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Polynomials whose coefficients are *k*-Fibonacci numbers

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Abstract

Let $\{a_n\}_{n\geq 0}$ denote the linear recursive sequence of order k ($k \geq 2$) defined by the initial values $a_0 = a_1 = \cdots = a_{k-2} = 0$ and $a_{k-1} = 1$ and the recursion $a_n = a_{n-1} + a_{n-2} + \cdots + a_{n-k}$ if $n \geq k$. The a_n are often called *k-Fibonacci* numbers and reduce to the usual *Fibonacci* numbers when $k = 2$. Let $P_{n,k}(x) = a_{k-1}x^n + a_kx^{n-1} + \cdots + a_{n+k-2}x + a_{n+k-1}$, which we will refer to as a *k-Fibonacci coefficient* polynomial. In this paper, we show for all k that the polynomial $P_{n,k}(x)$ has no real zeros if n is even and exactly one real zero if n is odd. This generalizes the known result for the $k = 2$ and $k = 3$ cases corresponding to Fibonacci and Tribonacci coefficient polynomials, respectively. It also improves upon a previous upper bound of approximately k for the number of real zeros of $P_{n,k}(x)$. Finally, we show for all k that the sequence of real zeros of the polynomials $P_{n,k}(x)$ when n is odd converges to the opposite of the positive zero of the characteristic polynomial associated with the sequence a_n. This generalizes a previous result for the case $k = 2$.

Keywords: k-Fibonacci sequence, zeros of polynomials, linear recurrences

MSC: 11C08, 13B25, 11B39, 05A20

1. Introduction

Let the recursive sequence $\{a_n\}_{n\geq 0}$ of order k ($k \geq 2$) be defined by the initial values $a_0 = a_1 = \cdots = a_{k-2} = 0$ and $a_{k-1} = 1$ and the linear recursion

$$a_n = a_{n-1} + a_{n-2} + \cdots + a_{n-k}, \qquad n \geq k. \tag{1.1}$$

The numbers a_n are sometimes referred to as *k-Fibonacci* numbers (or *generalized Fibonacci* numbers) and reduce to the usual *Fibonacci* numbers F_n when $k = 2$ and to the *Tribonacci* numbers T_n when $k = 3$. (See, e.g., A000045 and A000073 in [11].) The sequence a_n was first considered by Knuth [3] and has been a topic of study in enumerative combinatorics. See, for example, [1, Chapter 3] or [9] for interpretations of a_n in terms of linear tilings or k-filtering linear partitions, respectively, and see [10] for a q-generalization of a_n.

Garth, Mills, and Mitchell [2] introduced the definition of the Fibonacci coefficient polynomials $p_n(x) = F_1x^n + F_2x^{n-1} + \cdots + F_nx + F_{n+1}$ and–among other things–determined the number of real zeros of $p_n(x)$. In particular, they showed that $p_n(x)$ has no real zeros if n is even and exactly one real zero if n is odd. Later, this result was extended by Mátyás [5, 6] to more general second order recurrences. The same result also holds for the Tribonacci coefficient polynomials $q_n(x) = T_2x^n + T_3x^{n-1} + \cdots + T_{n+1}x + T_{n+2}$, which was shown by Mátyás and Szalay [8].

If $k \geq 2$ and $n \geq 1$, then define the polynomial $P_{n,k}(x)$ by

$$P_{n,k}(x) = a_{k-1}x^n + a_kx^{n-1} + \cdots + a_{n+k-2}x + a_{n+k-1}. \tag{1.2}$$

We will refer to $P_{n,k}(x)$ as a *k-Fibonacci coefficient* polynomial. Note that when $k = 2$ and $k = 3$, the $P_{n,k}(x)$ reduce to the Fibonacci and Tribonacci coefficient polynomials $p_n(x)$ and $q_n(x)$ mentioned above. In [7], the following result was obtained concerning the number of real zeros of $P_{n,k}(x)$ as a corollary to a more general result involving sequences defined by linear recurrences with non-negative integral weights.

Theorem 1.1. *Let h denote the number of real zeros of the polynomial $P_{n,k}(x)$ defined by* (1.2) *above. Then we have*
(i) $h = k - 2 - 2j$ for some $j = 0, 1, \ldots, (k-2)/2$, if k and n are even,
(ii) $h = k - 1 - 2j$ for some $j = 0, 1, \ldots, (k-2)/2$, if k is even and n is odd,
(iii) $h = k - 1 - 2j$ for some $j = 0, 1, \ldots, (k-1)/2$, if k is odd and n is even,
(iv) $h = k - 2j$ for some $j = 0, 1, \ldots, (k-1)/2$, if k and n are odd.

For example, Theorem 1.1 states when $k = 3$ that the number of real zeros of the polynomial $P_{n,3}(x)$ is either 0 or 2 if n is even or 1 or 3 if n is odd. As already mentioned, it was shown in [8] that $P_{n,3}(x)$ possesses no real zeros when n is even and exactly one real zero when n is odd.

In this paper, we show that the polynomial $P_{n,k}(x)$ possesses the smallest possible number of real zeros in every case and prove the following result.

Theorem 1.2. *Let $k \geq 2$ be a positive integer and $P_{n,k}(x)$ be defined by* (1.2) *above. Then we have the following:*
(i) If n is even, then $P_{n,k}(x)$ has no real zeros.
(ii) If n is odd, then $P_{n,k}(x)$ has exactly one real zero.

We prove Theorem 1.2 as a series of lemmas in the third and fourth sections below, and have considered separately the cases for even and odd k. Combining

Theorems 3.5 and 4.5 below gives Theorem 1.2. The crucial steps in our proofs of Theorems 3.5 and 4.5 are Lemmas 3.2 and 4.2, respectively, where we make a comparison of consecutive derivatives of a polynomial evaluated at the point $x = 1$. This allows us to show that there is exactly one zero when $x \leq -1$ in the case when n is odd. We remark that our proof, when specialized to the cases $k = 2$ and $k = 3$, provides an alternative proof to the ones given in [2] and [8], respectively, in these cases. In the final section, we show for all k that the sequence of real zeros of the polynomials $P_{n,k}(x)$ for n odd converges to $-\lambda$, where λ is the positive zero of the characteristic polynomial associated with the sequence a_n (see Theorem 5.5 below). This generalizes the result for the $k = 2$ case, which was shown in [2].

2. Preliminaries

We seek to determine the number of real zeros of the polynomial $P_{n,k}(x)$. By the following lemma, we may restrict our attention to the case when $x \leq -1$.

Lemma 2.1. *If $k \geq 2$ and $n \geq 1$, then the polynomial $P_{n,k}(x)$ has no zeros on the interval $(-1, \infty)$.*

Proof. Clearly, the equation $P_{n,k}(x) = 0$ has no roots if $x \geq 0$ since it has positive coefficients. Suppose $-1 < x < 0$. If n is odd, then

$$a_{k+2j-1}x^{n-2j} + a_{k+2j}x^{n-2j-1} > 0, \qquad 0 \leq j \leq (n-1)/2,$$

since $x^{n-2j-1} > -x^{n-2j} > 0$ if $-1 < x < 0$ and $a_{k+2j} \geq a_{k+2j-1} > 0$. This implies

$$P_{n,k}(x) = \sum_{j=0}^{\frac{n-1}{2}} (a_{k+2j-1}x^{n-2j} + a_{k+2j}x^{n-2j-1}) > 0.$$

Similarly, if n is even, then

$$P_{n,k}(x) = a_{k-1}x^n + \sum_{j=0}^{\frac{n-2}{2}} (a_{k+2j}x^{n-2j-1} + a_{k+2j+1}x^{n-2j-2}) > 0.$$

□

So we seek the zeros of $P_{n,k}(x)$ where $x \leq -1$, equivalently, the zeros of $P_{n,k}(-x)$ where $x \geq 1$. For this, it is more convenient to consider the zeros of $g_{n,k}(x)$ given by

$$g_{n,k}(x) := c_k(-x)P_{n,k}(-x), \tag{2.1}$$

see [7], where

$$c_k(x) := x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 \tag{2.2}$$

denotes the *characteristic polynomial* associated with the sequence a_n.

By [7, Lemma 2.1], we have

$$\begin{aligned} g_{n,k}(x) &= (-x)^{n+k} - a_{n+k}(-x)^{k-1} - (a_{n+1} + a_{n+2} + \cdots + a_{n+k-1})(-x)^{k-2} \\ &\quad - \cdots - (a_{n+k-2} + a_{n+k-1})(-x) - a_{n+k-1} \\ &= (-x)^{n+k} - a_{n+k}(-x)^{k-1} - \sum_{r=1}^{k-1} \left(\sum_{j=r}^{k-1} a_{n+j} \right) (-x)^{k-r-1}. \end{aligned} \tag{2.3}$$

We now wish to study the zeros of $g_{n,k}(x)$, where $x \geq 1$. In the subsequent two sections, we undertake such a study, considering separately the even and odd cases for k.

3. The case k even

Throughout this section, k will denote a positive even integer. We consider the zeros of the polynomial $g_{n,k}(x)$ where $x \geq 1$, and for this, it is more convenient to consider the zeros of the polynomial

$$f_{n,k}(x) := (1 + x)g_{n,k}(x), \tag{3.1}$$

where $x \geq 1$.

First suppose n is odd. Note that when k is even and n is odd, we have

$$\begin{aligned} f_{n,k}(x) &= -x^{n+k}(1 + x) + a_{n+k}x^k + a_n x^{k-1} - a_{n+1}x^{k-2} + a_{n+2}x^{k-3} \\ &\quad - \cdots + a_{n+k-2}x - a_{n+k-1} \\ &= -x^{n+k}(1 + x) + a_{n+k}x^k + \sum_{r=0}^{k-1} (-1)^r a_{n+r} x^{k-r-1}, \end{aligned} \tag{3.2}$$

by (2.3) and the recurrence for a_n. In the lemmas below, we ascertain the number of the zeros of the polynomial $f_{n,k}(x)$ when $x \geq 1$. We will need the following combinatorial inequality.

Lemma 3.1. *If $k \geq 4$ is even and $n \geq 1$, then*

$$a_{n+k+1} \geq \sum_{r=0}^{\frac{k}{2}-1} 2^{\frac{k}{2}-r} a_{n+2r+1}. \tag{3.3}$$

Proof. We have

$$\begin{aligned} a_{n+k+1} &= a_{n+k} + \sum_{r=1}^{k-1} a_{n+r} \geq 2 \sum_{r=1}^{k-1} a_{n+r} \\ &= 2a_{n+k-1} + 2_{a+k-2} + 2 \sum_{r=1}^{k-3} a_{n+r} \geq 2a_{n+k-1} + 4 \sum_{r=1}^{k-3} a_{n+r} \end{aligned}$$

$$
\begin{aligned}
&= 2a_{n+k-1} + 4a_{n+k-3} + 4a_{n+k-4} + 4\sum_{r=1}^{k-5} a_{n+r} \\
&\geq 2a_{n+k-1} + 4a_{n+k-3} + 8\sum_{r=1}^{k-5} a_{n+r} \\
&= \cdots \geq \sum_{r=i}^{\frac{k}{2}-1} 2^{\frac{k}{2}-r} a_{n+2r+1} + 2^{\frac{k}{2}-i+1} \sum_{r=1}^{2i-1} a_{n+r} \\
&= \sum_{r=i-1}^{\frac{k}{2}-1} 2^{\frac{k}{2}-r} a_{n+2r+1} + 2^{\frac{k}{2}-i+1} a_{n+2i-2} + 2^{\frac{k}{2}-i+1} \sum_{r=1}^{2i-3} a_{n+r} \\
&\geq \sum_{r=i-1}^{\frac{k}{2}-1} 2^{\frac{k}{2}-r} a_{n+2r+1} + 2^{\frac{k}{2}-i+2} \sum_{r=1}^{2i-3} a_{n+r} \\
&= \cdots \geq \sum_{r=0}^{\frac{k}{2}-1} 2^{\frac{k}{2}-r} a_{n+2r+1},
\end{aligned}
$$

which gives (3.3). □

The following lemma will allow us to determine the number of zeros of $f_{n,k}(x)$ for $x \geq 1$.

Lemma 3.2. *Suppose $k \geq 4$ is even and n is odd. If $1 \leq i \leq k-1$, then $f_{n,k}^{(i)}(1) < 0$ implies $f_{n,k}^{(i+1)}(1) < 0$, where $f_{n,k}^{(i)}$ denotes the i-th derivative of $f_{n,k}$.*

Proof. Let $f = f_{n,k}$ and $i = k-j$ for some $1 \leq j \leq k-1$. Then the assumption $f^{(k-j)}(1) < 0$ is equivalent to

$$
\frac{k!}{j!} a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} < \prod_{s=1}^{k-j} (n+j+s) + \prod_{s=1}^{k-j} (n+j+s+1). \quad (3.4)
$$

We will show that inequality (3.4) implies

$$
\frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} < \prod_{s=0}^{k-j} (n+j+s) + \prod_{s=0}^{k-j} (n+j+s+1). \quad (3.5)
$$

Observe first that the left-hand side of both inequalities (3.4) and (3.5) is positive as

$$
\begin{aligned}
&\frac{k!}{j!} a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \\
&\qquad = \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} > 0,
\end{aligned}
$$

since $a_{n+k}=\sum_{r=0}^{k-1}a_{n+r}$ and $\frac{k!}{j!}>\frac{(k-r-1)!}{(j-r-1)!}$. Note also that

$$\frac{\prod_{s=0}^{k-j}(n+j+s)+\prod_{s=0}^{k-j}(n+j+s+1)}{\prod_{s=1}^{k-j}(n+j+s)+\prod_{s=1}^{k-j}(n+j+s+1)}>n+j,$$

so to show (3.5), it suffices to show

$$\frac{k!}{(j-1)!}a_{n+k}+\sum_{r=0}^{j-2}(-1)^r\frac{(k-r-1)!}{(j-r-2)!}a_{n+r} \le (n+j)\left(\frac{k!}{j!}a_{n+k}+\sum_{r=0}^{j-1}(-1)^r\frac{(k-r-1)!}{(j-r-1)!}a_{n+r}\right). \tag{3.6}$$

For (3.6), it is enough to show

$$\frac{k!}{(j-1)!}a_{n+k}+\sum_{r=0}^{j-2}(-1)^r\frac{(k-r-1)!}{(j-r-2)!}a_{n+r} \le (j+1)\left(\frac{k!}{j!}a_{n+k}+\sum_{r=0}^{j-1}(-1)^r\frac{(k-r-1)!}{(j-r-1)!}a_{n+r}\right). \tag{3.7}$$

Starting with the left-hand side of (3.7), we have

$$\begin{aligned}
&\frac{k!}{(j-1)!}a_{n+k}+\sum_{r=0}^{j-2}(-1)^r\frac{(k-r-1)!}{(j-r-2)!}a_{n+r}\\
&=\frac{k!}{(j-1)!}\sum_{r=j-1}^{k-1}a_{n+r}+\sum_{r=0}^{j-2}\left(\frac{k!}{(j-1)!}+(-1)^r\frac{(k-r-1)!}{(j-r-2)!}\right)a_{n+r}\\
&=\frac{k!}{(j-1)!}\sum_{r=j}^{k-1}a_{n+r}+\sum_{r=0}^{j-1}\left(j\frac{k!}{j!}+(-1)^r(j-r-1)\frac{(k-r-1)!}{(j-r-1)!}\right)a_{n+r}\\
&=\frac{k!}{(j-1)!}\sum_{r=j}^{k-1}a_{n+r}+\sum_{r=0}^{j-1}j\left(\frac{k!}{j!}+(-1)^r\frac{(k-r-1)!}{(j-r-1)!}\right)a_{n+r}\\
&\quad+\sum_{r=0}^{j-1}(-1)^{r+1}(r+1)\frac{(k-r-1)!}{(j-r-1)!}a_{n+r}\\
&\le\frac{k!}{(j-1)!}\sum_{r=j}^{k-1}a_{n+r}+\sum_{r=0}^{j-1}j\left(\frac{k!}{j!}+(-1)^r\frac{(k-r-1)!}{(j-r-1)!}\right)a_{n+r}\\
&\quad+\sum_{r=0}^{\lfloor\frac{j-2}{2}\rfloor}(2r+2)\frac{(k-2r-2)!}{(j-2r-2)!}a_{n+2r+1}
\end{aligned}$$

$$= (j+1)\frac{k!}{j!}\sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j\left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!}\right) a_{n+r}$$
$$- \frac{k!}{j!}\sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{\lfloor\frac{j-2}{2}\rfloor} (2r+2)\frac{(k-2r-2)!}{(j-2r-2)!} a_{n+2r+1}.$$

Below we show

$$\sum_{r=0}^{\lfloor\frac{j-2}{2}\rfloor} (2r+2)\frac{(k-2r-2)!}{(j-2r-2)!} a_{n+2r+1} \le \frac{k!}{j!}\sum_{r=j}^{k-1} a_{n+r}. \tag{3.8}$$

Then from (3.8), we have

$$\frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r}$$
$$\le (j+1)\frac{k!}{j!}\sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j\left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!}\right) a_{n+r}$$
$$\le (j+1)\frac{k!}{j!}\sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} (j+1)\left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!}\right) a_{n+r}$$
$$= (j+1)\frac{k!}{j!} a_{n+k} + (j+1)\sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r},$$

which gives (3.7), as desired.

To finish the proof, we need to show (3.8). We may assume $j \ge 2$, since the inequality is trivial when $j = 1$. By Lemma 3.1 and the fact that $2^m \ge 2m$ if $m \ge 1$, we have

$$\sum_{r=j}^{k-1} a_{n+r} \ge a_{n+k-1}$$
$$\ge \sum_{r=0}^{\frac{k}{2}-2} 2^{\frac{k}{2}-r-1} a_{n+2r+1} \ge \sum_{r=0}^{\frac{k}{2}-2} (k-2r-2) a_{n+2r+1} \ge \sum_{r=0}^{\lfloor\frac{j-2}{2}\rfloor} (k-2r-2) a_{n+2r+1},$$

the last inequality holding since $j \le k-1$, with k even. So to show (3.8), it is enough to show

$$(k-2r-2)\frac{k!}{j!} \ge (2r+2)\frac{(k-2r-2)!}{(j-2r-2)!}, \qquad 0 \le r \le \lfloor (j-2)/2 \rfloor, \tag{3.9}$$

where $2 \le j \le k-1$. Since the ratio $\frac{k!/j!}{(k-2r-2)!/(j-2r-2)!}$ is decreasing in j for fixed k and r, one needs to verify (3.9) only when $j = k-1$, and it holds in this case since $2r+2 \le j < k$. This completes the proof. □

We now determine the number of zeros of $f_{n,k}(x)$ on the interval $[1,\infty)$.

Lemma 3.3. *Suppose $k \geq 4$ is even and n is odd. Then the polynomial $f_{n,k}(x)$ has exactly one zero on the interval $[1,\infty)$. Furthermore, this zero is simple.*

Proof. Let $f = f_{n,k}$, where we first assume $n \geq 3$. Then

$$f(1) = -2 + a_{n+k} + \sum_{r=0}^{k-1}(-1)^r a_{n+r} = -2 + 2\sum_{r=0}^{\frac{k}{2}-1} a_{n+2r} > 0,$$

since $a_{n+k-2} \geq a_{k+1} = 2$. Let ℓ be the smallest positive integer i such that $f^{(i)}(1) < 0$; note that $1 \leq \ell \leq k+1$ since $f^{(k+1)}(1) < 0$. Then

$$f^{(\ell+1)}(1), f^{(\ell+2)}(1), \ldots, f^{(k+1)}(1)$$

are all negative, by Lemma 3.2. Since $f^{(k+1)}(x) < 0$ for all $x \geq 1$, it follows that $f^{(\ell)}(x) < 0$ for all $x \geq 1$. To see this, note that if $\ell \leq k$, then $f^{(k)}(1) < 0$ implies $f^{(k)}(x) < 0$ for all $x \geq 1$, which in turn implies each of $f^{(k)}(x), f^{(k-1)}(x), \ldots, f^{(\ell)}(x)$ is negative for all $x \geq 1$.

If $\ell \geq 2$, then $f^{(\ell-1)}(1) \geq 0$ and $f^{(\ell)}(x) < 0$ for all $x \geq 1$. Since $f^{(\ell-1)}(1) \geq 0$ and $\lim_{x\to\infty} f^{(\ell-1)}(x) = -\infty$, we have either (i) $f^{(\ell-1)}(1) = 0$ and $f^{(\ell-1)}(x)$ has no zeros on the interval $(1,\infty)$ or (ii) $f^{(\ell-1)}(1) > 0$ and $f^{(\ell-1)}(x)$ has exactly one zero on the interval $(1,\infty)$. If $\ell \geq 3$, then $f^{(\ell-2)}(x)$ would also have at most one zero on $(1,\infty)$ since $f^{(\ell-2)}(1) \geq 0$, with $f^{(\ell-2)}(x)$ initially increasing up to some point $s \geq 1$ before it decreases monotonically to $-\infty$ (where $s = 1$ if $f^{(\ell-1)}(1) = 0$ and $s > 1$ if $f^{(\ell-1)}(1) > 0$). Note that each derivative of f of order ℓ or less is eventually negative. Continuing in this fashion, we then see that if $\ell \geq 2$, then $f'(x)$ has at most one zero on the interval $(1,\infty)$, with $f'(1) \geq 0$ and $f'(x)$ eventually negative. If $\ell = 1$, then $f'(x) < 0$ for all $x \geq 1$. Since $f(1) > 0$ and $\lim_{x\to\infty} f(x) = -\infty$, it follows in either case that f has exactly one zero on the interval $[1,\infty)$, which finishes the case when $n \geq 3$.

If $n = 1$, then $f_{1,k}(x) = -x^{k+1}(1+x) + 2x^k + x - 1$ so that $f_{1,k}(1) = 0$, with

$$\begin{aligned} f'_{1,k}(x) &= -(k+1)x^k - (k+2)x^{k+1} + 2kx^{k-1} + 1 \\ &\leq -(k+1)x^{k-1} - (k+2)x^{k-1} + 2kx^{k-1} + 1 = -3x^{k-1} + 1 < 0 \end{aligned}$$

for $x \geq 1$. Thus, there is exactly one zero on the interval $[1,\infty)$ in this case as well.

Let t be the root of the equation $f_{n,k}(x) = 0$ on $[1,\infty)$. We now show that t has multiplicity one. First assume $n \geq 3$. Then $t > 1$. We consider cases depending on the value of $f'(1)$. If $f'(1) < 0$, then $f'(x) < 0$ for all $x \geq 1$ and thus $f'(t) < 0$ is non-zero, implying t is a simple root. If $f'(1) > 0$, then $f'(t) < 0$ due to $f(1) > 0$ and the fact that $f'(x)$ would then have one root v on $(1,\infty)$ with $v < t$. Finally, if $f'(1) = 0$, then the proof of Lemma 3.2 above shows that $f''(1) < 0$ and thus $f''(x) < 0$ for all $x \geq 1$, which implies $f'(t) < 0$. If $n = 1$, then $t = 1$ and $f'_{1,k}(1) < 0$. Thus, t is a simple root in all cases, as desired, which completes the proof. □

We next consider the case when n is even.

Lemma 3.4. *Suppose $k \geq 4$ and n are even. Then $f_{n,k}(x)$ has no zeros on $[1, \infty)$.*

Proof. In this case, we have

$$f_{n,k}(x) = x^{n+k}(1+x) + a_{n+k}x^k + \sum_{r=0}^{k-1}(-1)^r a_{n+r}x^{k-r-1},$$

by (2.3) and (3.1). If $x \geq 1$, then $f_{n,k}(x) > 0$ since $a_{n+k} = \sum_{r=0}^{k-1} a_{n+r}$ and $x^k \geq x^{k-r-1}$ for $0 \leq r \leq k-1$. □

The main result of this section now follows rather quickly.

Theorem 3.5. *(i) If k is even and n is odd, then the polynomial $P_{n,k}(x)$ has one real zero q, and it is simple with $q \leq -1$.*
(ii) If k and n are even, then the polynomial $P_{n,k}(x)$ has no real zeros.

Proof. Note first that the preceding lemmas, where we assumed $k \geq 4$ is even, may be adjusted slightly and are also seen to hold in the case $k = 2$. First suppose n is odd. By Lemma 3.3, the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has one zero for $x \geq 1$, and it is simple. By [7, Lemma 2.3], the characteristic polynomial $c_k(x) = x^k - x^{k-1} - x^{k-2} - \cdots - 1$ has one negative real zero when k is even, and it is seen to lie in the interval $(-1, 0)$. Since $g_{n,k}(x) = c_k(-x)P_{n,k}(-x)$, it follows that $P_{n,k}(-x)$ has one zero for $x \geq 1$. Thus, $P_{n,k}(x)$ has one zero for $x \leq -1$, and it is simple. By Lemma 2.1, the polynomial $P_{n,k}(x)$ has exactly one real zero.

If n is even, then the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has no zeros for $x \geq 1$, by Lemma 3.4. By (2.1), it follows that $P_{n,k}(x)$ has no zeros for $x \leq -1$. By Lemma 2.1, $P_{n,k}(x)$ has no real zeros. □

4. The case k odd

Throughout this section, $k \geq 3$ will denote a positive odd integer. We study the zeros of the polynomial $g_{n,k}(x)$ when $x \geq 1$, and for this, it is again more convenient to consider the polynomial $f_{n,k}(x) := (1+x)g_{n,k}(x)$. First suppose n is odd. When k and n are both odd, note that

$$\begin{aligned} f_{n,k}(x) &= x^{n+k}(1+x) - a_{n+k}x^k - a_nx^{k-1} + a_{n+1}x^{k-2} - \cdots + a_{n+k-2}x - a_{n+k-1} \\ &= x^{n+k}(1+x) - a_{n+k}x^k + \sum_{r=0}^{k-1}(-1)^{r+1}a_{n+r}x^{k-r-1}, \end{aligned}$$

by (2.3) and the recurrence for a_n. In the lemmas below, we ascertain the number of zeros of the polynomial $f_{n,k}(x)$ when $x \geq 1$. We start with the following inequality.

Lemma 4.1. *Suppose $k \geq 3$ is odd and $n \geq 1$. If $1 \leq j \leq k-1$, then*

$$3\frac{k!}{j!}a_{n+k-1} \geq \sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} 2r\frac{(k-2r)!}{(j-2r)!}a_{n+2r-1}. \tag{4.1}$$

Proof. First note that we have the inequality

$$a_{n+k-1} \geq \sum_{r=1}^{\frac{k-3}{2}} 2^{\frac{k-1}{2}-r}a_{n+2r}. \tag{4.2}$$

To show (4.2), proceed as in the proof of Lemma 3.1 above and write

$$\begin{aligned} a_{n+k-1} &\geq a_{n+k-2} + \sum_{r=2}^{k-3} a_{n+r} \\ &\geq 2a_{n+k-3} + 2\sum_{r=2}^{k-4} a_{n+r} \\ &= 2a_{n+k-3} + 2a_{n+k-4} + 2\sum_{r=2}^{k-5} a_{n+r} \\ &\geq 2a_{2n+k-3} + 4a_{n+k-5} + 4\sum_{r=2}^{k-6} a_{n+r} \\ &= \cdots \geq \sum_{r=1}^{\frac{k-3}{2}} 2^{\frac{k-1}{2}-r}a_{n+2r}. \end{aligned}$$

Since $2^m \geq 2m$ if $m \geq 1$, we have

$$a_{n+k-1} \geq \sum_{r=1}^{\frac{k-3}{2}} 2^{\frac{k-1}{2}-r}a_{n+2r} \geq \sum_{r=1}^{\frac{k-3}{2}} (k-2r-1)a_{n+2r}. \tag{4.3}$$

First suppose $j \leq k-2$. In this case, we show

$$\frac{k!}{j!}a_{n+k-1} \geq \sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} r\frac{(k-2r)!}{(j-2r)!}a_{n+2r-1}, \tag{4.4}$$

which implies (4.1). And (4.4) is seen to hold since by (4.3),

$$\frac{k!}{j!}a_{n+k-1} \geq \sum_{r=1}^{\frac{k-3}{2}} \frac{(k-2r-1)k!}{j!}a_{n+2r} \geq \sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \frac{(k-2r-1)k!}{j!}a_{n+2r},$$

with $a_{n+2r} \geq a_{n+2r-1}$ and

$$\frac{(k-2r-1)k!}{r(k-2r)!} \geq \frac{(k-2)!}{(k-2r-2)!} \geq \frac{j!}{(j-2r)!}.$$

The $j=k-1$ case of (4.1) follows from noting

$$\begin{aligned}3ka_{n+k-1} &\geq ka_{n+k-1}+\sum_{r=1}^{\frac{k-3}{2}}2k(k-2r-1)a_{n+2r}\\ &\geq (k-1)a_{n+k-2}+\sum_{r=1}^{\frac{k-3}{2}}2r(k-2r)a_{n+2r-1}=\sum_{r=1}^{\frac{k-1}{2}}2r(k-2r)a_{n+2r-1},\end{aligned}$$

since $k(k-2r-1)\geq r(k-2r)$ if $1\leq r\leq \frac{k-3}{2}$. □

Lemma 4.2. *Suppose $k,n\geq 3$ are odd. If $1\leq i\leq k-1$, then $f_{n,k}^{(i)}(1)>0$ implies $f_{n,k}^{(i+1)}(1)>0$.*

Proof. Let $f=f_{n,k}$ and $i=k-j$ for some $1\leq j\leq k-1$. Then the assumption $f^{(k-j)}(1)>0$ is equivalent to

$$\frac{(n+k)!}{(n+j)!}+\frac{(n+k+1)!}{(n+j+1)!}>\frac{k!}{j!}a_{n+k}+\sum_{r=0}^{j-1}(-1)^r\frac{(k-r-1)!}{(j-r-1)!}a_{n+r}. \tag{4.5}$$

Using (4.5), we will show $f^{(k-j+1)}(1)>0$, i.e.,

$$\frac{(n+k)!}{(n+j-1)!}+\frac{(n+k+1)!}{(n+j)!}>\frac{k!}{(j-1)!}a_{n+k}+\sum_{r=0}^{j-2}(-1)^r\frac{(k-r-1)!}{(j-r-2)!}a_{n+r}. \tag{4.6}$$

Note that the right-hand side of both inequalities (4.5) and (4.6) is positive since $a_{n+k}=\sum_{r=0}^{k-1}a_{n+r}$. Since the left-hand side of (4.6) divided by the left-hand side of (4.5) is greater than $n+j$, it suffices to show

$$\begin{aligned}&\frac{k!}{(j-1)!}a_{n+k}+\sum_{r=0}^{j-2}(-1)^r\frac{(k-r-1)!}{(j-r-2)!}a_{n+r}\\ &\qquad\leq (n+j)\left(\frac{k!}{j!}a_{n+k}+\sum_{r=0}^{j-1}(-1)^r\frac{(k-r-1)!}{(j-r-1)!}a_{n+r}\right).\end{aligned} \tag{4.7}$$

For (4.7), it is enough to show

$$\begin{aligned}&\frac{k!}{(j-1)!}a_{n+k}+\sum_{r=0}^{j-2}(-1)^r\frac{(k-r-1)!}{(j-r-2)!}a_{n+r}\\ &\qquad\leq (j+3)\left(\frac{k!}{j!}a_{n+k}+\sum_{r=0}^{j-1}(-1)^r\frac{(k-r-1)!}{(j-r-1)!}a_{n+r}\right),\end{aligned} \tag{4.8}$$

since $n\geq 3$.

Starting with the left-hand-side of (4.8), and proceeding at this stage as in the proof of Lemma 3.2 above, we have

$$\begin{aligned}
&\frac{k!}{(j-1)!}a_{n+k}+\sum_{r=0}^{j-2}(-1)^r\frac{(k-r-1)!}{(j-r-2)!}a_{n+r}\\
&\le \frac{k!}{(j-1)!}\sum_{r=j}^{k-1}a_{n+r}+\sum_{r=0}^{j-1}j\left(\frac{k!}{j!}+(-1)^r\frac{(k-r-1)!}{(j-r-1)!}\right)a_{n+r}\\
&\quad+\sum_{r=1}^{\lfloor\frac{j}{2}\rfloor}2r\frac{(k-2r)!}{(j-2r)!}a_{n+2r-1}\\
&=(j+3)\frac{k!}{j!}\sum_{r=j}^{k-1}a_{n+r}+\sum_{r=0}^{j-1}j\left(\frac{k!}{j!}+(-1)^r\frac{(k-r-1)!}{(j-r-1)!}\right)a_{n+r}\\
&\quad-3\frac{k!}{j!}\sum_{r=j}^{k-1}a_{n+r}+\sum_{r=1}^{\lfloor\frac{j}{2}\rfloor}2r\frac{(k-2r)!}{(j-2r)!}a_{n+2r-1}\\
&\le (j+3)\frac{k!}{j!}\sum_{r=j}^{k-1}a_{n+r}+\sum_{r=0}^{j-1}j\left(\frac{k!}{j!}+(-1)^r\frac{(k-r-1)!}{(j-r-1)!}\right)a_{n+r},
\end{aligned}$$

where the last inequality follows from Lemma 4.1. Thus,

$$\begin{aligned}
&\frac{k!}{(j-1)!}a_{n+k}+\sum_{r=0}^{j-2}(-1)^r\frac{(k-r-1)!}{(j-r-2)!}a_{n+r}\\
&\le (j+3)\frac{k!}{j!}\sum_{r=j}^{k-1}a_{n+r}+\sum_{r=0}^{j-1}j\left(\frac{k!}{j!}+(-1)^r\frac{(k-r-1)!}{(j-r-1)!}\right)a_{n+r}\\
&\le (j+3)\frac{k!}{j!}\sum_{r=j}^{k-1}a_{n+r}+\sum_{r=0}^{j-1}(j+3)\left(\frac{k!}{j!}+(-1)^r\frac{(k-r-1)!}{(j-r-1)!}\right)a_{n+r}\\
&=(j+3)\frac{k!}{j!}a_{n+k}+(j+3)\sum_{r=0}^{j-1}(-1)^r\frac{(k-r-1)!}{(j-r-1)!}a_{n+r},
\end{aligned}$$

which gives (4.8) and completes the proof. □

We can now determine the number of zeros of $f_{n,k}(x)$ on the interval $[1,\infty)$.

Lemma 4.3. *Suppose $k\ge 3$ and n are odd. Then $f_{n,k}(x)$ has exactly one zero on the interval $[1,\infty)$ and it is simple.*

Proof. If $n\ge 3$, then use Lemma 4.2 and the same reasoning as in the proof of Lemma 3.3 above. Note that in this case we have

$$f_{n,k}(1)=2-a_{n+k}+\sum_{r=0}^{k-1}(-1)^{r+1}a_{n+r}=2-2\sum_{r=0}^{\frac{k-1}{2}}a_{n+2r}<0,$$

as $a_{n+k-1}, a_{n+k-3} > 0$. If $n = 1$, then $f_{1,k}(x) = x^{k+1}(1+x) - 2x^k + x - 1$ and the result also holds as $f_{1,k}(1) = 0$ with $f'_{1,k}(x) > 0$ if $x \geq 1$. □

We next consider the case when n is even.

Lemma 4.4. *If $k \geq 3$ is odd and n is even, then $f_{n,k}(x)$ has no zeros on $[1, \infty)$.*

Proof. In this case, we have

$$f_{n,k} = -x^{n+k}(1+x) - a_{n+k}x^k + \sum_{r=0}^{k-1}(-1)^{r+1}a_{n+r}x^{k-r-1}.$$

If $x \geq 1$, then $f_{n,k}(x) < 0$ since $a_{n+k} = \sum_{r=0}^{k-1} a_{n+r}$ and $-x^k \leq -x^{k-r-1}$ for $0 \leq r \leq k-1$. □

We now prove the main result of this section.

Theorem 4.5. *(i) If $k \geq 3$ and n are odd, then the polynomial $P_{n,k}(x)$ has one real zero q, and it is simple with $q \leq -1$.*
(ii) If $k \geq 3$ is odd and n is even, then the polynomial $P_{n,k}(x)$ has no real zeros.

Proof. First suppose n is odd. By Lemma 4.3, the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has one zero on $[1, \infty)$, and it is simple. By [7, Lemma 2.3], the characteristic polynomial $c_k(x) = x^k - x^{k-1} - x^{k-2} - \cdots - 1$ has no negative real zeros when k is odd. Since $g_{n,k}(x) = c_k(-x)P_{n,k}(-x)$, it follows that $P_{n,k}(x)$ has one zero for $x \leq -1$, and hence one real zero, by Lemma 2.1.

If n is even, then the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has no zeros for $x \geq 1$, by Lemma 4.4. Thus, neither does $P_{n,k}(-x)$, which implies it has no real zeros. □

5. Convergence of zeros

In this section, we show that for each fixed $k \geq 2$, the sequence of real zeros of $P_{n,k}(x)$ for n odd is convergent. Before proving this, we remind the reader of the following version of Rouché's Theorem which can be found in [4].

Theorem 5.1 (Rouché)**.** *If $p(z)$ and $q(z)$ are analytic interior to a simple closed Jordan curve $\mathcal{C}$, and are continuous on $\mathcal{C}$, with*

$$|p(z) - q(z)| < |q(z)|, \qquad z \in \mathcal{C},$$

then the functions $p(z)$ and $q(z)$ have the same number of zeros interior to $\mathcal{C}$.

We now give three preliminary lemmas.

Lemma 5.2. *(i) If $k \geq 2$, then the polynomial $c_k(x) = x^k - x^{k-1} - \cdots - x - 1$ has one positive real zero λ, with $\lambda > 1$. All of its other zeros have modulus strictly less than one.*
(ii) The zeros of $c_k(x)$, which we will denote by $\alpha_1 = \lambda, \alpha_2, \ldots, \alpha_k$, are distinct and thus

$$a_n = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_n\alpha_k^n, \qquad n \geq 0, \tag{5.1}$$

where $c_1, c_2, \ldots, c_k$ are constants.
(iii) The constant c_1 is a positive real number.

Proof. (i) It is more convenient to consider the polynomial $d_k(x) := (1-x)c_k(x)$. Note that

$$d_k(x) = (1-x)\left(x^k - \frac{1-x^k}{1-x}\right) = 2x^k - x^{k+1} - 1.$$

We regard $d_k(z)$ as a complex function. Since on the circle $|z| = 3$ in the complex plane holds

$$|2z^k| = 2 \cdot 3^k < 3^{k+1} - 1 = |-z^{k+1}| - 1 \leq |-z^{k+1} - 1|,$$

it follows from Rouché's Theorem that $d_k(z)$ has $k+1$ zeros in the disc $|z| < 3$ since the function $-z^{k+1} - 1$ has all of its zeros there. On the other hand, on the circle $|z| = 1 + \epsilon$, we have

$$|-z^{k+1}| = (1+\epsilon)^{k+1} < 2(1+\epsilon)^k - 1 \leq |2z^k - 1|,$$

which implies that the polynomial $d_k(z)$ has exactly k zeros in the disc $|z| < 1 + \epsilon$, for all $\epsilon > 0$ sufficiently small such that $-\frac{\ln(1-\epsilon)}{\ln(1+\epsilon)} < 2 \leq k$. Letting $\epsilon \to 0$, we see that there are k zeros for the polynomial $d_k(z)$ in the disc $|z| \leq 1$. But $z = 1$ is a zero of the polynomial $d_k(z) = (1-z)c_k(z)$ on the circle $|z| = 1$, and it is the only such zero since $d_k(z) = 0$ implies $|z|^k \cdot |2 - z| = 1$, or $|2 - z| = 1$, which is clearly satisfied by only $z = 1$. Hence, the polynomial $c_k(z)$ has $k - 1$ zeros in the disc $|z| < 1$ and exactly one zero in the domain $1 < |z| < 3$. Finally, by Descartes' rule of signs and since $c_k(1) < 0$, we see that $c_k(x)$ has exactly one positive real zero λ, with $1 < \lambda < 3$.

(ii) We'll prove only the first statement, as the second one follows from the first and the theory of linear recurrences. For this, first note that $d'_k(x) = 0$ implies $x = 0, \frac{2k}{k+1}$. Now the only possible rational roots of the equation $d_k(x) = 0$ are ± 1, by the rational root theorem. Thus $d_k\left(\frac{2k}{k+1}\right) = 0$ is impossible as $k \geq 2$, which implies $d_k(x)$ and $d'_k(x)$ cannot share a zero. Therefore, the zeros of $d_k(x)$, and hence of $c_k(x)$, are distinct.

(iii) Substitute $n = 0, 1, \ldots, k-1$ into (5.1), and recall that $a_0 = a_1 = \cdots = a_{k-2} = 0$ with $a_{k-1} = 1$, to obtain a system of linear equations in the variables $c_1, c_2, \ldots, c_k$. Let A be the coefficient matrix for this system (where the equations are understood to have been written in the natural order) and let A' be the matrix obtained from A by replacing the first column of A with the vector $(0, \ldots, 0, 1)$ of length k. Now the transpose of A and of the $(k-1) \times (k-1)$ matrix obtained from A' by deleting the first column and the last row are seen to be Vandermonde matrices. Therefore, by Cramer's rule, we have

$$\begin{aligned} c_1 = \frac{\det A'}{\det A} &= \frac{(-1)^{k+1} \prod_{2 \le i < j \le k} (\alpha_j - \alpha_i)}{\prod_{1 \le i < j \le k} (\alpha_j - \alpha_i)} \\ &= \frac{1}{(-1)^{k-1} \prod_{j=2}^{k} (\alpha_j - \alpha_1)} = \frac{1}{\prod_{j=2}^{k} (\alpha_1 - \alpha_j)}. \end{aligned}$$

If $j \ge 2$, then either $\alpha_j < 0$ or α_j and α_ℓ are complex conjugates for some ℓ. Note that $\alpha_1 - \alpha_j > 0$ in the first case and

$$(\alpha_1 - \alpha_j)(\alpha_1 - \alpha_\ell) = (\alpha_1 - a)^2 + b^2 > 0$$

in the second, where $\alpha_j = a + bi$. Since all of the complex zeros of $c_k(x)$ which aren't real come in conjugate pairs, it follows that c_1 is a positive real number. □

We give the zeros of $c_k(z)$ for $2 \le k \le 5$ as well as the value of the constant c_1 in Table 1 below, where $\overline{z}$ denotes the complex conjugate of z.

k	The zeros of $c_k(z)$	The constant c_1
2	1.61803, −0.61803	0.44721
3	1.83928, $r_1 = -0.41964 + 0.60629i$, $\overline{r_1}$	0.18280
4	1.92756, −0.77480, $r_1 = -0.07637 + 0.81470i$, $\overline{r_1}$	0.07907
5	1.96594, $r_1 = 0.19537 + 0.84885i$, $r_2 = -0.67835 + 0.45853i$, $\overline{r_1}$, $\overline{r_2}$	0.03601

Table 1: The zeros of $c_k(z)$ and the constant c_1.

The next lemma concerns the location of the positive zero of the k-th derivative of $f_{n,k}(x)$.

Lemma 5.3. *Suppose $k \ge 2$ is fixed and n is odd. Let s_n $(= s_{n,k})$ be the zero of $f_{n,k}(x)$ on $[1, \infty)$, where $f_{n,k}(x)$ is given by (3.1), and let t_n $(= t_{n,k})$ be the positive zero of the k-th derivative of $f_{n,k}(x)$. Let λ be the positive zero of $c_k(x)$. Then we have*

(i) $t_n < s_n$ for all odd n, and

(ii) $t_n \to \lambda$ as n odd increases without bound.

Proof. Suppose k is even, the proof when k is odd being similar. Then $f_{n,k}$ is given by (3.2) above. Throughout the following proof, n will always represent an odd integer and $f = f_{n,k}$. Recall from Lemma 3.3 that f has exactly one zero on the interval $[1, \infty)$.

(i) By Descartes' rule of signs, the polynomial $f^{(k)}(x)$ has one positive real zero t_n. If $t_n < 1 \le s_n$, then we are done, so let us assume $t_n \ge 1$. The condition $t_n \ge 1$, or equivalently $f^{(k)}(1) \geq 0$, then implies $n \geq 3$, and thus $f(1) > 0$. (Indeed, $t_n \geq 1$ for all n sufficiently large since $a_{n+k} \sim c_1\lambda^{n+k}$, with $\lambda > 1$.)

Now observe that $f^{(k)}(1) \geq 0$ implies $f^{(i)}(1) > 0$ for $1 \leq i \leq k-1$, as the proof of Lemma 3.2 above shows in fact that $f^{(i)}(1) \leq 0$ implies $f^{(i+1)}(1) < 0$. Since $f^{(i)}(1) > 0$ for $0 \leq i \leq k-1$ and $f^{(k)}(1) \geq 0$, it follows that each of the polynomials $f(x), f'(x), \ldots, f^{(k)}(x)$ has exactly one zero on $[1, \infty)$ since $f^{(k+1)}(x) < 0$ for all $x \geq 1$. Furthermore, the zero of $f^{(i)}(x)$ on $[1, \infty)$ is strictly larger than the zero of $f^{(i+1)}(x)$ on $[1, \infty)$ for $0 \leq i \leq k-1$. In particular, the zero of $f(x)$ is strictly larger than the zero of $f^{(k)}(x)$, which establishes the first statement.

(ii) Let us assume n is large enough to ensure $t_n \geq 1$. Note that

$$\frac{f^{(k)}(x)}{k!} = -\binom{n+k}{k}x^n - \binom{n+k+1}{k}x^{n+1} + a_{n,k}$$

so that

$$-2\binom{n+k+1}{k}x^{n+1} + a_{n,k} \leq \frac{f^{(k)}(x)}{k!} \leq -2\binom{n+k}{k}x^n + a_{n,k}, \qquad x \geq 1. \quad (5.2)$$

Setting $x = t_n$ in (5.2), and rearranging, then gives

$$\left(\frac{a_{n+k}}{2\binom{n+k+1}{k}}\right)^{1/(n+1)} \leq t_n \leq \left(\frac{a_{n+k}}{2\binom{n+k}{k}}\right)^{1/n}. \quad (5.3)$$

The second statement then follows from letting n tend to infinity in (5.3) and noting $\lim_{n\to\infty}(a_{n+k})^{1/n} = \lambda$ (as $a_{n+k} \sim c_1\lambda^{n+k}$, by Lemma 5.2). □

We will also need the following formula for an expression involving the zeros of $c_k(x)$.

Lemma 5.4. *If $\alpha_1 = \lambda, \alpha_2, \ldots, \alpha_k$ are the zeros of $c_k(x)$, then*

$$\sum_{j=0}^{k-1}(-1)^j\lambda^{k-j-1}\mathcal{S}_j\{\alpha_2, \alpha_3, \ldots, \alpha_k\}$$
$$= \frac{k\lambda^{k+2} - (2k-1)\lambda^{k+1} - (k-1)\lambda^k + 2k\lambda^{k-1} - \lambda - 1}{(\lambda-1)^2(\lambda+1)}, \quad (5.4)$$

where $\mathcal{S}_j\{\alpha_2, \alpha_3, \ldots, \alpha_k\}$ denotes the j-th symmetric function in the quantities $\alpha_2, \alpha_3, \ldots, \alpha_k$ if $1 \leq j \leq k-1$, with $\mathcal{S}_0\{\alpha_2, \alpha_3, \ldots, \alpha_k\} := 1$.

Proof. Let us assume k is even, the proof in the odd case being similar. First note that

$$(-1)^{i+1} = \mathcal{S}_i\{\alpha_1, \alpha_2, \ldots, \alpha_k\} = \mathcal{S}_i\{\alpha_2, \ldots, \alpha_k\} + \lambda\mathcal{S}_{i-1}\{\alpha_2, \ldots, \alpha_k\}, \quad 1 \leq i \leq k,$$

which gives the recurrences

$$\mathcal{S}_{2r}\{\alpha_2, \ldots, \alpha_k\} = -1 - \lambda\mathcal{S}_{2r-1}\{\alpha_2, \ldots, \alpha_k\}, \qquad 1 \leq r \leq (k-2)/2, \quad (5.5)$$

and

$$\mathcal{S}_{2r+1}\{\alpha_2, \ldots, \alpha_k\} = 1 - \lambda \mathcal{S}_{2r}\{\alpha_2, \ldots, \alpha_k\}, \qquad 0 \le r \le (k-2)/2. \tag{5.6}$$

Iterating (5.5) and (5.6) yields

$$\begin{aligned}\mathcal{S}_{2r}\{\alpha_2, \ldots, \alpha_k\} &= -(1 + \lambda + \cdots + \lambda^{2r-1}) + \lambda^{2r} \\ &= -\frac{1 - 2\lambda^{2r} + \lambda^{2r+1}}{1-\lambda}, \quad 1 \le r \le (k-2)/2,\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}\mathcal{S}_{2r+1}\{\alpha_2, \ldots, \alpha_k\} &= (1 + \lambda + \cdots + \lambda^{2r}) - \lambda^{2r+1} \\ &= \frac{1 - 2\lambda^{2r+1} + \lambda^{2r+2}}{1-\lambda}, \quad 0 \le r \le (k-2)/2.\end{aligned} \tag{5.8}$$

Note that (5.7) also holds in the case when $r = 0$.

By (5.7) and (5.8), we then have

$$\begin{aligned}&\sum_{j=0}^{k-1} (-1)^j \lambda^{k-j-1} \mathcal{S}_j\{\alpha_2, \alpha_3, \ldots, \alpha_k\} \\ &= -\sum_{r=0}^{\frac{k}{2}-1} \lambda^{k-2r-1}\left(\frac{1 - 2\lambda^{2r} + \lambda^{2r+1}}{1-\lambda}\right) - \sum_{r=0}^{\frac{k}{2}-1} \lambda^{k-2r-2}\left(\frac{1 - 2\lambda^{2r+1} + \lambda^{2r+2}}{1-\lambda}\right) \\ &= \frac{1}{\lambda-1}\sum_{r=0}^{\frac{k}{2}-1}(\lambda^{k-2r-1} - 2\lambda^{k-1} + \lambda^k) + \frac{1}{\lambda-1}\sum_{r=0}^{\frac{k}{2}-1}(\lambda^{k-2r-2} - 2\lambda^{k-1} + \lambda^k) \\ &= \frac{\lambda}{\lambda-1}\left(\frac{\lambda^k - 1}{\lambda^2 - 1}\right) + \frac{1}{\lambda-1}\left(\frac{\lambda^k - 1}{\lambda^2 - 1}\right) - \frac{2k\lambda^{k-1}}{\lambda-1} + \frac{k\lambda^k}{\lambda-1},\end{aligned}$$

which gives (5.4). □

We now can prove the main result of this section.

Theorem 5.5. *Suppose $k \ge 2$ and n is odd. Let r_n $(= r_{n,k})$ denote the real zero of the polynomial $P_{n,k}(x)$ defined by (1.2) above. Then $r_n \to -\lambda$ as $n \to \infty$.*

Proof. Let n denote an odd integer throughout. We first consider the case when k is even. Equivalently, we show that $s_n \to \lambda$ as $n \to \infty$, where s_n denotes the zero of $f_{n,k}(x)$ on the interval $[1, \infty)$. By Lemma 5.3, we have $t_n < s_n$ for all n with $t_n \to \lambda$ as $n \to \infty$, where t_n is the positive zero of the k-th derivative of $f_{n,k}(x)$. So it is enough to show $s_n < \lambda$ for all n sufficiently large, i.e., $f_{n,k}(\lambda) < 0$.

By Lemma 5.2, we have

$$f_{n,k}(\lambda) = -\lambda^{n+k}(1+\lambda) + a_{n,k}\lambda^k + \sum_{r=0}^{k-1}(-1)^r a_{n+r}\lambda^{k-r-1}$$

$$\sim -\lambda^{n+k}(1+\lambda) + c_1\lambda^{n+2k} + \sum_{r=0}^{k-1}(-1)^r c_1\lambda^{n+k-1}$$
$$= \lambda^{n+k}(-1-\lambda+c_1\lambda^k),$$

so that $f_{n,k}(\lambda) < 0$ for large n if $-1-\lambda+c_1\lambda^k < 0$, i.e.,

$$\lambda^k < \frac{1+\lambda}{c_1}. \tag{5.9}$$

So to complete the proof, we must show (5.9). By Lemmas 5.2 and 5.4, we have

$$\frac{1}{c_1} = \prod_{j=2}^{k}(\lambda-\alpha_j) = \sum_{j=0}^{k-1}(-1)^j\lambda^{k-j-1}\mathcal{S}_j\{\alpha_2,\alpha_3,\dots,\alpha_k\}$$
$$= \frac{k\lambda^{k+2}-(2k-1)\lambda^{k+1}-(k-1)\lambda^k+2k\lambda^{k-1}-\lambda-1}{(\lambda-1)^2(\lambda+1)},$$

so that (5.9) holds if and only

$$\lambda^k(\lambda-1)^2 < k\lambda^{k+2}-(2k-1)\lambda^{k+1}-(k-1)\lambda^k+2k\lambda^{k-1}-\lambda-1,$$

i.e.,

$$1+\lambda+k\lambda^k+(2k-3)\lambda^{k+1} < 2k\lambda^{k-1}+(k-1)\lambda^{k+2}. \tag{5.10}$$

Recall from the proof of Lemma 5.2 that $2\lambda^k = 1+\lambda^{k+1}$. Substituting $\lambda^{k+1} = \frac{\lambda+\lambda^{k+2}}{2}$,

$$\lambda^k = \frac{1+\frac{\lambda+\lambda^{k+2}}{2}}{2} = \frac{2+\lambda+\lambda^{k+2}}{4},$$

and

$$\lambda^{k-1} = \frac{\lambda^k}{\lambda} = \frac{2+\lambda+\lambda^{k+2}}{4\lambda}$$

into (5.10), and rearranging, then gives

$$\left(1-\frac{\lambda}{2}-\frac{k}{\lambda}\right)+\frac{5k\lambda}{4} < \lambda^{k+2}\left(\frac{k}{2\lambda}-\frac{k}{4}+\frac{1}{2}\right). \tag{5.11}$$

For (5.11), note first that $c_k(2) > 0$ as $2^k > 2^k-1 = 2^{k-1}+\cdots+1$, which implies $\lambda < 2 \le k$ and thus $1-\frac{\lambda}{2}-\frac{k}{\lambda} < 0$. So to show (5.11), it is enough to show

$$\frac{5k}{4} < \lambda^{k+1}\left(\frac{k}{2\lambda}-\frac{k}{4}+\frac{1}{2}\right). \tag{5.12}$$

For (5.12), we'll consider the cases $k=2$ and $k \ge 4$. If $k=2$, then $\lambda = \theta = \frac{1+\sqrt{5}}{2}$, so that (5.12) reduces in this case to $\frac{5}{2} < \theta^2 = \theta+1$, which is true. Now suppose

$k \geq 4$ is even. First observe that $c_k\left(\frac{5}{3}\right) < 0$, whence $\lambda > \frac{5}{3}$, as $d_k\left(\frac{5}{3}\right) > 0$ since $\left(\frac{5}{3}\right)^k\left(2-\frac{5}{3}\right) > 1$ for *all* $k \geq 3$. Thus, we have

$$\begin{aligned}\lambda^k &= \left(\lambda^{k-1}+1\right)+\lambda^{k-2}+\lambda^{k-3}+\cdots+\lambda \\ &> 2\lambda^{\frac{k-1}{2}}+\lambda^{k-2}+\lambda^{k-3}+\cdots+\lambda > 2\cdot\frac{5}{3}+\frac{5(k-2)}{3}=\frac{5k}{3}.\end{aligned}$$

So to show (5.12) when $k \geq 4$, it suffices to show

$$0 < \lambda\left(\frac{k}{2\lambda}-\frac{k}{4}+\frac{1}{2}\right)-\frac{3}{4}=\frac{k(2-\lambda)}{4}+\frac{2\lambda-3}{4},$$

which is true as $\frac{5}{3} < \lambda < 2$. This completes the proof in the even case.

If k is odd, then we proceed in a similar manner. Instead of inequality (5.9), we get

$$\lambda^k+\frac{1}{\lambda} < \frac{1+\lambda}{c_1}, \tag{5.13}$$

which is equivalent to

$$\left(1-\frac{\lambda}{2}-\frac{k}{\lambda}+\frac{(\lambda-1)^2}{\lambda}\right)+\frac{5k\lambda}{4} < \lambda^{k+2}\left(\frac{k}{2\lambda}-\frac{k}{4}+\frac{1}{2}\right). \tag{5.14}$$

Note that the sum of the first four terms on the left-hand side of (5.14) is negative since $1-\frac{k}{\lambda} < 0$ and $-\frac{\lambda}{2}+\frac{(\lambda-1)^2}{\lambda} < 0$ as $\frac{5}{3} < \lambda < 2$ for $k \geq 3$. Thus, it suffices to show (5.12) in the case when $k \geq 3$ is odd, which has already been done since the proof given above for it applies to *all* $k \geq 3$. □

$n\backslash k$	2	3	4	5
1	1	1	1	1
5	1.39118	1.59674	1.61156	1.64627
9	1.48442	1.69002	1.73834	1.77122
49	1.59187	1.80885	1.88958	1.92625
99	1.60498	1.82403	1.90856	1.94605
199	1.61151	1.83165	1.91805	1.95599
λ	1.61803	1.83928	1.92756	1.96594

Table 2: Some real zeros of $P_{n,k}(-x)$, where λ is the positive zero of $c_k(x)$.

Perhaps the proofs presented here of Theorems 1.2 and 5.5 could be generalized to show comparable results for polynomials associated with linear recurrent sequences having various non-negative real weights, though the results are not true for all linear recurrences having such weights, as can be seen numerically in the case $k = 3$. Furthermore, numerical evidence (see Table 2 below) suggests that the sequence of zeros in Theorem 5.5 decreases monotonically for all k, as is true in the $k = 2$ case (see [2, Theorem 3.1]).

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