitrary order $-m$ such an explicit expression is not known. However, using generalized Stirling numbers (or Bell polynomials) we can write (3.8) in another fashion. For this let us recall that generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k)$ were introduced in [13–15] as *normal ordering coefficients* of $(VU)^n$ in the variables U,V satisfying (2.11), i.e., one has

$$(VU)^n=\sum_{k=0}^{n}\mathfrak{S}_{s;h}(n,k)V^{s(n-k)+k}U^k. \tag{3.9}$$

The special case $s=0$ and $h=1$ corresponds to variables U,V satisfying $UV=VU+1$ (i.e., the Weyl algebra) and it follows that $\mathfrak{S}_{0;1}(n,k)=S(n,k)$, the conventional Stirling numbers of the second kind. The corresponding generalized Bell polynomials are defined as usual, i.e.,

$$\mathfrak{B}_{s;h|n}(x):=\sum_{k=0}^{n}\mathfrak{S}_{s;h}(n,k)x^k.$$

It was shown in [15, Theorem 6.8] that one has the relation

$$T_n^{(-m)}(x)=x^{-(m+1)n}\mathfrak{B}_{\frac{m+1}{m};-m|n}(x). \tag{3.10}$$

Therefore, we can write (3.8) as

$$(V+U)^k=\sum_{r=0}^{k}\binom{k}{r}V^{\frac{(m+1)(k-r)}{m}}\mathfrak{B}_{\frac{m+1}{m};-m|k-r}(V^{-\frac{1}{m}})U^r.$$

This shows the following theorem.

Theorem 3.3. *Let U,V be variables satisfying the commutation relation $UV=VU-mV^{\frac{2m+1}{m}}$ with $m\in\mathbb{N}$. Then the following noncommutative binomial theorem holds true for any $k\in\mathbb{N}_0$*

$$(V+U)^k=\sum_{r=0}^{k}\binom{k}{r}V^{k-r}\mathcal{B}_{k-r}^{(m)}\left(\sqrt[m]{V}\right)U^r, \tag{3.11}$$

where the polynomial $\mathcal{B}_\ell^{(m)}(x)$ is given explicitly by

$$\mathcal{B}_\ell^{(m)}(x)=x^\ell\mathfrak{B}_{\frac{m+1}{m};-m|\ell}(x^{-1})=\sum_{s=0}^{\ell}\mathfrak{S}_{\frac{m+1}{m};-m}(\ell,s)x^{\ell-s}. \tag{3.12}$$

Theorem 3.3 is the generalization of Theorem 2.3 to arbitrary negative order $-m$ with $m\in\mathbb{N}$. The polynomials $\mathcal{B}_\ell^{(m)}(x)$ - which measure the influence of the noncommutativity of the variables U,V - seem to be rather poorly understood, in contrast to the special case $m=1$ where Bessel polynomials appear.

Remark 3.4. The highest power of V which appears in the normal ordering of $(U+V)^k$ is $V^{\frac{(m+1)k-1}{m}}$ (which reduces for $m=1$ to V^{2k-1}). To see this, it is clear that in the sum of the right-hand side of (3.11) only the summand $r=0$ has to be considered, i.e., $V^k\mathcal{B}_k^{(m)}\left(\sqrt[m]{V}\right)$. From the explicit expression given in (3.12) and the fact that $\mathfrak{S}_{\frac{m+1}{m};-m}(\ell,k)=\delta_{\ell,k}$ we find $\mathcal{B}_k^{(m)}\left(\sqrt[m]{V}\right)\sim\left(\sqrt[m]{V}\right)^{k-1}$, showing the assertion. This also fits the general formula given in (2.14). Note that on the right-hand side of (2.14) the highest power of V which appears is given in the case $s>2$ by $V^{k+(k-1)(s-2)}$. In the case at hand we have $s=\frac{2m+1}{m}>2$ and it follows that $V^{k+(k-1)(s-2)}=V^{\frac{(m+1)k-1}{m}}$.

The same arguments as above can be used for the generalized Touchard polynomials $T_n^{(m)}(x)$ with $m\in\mathbb{N}$. Here we have [6, Eq. (39)]

$$(x^m+x^m\partial_x)=\sum_{r=0}^{k}\binom{k}{r}T_{k-r}^{(m)}(x)(x^m\partial_x)^r.$$

As above, $x^m\mapsto V$ and $x^m\partial_x\mapsto U$ corresponds to variables U,V satisfying $UV=VU+mV^{\frac{2m-1}{m}}$. For these variables one has thus $(V+U)^k=\sum_{r=0}^{k}\binom{k}{r}T_{k-r}^{(m)}(V^{\frac{1}{m}})U^r$. Using (4.2), this equals

$$(V+U)^k = \sum_{r=0}^{k} \binom{k}{r} V^{k-r} \left(\sqrt[m]{V}\right)^{-(k-r)} \mathfrak{B}_{\frac{m-1}{m};m|k-r}\left(\sqrt[m]{V}\right) U^r.$$

Now, we can formulate the analog to Theorem 3.3.

Theorem 3.5. *Let U, V be variables satisfying the commutation relation $UV = VU + mV^{\frac{2m-1}{m}}$ with $m \in \mathbb{N}$. Then the following noncommutative binomial theorem holds true for any $k \in \mathbb{N}_0$*

$$(V+U)^k = \sum_{r=0}^{k} \binom{k}{r} V^{k-r} \mathcal{C}_{k-r}^{(m)}\left(\sqrt[m]{V}\right) U^r, \tag{3.13}$$

where $\mathcal{C}_\ell^{(m)}(x)$ is given explicitly by

$$\mathcal{C}_\ell^{(m)}(x) = x^{-\ell} \mathfrak{B}_{\frac{m-1}{m};m|\ell}(x) = \sum_{s=0}^{\ell} \mathfrak{S}_{\frac{m-1}{m};m}(\ell, s) x^{s-\ell}. \tag{3.14}$$

Remark 3.6. The function $\mathcal{C}_\ell^{(m)}(x)$ of Theorem 3.5 results from the polynomial $\mathcal{B}_\ell^{(m)}(x)$ appearing in Theorem 3.3 by switching formally from m to $-m$ as well as from x to x^{-1}, i.e.,

$$\mathcal{C}_\ell^{(m)}(x) = \mathcal{B}_\ell^{(-m)}(x^{-1}).$$

4. A recursion relation of the generalized Touchard polynomials

For the conventional Touchard polynomials one can easily derive a recursion relation due to the identification (1.2) with the Bell polynomials. Using that the exponential generating function of the Bell polynomials is given by $e^{x(e^z-1)}$, we obtain

$$\sum_{n \geqslant 0} T_n(x) \frac{z^n}{n!} = e^{x(e^z-1)}.$$

Taking a derivative with respect to z, this shows

$$\sum_{n \geqslant 0} T_{n+1}(x) \frac{z^n}{n!} = xe^z e^{x(e^z-1)} = x\left(\sum_{l \geqslant 0} \frac{z^l}{l!}\right) \cdot \left(\sum_{m \geqslant 0} T_m(x) \frac{z^m}{m!}\right) = x \sum_{n \geqslant 0} \left\{\sum_{k=0}^{n} \binom{n}{k} T_k(x)\right\} \frac{z^n}{n!}.$$

Comparing coefficients gives the recursion relation

$$T_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} T_k(x), \tag{4.1}$$

which can be found in exactly this form in [2, Eq. (2.14)]. Now, we want to derive an analogous recursion relation for the generalized Touchard polynomials. For this we consider first the case $T_n^{(m)}(x)$ with $m \in \mathbb{N}$. Here we have the following relation to the generalized Bell polynomials [15, Theorem 6.1]

$$T_n^{(m)}(x) = x^{(m-1)n} \mathfrak{B}_{\frac{m-1}{m};m|n}(x), \tag{4.2}$$

and the main task is to establish a recursion relation for the generalized Bell polynomials. In [14, Corollary 4.1] the exponential generating function of the generalized Bell polynomials $\mathfrak{B}_{s;h|n}(x)$ is given in a slightly implicit form. There one has to consider the cases $s = 0, s = 1$ and $s \in \mathbb{R} \setminus \{0, 1\}$ separately. In the case we are interested in we have $s(m) = \frac{m-1}{m}$, so that $s(1) = 0, s(2) = \frac{1}{2}$ and $s(m) \in (0, 1)$ for $m \in \{2, 3, \ldots\}$. Since the case $s(m) = 0$ (i.e., $m = 1$) is the conventional case, we restrict to the case $m \geqslant 2$. Then we can use [14, Corollary 4.1], giving the exponential generating function of the generalized Bell polynomials

$$\sum_{n \geqslant 0} \mathfrak{B}_{s;h|n}(x) \frac{z^n}{n!} = e^{\frac{x}{h(s-1)}\left\{1-(1-hsz)^{\frac{s-1}{s}}\right\}}. \tag{4.3}$$

Following the strategy from above, we take a derivative with respect to z and obtain

$$\sum_{n \geqslant 0} \mathfrak{B}_{s;h|n+1}(x) \frac{z^n}{n!} = x(1-hsz)^{-\frac{1}{s}} e^{\frac{x}{h(s-1)}\left\{1-(1-hsz)^{\frac{s-1}{s}}\right\}} = x(1-hsz)^{-\frac{1}{s}} \sum_{m \geqslant 0} \mathfrak{B}_{s;h|m}(x) \frac{z^m}{m!}.$$

Using

$$(1-hsz)^{-\frac{1}{s}} = \sum_{r \geqslant 0} \binom{r + \frac{1}{s} - 1}{r} (hs)^r z^r = \sum_{r \geqslant 0} \frac{\Gamma(r + \frac{1}{s})}{\Gamma(\frac{1}{s})} (hs)^r \frac{z^r}{r!},$$

one finds

$$\sum_{n\geqslant 0}\mathfrak{B}_{s;h|n+1}(x)\frac{z^n}{n!}=x\left(\sum_{r\geqslant 0}\frac{\Gamma\left(r+\frac{1}{s}\right)}{\Gamma\left(\frac{1}{s}\right)}(hs)^r\frac{z^r}{r!}\right)\cdot\left(\sum_{m\geqslant 0}\mathfrak{B}_{s;h|m}(x)\frac{z^m}{m!}\right).$$

Comparing coefficients shows the following theorem.

Theorem 4.1. *Let $h\neq 0$ and $s\in\mathbb{R}\setminus\{0,1\}$. The generalized Bell polynomials satisfy the recursion relation*

$$\mathfrak{B}_{s;h|n+1}(x)=x\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma\left(n-k+\frac{1}{s}\right)}{\Gamma\left(\frac{1}{s}\right)}(hs)^{n-k}\mathfrak{B}_{s;h|k}(x).$$

To obtain the recursion relation for the generalized Touchard polynomials, it remains to combine (4.2) and Theorem 4.1:

$$\begin{aligned}T_{n+1}^{(m)}(x)&=x^{(m-1)(n+1)}\mathfrak{B}_{\frac{m-1}{m};m|n+1}(x)=x^{m-1}x^{(m-1)n}x\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n-k+\frac{m}{m-1})}{\Gamma(\frac{m}{m-1})}(m-1)^{n-k}\mathfrak{B}_{\frac{m-1}{m};m|k}(x)\\&=x^m\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n-k+\frac{m}{m-1})}{\Gamma(\frac{m}{m-1})}[(m-1)x^{(m-1)}]^{n-k}T_k^{(m)}(x).\end{aligned}$$

The last equation is the sought-for recursion relation in the case $m\geqslant 2$. Let us turn to the generalized Touchard polynomials of negative order, i.e., to $T_n^{(-m)}(x)$ with $m\in\mathbb{N}$. Using (3.10), we see that the parameter $s(-m)$ of the corresponding generalized Bell polynomial is given by $s(-m)=\frac{m+1}{m}\in(1,2]$ so that one can use Theorem 4.1 as in the case of positive order. A calculation similar to the one above shows that

$$T_{n+1}^{(-m)}(x)=x^{-m}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma\left(n-k+\frac{m}{m+1}\right)}{\Gamma\left(\frac{m}{m+1}\right)}[-(m+1)x^{-(m+1)}]^{n-k}T_k^{(-m)}(x).$$

Note that this recursion relation is the same as the one which results from the one given for $T_n^{(m)}(x)$ above by switching formally from m to $-m$! Thus, we can combine these two relations into one relation for arbitrary integer order.

Theorem 4.2. *Let $r\in\mathbb{Z}\setminus\{0,1\}$. The generalized Touchard polynomials of order r satisfy the recursion relation*

$$T_{n+1}^{(r)}(x)=x^r\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n-k+\frac{r}{r-1})}{\Gamma(\frac{r}{r-1})}[(r-1)x^{(r-1)}]^{n-k}T_k^{(r)}(x).\tag{4.4}$$

Remark 4.3. Let us discuss briefly the two cases $r=0,1$ excluded in Theorem 4.2. In the trivial case $r=0$ one has $T_n^{(0)}(x)=1$ for all n, implying $T_{n+1}^{(0)}(x)=T_n^{(0)}(x)$. The case $r=1$ corresponds to the conventional case with recursion relation (4.1). Although we have excluded this case in the derivation of Theorem 4.2, we can nevertheless see what happens in (4.4) when "$r\to 1$". Since there are terms containing $\frac{1}{r-1}$, we cannot insert $r=1$ directly. However, if we define for $r\in\mathbb{R}$ with $r>1$

$$A_{n,k}(r,x):=\frac{\Gamma\left(n-k+\frac{r}{r-1}\right)}{\Gamma\left(\frac{r}{r-1}\right)}[(r-1)x^{(r-1)}]^{n-k},$$

then we can write (4.4) as

$$T_{n+1}^{(r)}(x)=x^r\sum_{k=0}^{n}\binom{n}{k}A_{n,k}(r,x)T_k^{(r)}(x).$$

Since one has for fixed n,k,x that

$$\lim_{r\to 1}A_{n,k}(r,x)=1,$$

this shows that – in a certain sense – (4.4) reduces to (4.1) for $r\to 1$.

Before closing this section, let us point out that one can switch in the definition (1.1) to another variable as follows. If we let $x(z)=e^z$, then one finds due to the transformation $x\frac{d}{dx}\rightsquigarrow x(z)\left(\frac{dx(z)}{dz}\right)^{-1}\frac{d}{dz}$ that the operator $x\frac{d}{dx}$ becomes the operator $\frac{d}{dz}$. Thus, one can write (1.1) equivalently as the well-known Rodriguez-like formula

$$T_n(e^z)=e^{-e^z}\left(\frac{d}{dz}\right)^n e^{e^z}.\tag{4.5}$$

For $T_n^{(r)}(x)$ with arbitrary $r\in\mathbb{Z}$ one can do the same thing (a closely related argument was used in [6]). Here we want to find a function $x_r(z)$ such that $x^r\frac{d}{dx}$ transforms into $\frac{d}{dz}$. In general, one obtains by the change of variable $x\rightsquigarrow x_r(z)$ for $x^r\frac{d}{dx}$ the operator

$\{x_r(z)\}^r\left(\frac{dx_r(z)}{dz}\right)^{-1}\frac{d}{dz}$ so that we have to solve the differential equation $\frac{dx_r(z)}{dz}=\{x_r(z)\}^r$. For $r=1$ we obtain the solution $x_1(z)=e^z$ from above and are done. For $r\in\mathbb{Z}\setminus\{1\}$ we obtain the solution $x_r(z)=\{(1-r)z\}^{-\frac{1}{r-1}}=\frac{1}{\sqrt[r-1]{(1-r)z}}$ and obtain as analog to (4.5) the following proposition.

Proposition 4.4. *The generalized Touchard polynomials $T_n^{(r)}$ satisfy for $r\in\mathbb{Z}\setminus\{1\}$ the Rodriguez-like formula*

$$T_n^{(r)}\left(\frac{1}{\sqrt[r-1]{(1-r)z}}\right)=e^{-\frac{1}{\sqrt[r-1]{(1-r)z}}}\left(\frac{d}{dz}\right)^n e^{\frac{1}{\sqrt[r-1]{(1-r)z}}}. \tag{4.6}$$

Example 4.5. Let us consider $r=2$. Here we get $x_2(z)=-\frac{1}{z}$ and, therefore, the nice formula

$$T_n^{(2)}\left(-\frac{1}{z}\right)=e^{\frac{1}{z}}\left(\frac{d}{dz}\right)^n e^{-\frac{1}{z}}.$$

As another example we choose $r=-1$. Here we get $x_{-1}(z)=\sqrt{2z}$ and, consequently,

$$T_n^{(-1)}\left(\sqrt{2z}\right)=e^{-\sqrt{2z}}\left(\frac{d}{dz}\right)^n e^{\sqrt{2z}}.$$

5. The Touchard functions of arbitrary real order

Recall that we defined for any $m\in\mathbb{Z}\setminus\{0\}$ the generalized Touchard polynomials of order m by

$$T_n^{(m)}(x)=e^{-x}(x^m\partial_x)^n e^x,$$

(which for $m<0$ are polynomials in x^{-1}). It is then tempting to introduce for any $\alpha\in\mathbb{R}\setminus\{0\}$ *generalized Touchard functions of order* α by exactly the same formula, i.e.,

$$T_n^{(\alpha)}(x):=e^{-x}(x^\alpha\partial_x)^n e^x. \tag{5.1}$$

As above there exist two natural pairs of associated noncommutative variables. On the one hand one can consider $V_1=X^\alpha$ and $U_1=\partial_x$, implying the commutation relation $U_1V_1=V_1U_1+\alpha V_1^{\frac{\alpha-1}{\alpha}}$. Thus, this gives an example of (2.11) with $s=\frac{\alpha-1}{\alpha}$ and $h=\alpha$. It follows from (3.9) that

$$(V_1U_1)^n=\sum_{k=0}^n \mathfrak{S}_{\frac{\alpha-1}{\alpha};\alpha}(n,k)V_1^{\frac{\alpha-1}{\alpha}(n-k)+k}U_1^k,$$

implying

$$T_n^{(\alpha)}(x)=x^{(\alpha-1)n}\sum_{k=0}^n \mathfrak{S}_{\frac{\alpha-1}{\alpha};\alpha}(n,k)x^k=x^{(\alpha-1)n}\mathfrak{B}_{\frac{\alpha-1}{\alpha};\alpha|n}(x). \tag{5.2}$$

On the other hand, one can consider the pair $U_2=X^\alpha\partial_x$ and $V_2=X^\alpha$ with the commutation relation $U_2V_2=V_2U_2+\alpha V_2^{\frac{2\alpha-1}{\alpha}}$. Thus, if we want to consider $(x^\alpha+x^\alpha\partial_x)^k$ in analogy to above, we are led to the pair $\{V_2,U_2\}$ of noncommuting variables which is a particular case of (2.11) with $(s_\alpha,h_\alpha)=\left(\frac{2\alpha-1}{\alpha},\alpha\right)$.

Example 5.1. For the choice $\alpha=\frac{1}{2}$ one finds $(s_{\frac{1}{2}},h_{\frac{1}{2}})=\left(0,\frac{1}{2}\right)$, i.e., the variables V_2 and U_2 satisfy the (scaled) Weyl algebra $U_2V_2=V_2U_2+\frac{1}{2}$. In this case the explicit formula for $(V_2+U_2)^k$ has been established already long ago, see the remarks and literature given in [13]. To be more concrete, consider the operators $\mathcal{X}$ and $\mathfrak{D}_h$ (with $h\in\mathbb{R}$) satisfying the commutation relation $\mathfrak{D}_h\mathcal{X}=\mathcal{X}\mathfrak{D}_h+h$. Then one can write

$$(\mathcal{X}+\mathfrak{D}_h)^k=\sum_{r=0}^k\binom{k}{r}H_{k-r}(\mathcal{X},h)\mathfrak{D}_h^r, \tag{5.3}$$

where the polynomials $H_m(x,h)$ are a variant of the Hermite polynomials, see the discussion in [13]. In an equivalent form this identity was already known to Burchnall [3]. In [9] a variant of the Burchnall identity was discussed in the context of generalized shift operators $\hat{E}=e^{\lambda q(x)(d/dx)}=e^{\lambda\hat{\mathcal{T}}_x}$ with $\hat{\mathcal{T}}_x=q(x)\frac{d}{dx}$. Defining $F_q(x)=\int^x\frac{d\zeta}{q(\zeta)}$, one obtains the identity

$$(F_q(x)+2y\hat{\mathcal{T}}_x)^k=\sum_{r=0}^k\binom{k}{r}(2y)^r h_{k-r}^{(2)}(x,y)(\hat{\mathcal{T}}_x)^r,$$

where the functions $h_m^{(2)}(x,y)$ are called pseudo-Hermite–Kampé de Feriet polynomials [9]. Here one has the commutation relation $[2y\hat{\mathcal{T}}_x,F_q(x)]=y$, i.e., $F_q(x)$ and $2y\hat{\mathcal{T}}_x$ satisfy a (scaled) Weyl algebra. Choosing $y=1$ and $q(x)=1$, i.e., $F_q(x)=x$ and $\hat{\mathcal{T}}_x=\frac{d}{dx}$, one obtains the conventional Burchnall identity.

Comparing the identity mentioned in the preceding example with (3.8), one expects that the Touchard functions of order $\alpha = \frac{1}{2}$ are given by Hermite polynomials. This is what we will show in Theorem 5.3 after having established a necessary Lemma.

Lemma 5.2. *Let $h \in \mathbb{C} \setminus \{0\}$. The generalized Bell polynomials satisfy*

$$\mathfrak{B}_{s;h|n}(x) = h^n \mathfrak{B}_{s;1|n}\left(\frac{x}{h}\right).$$

Proof. Using the relation $\mathfrak{S}_{s;h}(n,k) = h^{n-k}\mathfrak{S}_{s;1}(n,k)$ of the generalized Stirling numbers [14], the assertion follows directly from the definition of the generalized Bell polynomials since

$$\mathfrak{B}_{s;h|n}(x) = \sum_{k=0}^{n} \mathfrak{S}_{s;h}(n,k)x^k = h^n \sum_{k=0}^{n} \mathfrak{S}_{s;1}(n,k)\left(\frac{x}{h}\right)^k = h^n \mathfrak{B}_{s;1|n}\left(\frac{x}{h}\right),$$

as requested. $\square$

Recall that the *Hermite polynomials* $H_n(x)$ can be defined by their exponential generating function [5, p. 50],

$$e^{2tz-t^2} = \sum_{n \geqslant 0} H_n(z)\frac{t^n}{n!}. \tag{5.4}$$

Theorem 5.3. *The Touchard functions of order $\frac{1}{2}$ can be expressed by Hermite polynomials, i.e.,*

$$T_n^{(\frac{1}{2})}(x) = \left(\frac{i}{2}\right)^n H_n(-i\sqrt{x}). \tag{5.5}$$

Proof. From (5.2) we immediately obtain

$$T_n^{(\frac{1}{2})}(x) = x^{-\frac{n}{2}}\mathfrak{B}_{-1;\frac{1}{2}|n}(x) = \left(\frac{1}{2\sqrt{x}}\right)^n \mathfrak{B}_{-1;1|n}(2x),$$

where we have used Lemma 5.2 in the second equation. In [15, Proposition 6.13] it was shown that

$$\mathfrak{B}_{-1;1|n}(y) = \left(\frac{i\sqrt{y}}{\sqrt{2}}\right)^n H_n\left(\frac{\sqrt{y}}{i\sqrt{2}}\right).$$

Inserting this into the above equation yields the assertion. $\square$

Remark 5.4. The connection between $T_n^{(\frac{1}{2})}(x)$ and $H_n(x)$ can also be seen as follows. Using that the formula given in Proposition 4.4 also holds for $r \in \mathbb{R} \setminus \{1\}$, we can choose $r = \frac{1}{2}$ to find $x_{1/2}(z) = \left(\frac{z}{2}\right)^2$ and, therefore,

$$T_n^{(\frac{1}{2})}\left(\left(\frac{z}{2}\right)^2\right) = e^{-\left(\frac{z}{2}\right)^2}\left(\frac{d}{dz}\right)^n e^{\left(\frac{z}{2}\right)^2}.$$

Recalling that the classical Rodriguez formula for $H_n(x)$ is given by $H_n(x) = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}$ [17, p. 45], this shows the connection in an alternative way.

Above we have seen that the variables U_2, V_2 corresponding to α satisfy (2.11) with $s_\alpha = \frac{2\alpha-1}{\alpha}$ (and $h_\alpha = \alpha$). Let us check when $s_\alpha \in \mathbb{Z} \setminus \{2\}$. If $s_{\alpha_n} = n$ then $\alpha_n = \frac{1}{2-n}$. Thus, $\alpha_0 = \frac{1}{2}, \alpha_1 = 1, \alpha_3 = -1, \alpha_4 = -\frac{1}{2}, \alpha_5 = -\frac{1}{3}, \ldots,$ whereas $\alpha_{-1} = \frac{1}{3}, \alpha_{-2} = \frac{1}{4}, \alpha_{-3} = \frac{1}{5}, \ldots$ For example, if $\alpha = -1$ then $\{x^{-1}, x^{-1}\frac{d}{dx}\}$ corresponds to variables $\{V_2, U_2\}$ satisfying $U_2V_2 = V_2U_2 - V_2^3$ as discussed in Section 2. If $\alpha = -1/3$ then $\{x^{-1/3}, x^{-1/3}\frac{d}{dx}\}$ corresponds to variables $\{V_2, U_2\}$ satisfying $U_2V_2 = V_2U_2 - \frac{1}{3}V_2^5$. To derive a binomial formula for these variables in analogy to above one needs information about, e.g., the exponential generating function of $T_n^{(\alpha)}(x)$. However, since we cannot cite the relevant properties (as in the case of order m with $m \in \mathbb{Z}$) we refrain from a closer study of these functions and turn instead to another generalization in the next section.

Remark 5.5. Let us point out that in [1] certain analogs of the Touchard (or exponential) polynomials are considered. Recalling the definition $T_n(x) = e^{-x}\left(x\frac{d}{dx}\right)^n e^x = \sum_{k=0}^{n} S(n,k)x^k$, one can consider instead of $e^x = \sum_{k \geqslant 0}\frac{x^k}{k!}$ the function $\frac{1}{1-x} = \sum_{k \geqslant 0} x^k$ and define in analogy to $T_n(x)$ the functions

$$U_n(x) := (1-x)\left(x\frac{d}{dx}\right)^n\left\{\frac{1}{1-x}\right\}.$$

Introducing the geometric polynomials by $\omega_n(x) = \sum_{k=0}^{n} S(n,k)k!x^k$ [1, Eq. (3.3)], one can show that $\left(x\frac{d}{dx}\right)^n\left\{\frac{1}{1-x}\right\} = \frac{1}{1-x}\omega_n\left(\frac{x}{1-x}\right)$ [1, Eq. (3.8)]. Thus,

$$U_n(x) = \omega_n\left(\frac{x}{1-x}\right).$$

If we denote by $A_n(x)$ the *Eulerian polynomials* [5], one has $A_n(x) = (1-x)^n\omega_n\left(\frac{x}{1-x}\right)$ [1, Eq. (3.18)], implying $U_n(x) = (1-x)^{-n}A_n(x)$. Following the idea of the present section, one should define for $\alpha \in \mathbb{R}$ the functions $U_n^{(\alpha)}(x) := (1-x)\left(x^\alpha\frac{d}{dx}\right)^n\left\{\frac{1}{1-x}\right\}$ and consider their properties.

6. Outlook: a further generalization of Touchard functions

In this section we want to sketch a possible further generalization of the Touchard functions considered above. Recall that we defined Touchard functions $T_n^{(\alpha)}(x)$ for arbitrary $\alpha \in \mathbb{R}$ by $T_n^{(\alpha)}(x) = e^{-x}\left(x^\alpha\frac{d}{dx}\right)^n e^x$. The operator $x^\alpha\frac{d}{dx}$ can be interpreted in an algebraic fashion as derivation (and in a more geometric fashion as a vector field). Its exponential thus represents an automorphism (and is also called *generalized shift operator* [9] or *exponential operator* [8]). A general derivation can be written in the form

$$\hat{\mathcal{T}}_x \equiv g(x)\frac{d}{dx},$$

where the function g is assumed to be "sufficiently smooth" (e.g., analytic in an open interval). Using such a general derivation instead of $x^\alpha\frac{d}{dx}$, one is led to the following definition.

Definition 6.1 (*Comtet–Touchard function associated to g*). *Let g be a smooth function. The Comtet–Touchard functions* $T_n^{(g)}(x)$ *associated to g are defined for* $n \in \mathbb{N}$ *by*

$$T_n^{(g)}(x) := e^{-x}\left(g(x)\frac{d}{dx}\right)^n e^x. \tag{6.1}$$

Clearly, for $g(x) = x^\alpha$ this definition gives back the Touchard functions considered in previous sections. The reason for calling these functions Comtet–Touchard functions is that L. Comtet considered in 1973 [4] expressions of the form $(g(x)\frac{d}{dx})^n$ in detail and obtained in particular the following theorem.

Theorem 6.2 (*Comtet*). *Let g be a smooth function. Then one has for any* $n \in \mathbb{N}$ *the expansion*

$$\left(g(x)\frac{d}{dx}\right)^n = \sum_{l=1}^{n} T_{n,l}^{(g)}(x)\left(\frac{d}{dx}\right)^l,$$

where for $1 \leqslant l \leqslant n$

$$T_{n,l}^{(g)}(x) = \sum_{\substack{k_1+\cdots+k_{n-1}=n-l\,(k_i\geqslant 0)\\ k_1+\cdots+k_i\leqslant i\,(1\leqslant i<n)}} \frac{g(x)}{l!}\prod_{j=1}^{n-1}(j+1-k_1-\cdots-k_j)\frac{g_{k_j}(x)}{k_j!},$$

where $g_{k_j}(x)$ *is the* k_j*-th derivative function of* $g(x)$.

The numbers appearing as coefficients in the above expressions for $T_{n,l}^{(g)}(x)$ are related to the number of rooted trees and can be found as A139605 in OEIS [18]. Let us give the cases $n = 2, 3$ explicitly where we write briefly $g = g(x)$:

$$\left(g\frac{d}{dx}\right)^2 = gg_1\left(\frac{d}{dx}\right) + g^2\left(\frac{d}{dx}\right)^2,$$
$$\left(g\frac{d}{dx}\right)^3 = (gg_1^2 + g^2g_2)\left(\frac{d}{dx}\right) + 3g^2g_1\left(\frac{d}{dx}\right)^2 + g^3\left(\frac{d}{dx}\right)^3.$$

Thus, one has for $n = 1$ trivially $T_{1,1}^{(g)} = g$, for $n = 2$ that $T_{2,l}^{(g)} = gg_1, T_{2,2}^{(g)} = g^2$ and for $n = 3$ that $T_{3,1}^{(g)} = gg_1^2 + g^2g_2, T_{3,2}^{(g)} = 3g^2g_1, T_{3,3}^{(g)} = g^3$. It is easy to see that one has in general $T_{n,n}^{(g)} = g^n$.

Proposition 6.3. *The Comtet–Touchard functions associated to g are given for any* $n \in \mathbb{N}$ *by*

$$T_n^{(g)}(x) = \sum_{l=1}^{n} T_{n,l}^{(g)}(x), \tag{6.2}$$

where $T_{n,l}^{(g)}(x)$ *are given by Theorem 6.2.*

Proof. Inserting the expansion given in Theorem 6.2 into (6.1) shows the assertion. $\square$

Let us give the first few Comtet–Touchard functions explicitly where we denote the derivative with respect to x by a prime:

$$T_1^{(g)}(x) = g(x),$$
$$T_2^{(g)}(x) = g(x)\{g'(x) + g(x)\},$$
$$T_3^{(g)}(x) = g(x)\{(g'(x))^2 + g(x)g''(x) + 3g(x)g'(x) + g(x)^2\}.$$

If we follow the procedure of the previous sections, we should consider the exponential generating function of the Comtet–Touchard functions. Using the operational method, one obtains from the definition

$$\sum_{n\geqslant 0}\frac{t^n}{n!}T_n^{(g)}(x) = \sum_{n\geqslant 0}\frac{t^n}{n!}e^{-x}\left(g(x)\frac{d}{dx}\right)^n e^x = e^{-x}\left(\sum_{n\geqslant 0}\frac{t^n}{n!}\left(g(x)\frac{d}{dx}\right)^n\right)e^x = e^{-x}e^{tg(x)\frac{d}{dx}}e^x.$$

Following [8,9] (see also [11]), we introduce

$$F_g(x) := \int^x \frac{d\zeta}{g(\zeta)},$$

and denote by F_g^{-1} its inverse. Then one can write [8, Eq. (I.2.7)] for the associated exponential operator

$$e^{tg(x)\frac{d}{dx}}f(x) = f\{F_g^{-1}(F_g(x) + t)\}. \tag{6.3}$$

Thus, we have shown the following proposition.

Proposition 6.4. *The exponential generating function of the Comtet–Touchard functions associated to g is given by*

$$\sum_{n\geqslant 0}\frac{t^n}{n!}T_n^{(g)}(x) = e^{F_g^{-1}\{F_g(x)+t\}-x}, \tag{6.4}$$

where $F_g(x) := \int^x \frac{d\zeta}{g(\zeta)}$ *and* F_g^{-1} *denotes its inverse.*

The expression given in (6.4) hides the complexity that one has to solve an integral for F_g and find the inverse function F_g^{-1}. Clearly, if we choose $g(x) = x^\alpha$ (in particular with $\alpha \in \mathbb{Z}$) we obtain the results discussed in previous sections. In the general framework one could also be interested in other cases, for example polynomials instead of monomials.

Example 6.5. Let us consider $g(x) = x^2 + 1$. It follows that $F_g(x) = \int^x \frac{d\zeta}{1+\zeta^2} = \arctan(x)$ as well as $F_g^{-1}(x) = \tan(x)$. Thus, from (6.4) one finds

$$\sum_{n\geqslant 0}\frac{t^n}{n!}T_n^{(g)}(x) = e^{\tan\{\arctan(x)+t\}-x}.$$

Using $\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$, one can get rid of the *arctan* and obtains

$$\sum_{n\geqslant 0}\frac{t^n}{n!}T_n^{(g)}(x) = \exp\left(\frac{(1+x)\tan(t)}{1-x\tan(t)}\right).$$

In the same fashion one can use (6.3) to obtain for the action of the associated exponential operator

$$e^{t(1+x^2)\frac{d}{dx}}f(x) = f\left(\frac{x\cos(t)+\sin(t)}{\cos(t)-x\sin(t)}\right),$$

a relation which was already given in [8, Eq. (I.2.7)]. Analogous formulas for the case $g(x) = \sqrt{x^2+1}$ can also be found in [8].

It is worth pointing out that in [8] a wealth of information concerning exponential operators and operational rules can be found, e.g., when the exponent is generalized to an arbitrary first order differential operator or to (special) operators containing higher order derivatives. Ihara considered similar problems in a closely related context in a more algebraic fashion [11]. A generalization of Theorem 6.2 to the case of several dimensions has been established by Ginocchio [10]. In the following remark the consideration of $T_n^{(g)}(x)$ from above is generalized further, but only in principle (due to the increasing complexity it will be extremely difficult to obtain explicit expressions).

Remark 6.6. Recall that we defined the Comtet–Touchard functions associated to g by (6.1). Here we have used the derivation $g(x)\frac{d}{dx}$ associated to g. More generally, we can also consider a general differential operator of order one, i.e., $\mathfrak{D}_{g,v} = g(x)\frac{d}{dx} + v(x)$. Thus, we introduce the following Comtet–Touchard functions associated to (g, v) by

$$T_n^{(g,v)}(x) := e^{-x}(\mathfrak{D}_{g,v})^n e^x = e^{-x}\left(g(x)\frac{d}{dx} + v(x)\right)^n e^x. \tag{6.5}$$

It is clear that one can generalize this to differential operators $\mathfrak{D}_{\mathbf{a}} = a_m(x)\frac{d^m}{dx^m} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)$ (with $\mathbf{a} = (a_m, a_{m-1}, \ldots, a_1, a_0)$) of arbitrary order by setting $T_n^{\mathbf{a}}(x) := e^{-x}(\mathfrak{D}_{\mathbf{a}})^n e^x$. However, since the resulting expressions will become quickly messy, we restrict to the first order case. We will not attempt to give an explicit expression for $\{g(x)\frac{d}{dx} + v(x)\}^n$, thereby generalizing Theorem 6.2 of Comtet. Instead, we consider the exponential generating function of $T_n^{(g,v)}(x)$ and the same argument as above shows that

$$\sum_{n\geqslant 0} \frac{t^n}{n!} T_n^{(g,v)}(x) = e^{-x} e^{t\{g(x)\frac{d}{dx}+v(x)\}} e^x.$$

The action of the exponential operator can be described as follows [8, p. 6]. If we let $x(t)$ and $k(t)$ be the solutions of the system of first order differential equations

$$\frac{d}{dt}x(t) = g(x(t)), x(0) = x,$$
$$\frac{d}{dt}k(t) = v(x(t))k(t), k(0) = 1,$$

then one has the relation [8, Eq. (I.2.25)]:

$$e^{t\{g(x)\frac{d}{dx}+v(x)\}} f(x) = f(x(t))k(t). \tag{6.6}$$

(The notation is slightly misleading since $k(t)$ can also depend on x, see the example below.) Note that in the case $v = 0$ the second differential equation implies $k(t) = 1$ for all t, so that we get back the result (6.3), i.e., $x(t) = F_g^{-1}\{F_g(x) + t\}$ in the notation from above. Thus, using these notations, we can write

$$\sum_{n\geqslant 0} \frac{t^n}{n!} T_n^{(g,v)}(x) = e^{x(t)-x}k(t) = e^{F_g^{-1}\{F_g(x)+t\}-x}k(t).$$

As a particular example, let us consider the operator $\frac{d}{dx} + x$, i.e., $g(x) = 1$ as well as $v(x) = x$. The first differential equation reduces to $x'(t) = 1$ with $x(0) = x$, yielding $x(t) = x + t$. Therefore, the second differential equation reduces to $k'(t) = (x + t)k(t)$ with $k(0) = 1$, yielding $k(t) = e^{tx+t^2/2}$. Thus,

$$\sum_{n\geqslant 0} \frac{t^n}{n!} T_n^{(1,\mathrm{id})}(x) = e^t e^{tx+\frac{t^2}{2}}.$$

The right-hand side is very similar to the exponential generating function of the Hermite polynomials, see (5.4). This is not surprising since we can use the Burchnall identity (5.3) to obtain

$$T_n^{(1,\mathrm{id})}(x) = e^{-x}\left(\frac{d}{dx} + x\right)^n e^x = e^{-x}\left(\sum_{r=0}^{n} \binom{n}{r} H_{n-r}(x,1)\left(\frac{d}{dx}\right)^r\right)e^x = \sum_{r=0}^{n} \binom{n}{r} H_{n-r}(x,1).$$

7. Conclusions

We discussed several properties of the generalized Touchard polynomials, e.g., a recursion relation generalizing the one of the conventional Touchard polynomials. Furthermore, an interpretation of some operational formulas was given in terms of a binomial theorem for particular noncommuting variables. It was suggested to consider generalized Touchard functions associated to $x^\alpha \frac{d}{dx}$ with arbitrary $\alpha \in \mathbb{R}$ and as a first example it was shown that the resulting Touchard functions of order $\alpha = 1/2$ are given by Hermite polynomials. Generalizing still further, we introduced so called Comtet–Touchard functions associated to arbitrary derivations $g(x)\frac{d}{dx}$ and showed first properties of these functions, using in particular Comtet's result about powers of $g(x)\frac{d}{dx}$ (hence the name Comtet–Touchard). We believe that the class of Comtet–Touchard functions provides a natural and unifying framework which merits closer study.

Let us mention some possible avenues for future investigations. Apart from the already mentioned closer study of the Comtet–Touchard functions, we think it might be worthwhile to consider a q-deformed version of the generalized Touchard polynomials. Let us give some details. If we denote by $S_q(n,k)$ the q-deformed Stirling numbers of the second kind and by D_q the Jackson-derivative $D_q f(x) = \frac{f(qx)-f(x)}{q(x-1)}$, one has the well-known operational relation $(XD_q)^n = \sum_{k=0}^n S_q(n,k)X^k D_q^k$. Using the basic numbers $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$, we introduce the two classic q-deformed exponential functions

$$e_q(x) = \sum_{n\geqslant 0} \frac{x^n}{[n]_q!}, \quad E_q(x) = \sum_{n\geqslant 0} \frac{q^{\binom{n}{2}} x^n}{[n]_q!}.$$

Recalling that e_q is an eigenfunction for D_q and that $E_q(-x) = (e_q(x))^{-1}$, we may introduce the q-deformed generalized Touchard polynomials of order $m \in \mathbb{Z}$ (and $n \in \mathbb{N}$) by

$$T_{n;q}^{(m)}(x) := E_q(-x)(x^m D_q)^n e_q(x).$$

Clearly, considering $q = 1$ gives back the generalized Touchard polynomials studied in the present paper. For $m = 1$ we write $T^{(1)}_{n;q}(x) = T_{n;q}(x)$ and obtain in close analogy to (1.2) that $T_{n;q}(x) = \sum_{k=0}^{n} S_q(n,k)x^k = B_{n;q}(x)$. One may then hope that it is possible to generalize the results of the present paper to this q-deformed situation.

In a different direction, the functions $\mathcal{B}^{(m)}_{\ell}(x)$ appearing in Theorem 3.5 and describing the influence of the noncommutativity of the variables U, V (satisfying $UV = VU - mV^{\frac{2m-1}{m}}$) in the binomial formula should be understood better. In the case $m = 1$ the same role is played by the Bessel polynomials, so the functions $\mathcal{B}^{(m)}_{\ell}(x)$ should be considered as some kind of "higher order analog" to the Bessel polynomials. The coefficients of the Bessel polynomials – the Bessel numbers – have a nice combinatorial interpretation, so one should consider the coefficients of $\mathcal{B}^{(m)}_{\ell}(x)$ in an analogous fashion and find out whether they have a nice combinatorial interpretation, too.

As a final point we would like to mention the consideration of the generalized Touchard functions of order $\alpha \in \mathbb{R}$. We considered the particular case $\alpha = 1/2$ and showed that $T^{(\frac{1}{2})}_{n}(x)$ is given by a Hermite polynomial. It would be interesting to find out whether one has a relation to other well-known polynomials for appropriate choices of α. Furthermore, one should also consider the associated exponential generating function, the recursion relation, etc.

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