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Polynomials whose coefficients are generalized Tribonacci numbers

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ABSTRACT

Let a_n denote the third order linear recursive sequence defined by the initial values $a_0 = a_1 = 0$ and $a_2 = 1$ and the recursion $a_n = pa_{n-1} + qa_{n-2} + ra_{n-3}$ if $n \geqslant 3$, where p, q, and r are constants. The a_n are *generalized Tribonacci* numbers and reduce to the usual *Tribonacci* numbers when $p = q = r = 1$ and to the *3-bonacci* numbers when $p = r = 1$ and $q = 0$. Let $Q_n(x) = a_2x^n + a_3x^{n-1} + \cdots + a_{n+1}x + a_{n+2}$, which we will refer to as a *generalized Tribonacci coefficient polynomial*. In this paper, we show that the polynomial $Q_n(x)$ has no real zeros if n is even and exactly one real zero if n is odd, under the assumption that p and q are non-negative real numbers with $p \geqslant \max\{1, q\}$. This generalizes the known result when $p = q = r = 1$ and seems to be new in the case when $p = r = 1$ and $q = 0$. Our argument when specialized to the former case provides an alternative proof of that result. We also show, under the same assumptions for p and q, that the sequence of real zeros of the polynomials $Q_n(x)$ when n is odd converges to the opposite of the positive zero of the characteristic polynomial associated with the sequence a_n. In the case $p = q = r = 1$, this convergence is monotonic. Finally, we are able to show the convergence in modulus of all the zeros of $Q_n(x)$ when $p \geqslant 1 \geqslant q \geqslant 0$.

1. Introduction

Let the recursive sequence $\{a_n\}_{n\geqslant 0}$ be defined by the initial values $a_0 = a_1 = 0$ and $a_2 = 1$ and the linear recurrence

$$a_n = pa_{n-1} + qa_{n-2} + ra_{n-3}, \quad n \geqslant 3. \tag{1}$$

The numbers a_n could be referred to as *generalized Tribonacci numbers*, and reduce when $p = q = r = 1$ to the usual *Tribonacci* numbers T_n (see A000073 in [10]). When $p = r = 1$ and $q = 0$, the a_n reduce to what are termed the *3-bonacci* numbers by Benjamin and Quinn [1, p. 41] (see also A000930 in [10]). The sequence a_n has been studied in enumerative combinatorics. For instance, when p, q, and r are positive integers, then a_n is seen to count the linear tilings of length $n - 2$ using squares, dominos, and trominos, where the various pieces are assigned colors in one of p, q, or r ways, respectively (see, e.g., [1, p. 36]). Explicit formulas involving binomial coefficients were derived for a_n in [9]. See also Knuth [3], who considered similar linear recurrences of arbitrary order.

Garth et al. [2] introduced the definition of the Fibonacci coefficient polynomials $p_n(x) = F_1x^n + F_2x^{n-1} + \cdots + F_nx + F_{n+1}$ and – among other things – determined the number of real zeros of $p_n(x)$. In particular, they showed that $p_n(x)$ has no real zeros if n is even and exactly one real zero if n is odd. Later, this result was extended by Mátyás [5,6] to more general second order recurrences. Mátyás and Szalay [8] showed that the same result also holds for the Tribonacci coefficient polynomials $q_n(x) = T_2x^n + T_3x^{n-1} + \cdots + T_{n+1}x + T_{n+2}$.

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If $n \geqslant 1$, then define the polynomial $Q_n(x)$ by

$$Q_n(x) = a_2x^n + a_3x^{n-1} + \cdots + a_{n+1}x + a_{n+2}. \tag{2}$$

One could refer to $Q_n(x)$ as a *generalized Tribonacci coefficient* polynomial. Note that when $p = q = r = 1$, the $Q_n(x)$ reduce to the Tribonacci coefficient polynomials $q_n(x)$. We wish to study the real zeros of $Q_n(x)$. By Proposition 2.1 below, we may assume $r = 1$ in (1) without loss of generality when studying the zeros of $Q_n(x)$.

From [7, Theorem 3.1], we know that the polynomials $Q_n(x)$, assuming p, q, and r are positive numbers, have either 0 or 2 real zeros if n is even and either 1 or 3 real zeros if n is odd. Our main result is as follows, where we show, under a general assumption for p and q, that the latter alternative in each case is never possible.

Theorem 1.1. *Let $Q_n(x)$ be defined by (2) above, and suppose p and q are non-negative real numbers with $p \geqslant \max\{1, q\}$ and $r = 1$. Then we have:*

(i) If n is even, then $Q_n(x)$ has no real zeros.
(ii) If n is odd, then $Q_n(x)$ has exactly one real zero.

We will prove Theorem 1.1 in the next two sections. Note that Theorem 1.1 follows from Lemma 2.2 and Theorem 3.4 below. We remark that our proof for Theorem 1.1, when specialized to the case $p = q = 1$, supplies an alternative proof to the one given in [8] for that case (which we attempted, unsuccessfully, to generalize). Furthermore, taking $p = 1$ and $q = 0$ provides a seemingly new result concerning the zeros of polynomials whose coefficients are 3-bonacci numbers.

We also show, under the same assumptions for p, q, and r, that the sequence of real zeros r_n of the polynomials $Q_n(x)$ for n odd converges to $-\lambda$, where λ is the positive zero of the characteristic polynomial associated with the sequence a_n (see Theorem 3.5 below). When $p = q = 1$, this provides a further result concerning the polynomials $q_n(x)$, and we note here that the comparable result for $p_n(x)$ was shown in [2]. In the final section, we show that the condition that the sequence of real zeros r_n be decreasing is equivalent to an inequality involving terms of the sequence a_n. Furthermore, when $p = q = 1$, we are able to show this inequality and thus establish the monotonicity of the real zeros of the Tribonacci coefficient polynomials $q_n(x)$ for n odd (see Theorem 4.3). Our proof of this is of a more computational nature and utilizes estimates for the roots of the equation $x^3 - x^2 - x - 1 = 0$, but its steps could in principle be performed for any given p and q. Finally, we show the convergence in modulus of all the zeros of $Q_n(x)$ as n increases without bound to the positive root of the equation $x^3 - px^2 - qx - 1 = 0$ when $p \geqslant 1 \geqslant q \geqslant 0$.

2. Preliminaries

We wish to study the real zeros of the polynomial $Q_n(x)$. By the following proposition, we may assume $r = 1$ in (1).

Proposition 2.1. *When considering the zeros of the polynomial $Q_n(x)$, there is no loss in generality if one assumes $r = 1$ in (1).*

Proof. Let $Q_n(x) = \sum_{i=0}^{n} a_{i+2}x^{n-i}$, where a_i is given by (1) above. Dividing both sides of (1) by $r^{(n-2)/3}$ (we assume $r \neq 0$), and letting

$$b_n = \frac{a_n}{r^{(n-2)/3}}, \quad p' = \frac{p}{r^{1/3}}, \quad \text{and} \quad q' = \frac{q}{r^{2/3}},$$

we obtain the recurrence

$$b_n = p'b_{n-1} + q'b_{n-2} + b_{n-3}, \quad n \geqslant 3, \tag{3}$$

with the initial values $b_0 = b_1 = 0$ and $b_2 = 1$. Let $R_n(x) = \sum_{i=0}^{n} b_{i+2}x^{n-i}$. Note that

$$Q_n(r^{1/3}x) = \sum_{i=0}^{b} a_{i+2}(r^{1/3}x)^{n-i} = \sum_{i=0}^{n} r^{i/3}b_{i+2}(r^{1/3}x)^{n-i} = r^{n/3}\sum_{i=0}^{n} b_{i+2}x^{n-i} = r^{n/3}R_n(x).$$

Therefore, $R_n(x)$ has a zero at $x = c$ if and only if $Q_n(x)$ has a zero at $x = r^{1/3}c$. $\square$

From this point onward, we will always assume $r = 1$ in (1) when considering the zeros of $Q_n(x)$. By the following lemma, we may restrict our attention to the case when $x \leqslant -1$.

Lemma 2.2. *Suppose $p \geqslant 1$ and $q \geqslant 0$ are real numbers. If $n \geqslant 1$, then the polynomial $Q_n(x)$ has no zeros on the interval $(-1, \infty)$.*

Proof. Clearly, the equation $Q_n(x) = 0$ has no roots if $x \geqslant 0$ since it has positive coefficients. Suppose $-1 < x < 0$. If n is odd, then

$$a_{2j+2}x^{n-2j} + a_{2j+3}x^{n-2j-1} > 0, \quad 0 \leqslant j \leqslant (n-1)/2,$$

since $x^{n-2j-1} > -x^{n-2j} > 0$ if $-1 < x < 0$ and since $a_{2j+3} \geqslant a_{2j+2} > 0$ as $p \geqslant 1$ and $q \geqslant 0$. This implies

$$Q_n(x) = \sum_{j=0}^{\frac{n-1}{2}} (a_{2j+2}x^{n-2j} + a_{2j+3}x^{n-2j-1}) > 0.$$

Similarly, if n is even, then

$$Q_n(x) = a_2x^n + \sum_{j=0}^{\frac{n-2}{2}} (a_{2j+3}x^{n-2j-1} + a_{2j+4}x^{n-2j-2}) > 0. \qquad \square$$

So we seek the zeros of $Q_n(x)$, where $x \leqslant -1$, equivalently, the zeros of $Q_n(-x)$, where $x \geqslant 1$. For this, it is easier to consider zeros of the polynomial $f_n(x)$ given by

$$f_n(x) := c(-x)Q_n(-x), \tag{4}$$

see [7], where

$$c(x) := x^3 - px^2 - qx - 1 \tag{5}$$

denotes the *characteristic polynomial* associated with the sequence a_n.

Performing the multiplication in (4) and using the recurrence for a_n (or referring to [7, Lemma 2.1]), we have

$$f_n(x) = (-x)^{n+3} - a_{n+3}x^2 + (a_{n+1} + qa_{n+2})x - a_{n+2}. \tag{6}$$

In the following section, we study the zeros of $f_n(x)$ when $x \geqslant 1$, and hence of $Q_n(x)$ when $x \leqslant -1$.

3. Main results

Throughout this section, we assume p and q are non-negative real numbers with $p \geqslant \max\{1, q\}$. We first consider the case when n is odd. By (6), we have

$$f_n(x) = x^{n+3} - a_{n+3}x^2 + (a_{n+1} + qa_{n+2})x - a_{n+2}$$

for n odd. The first lemma concerns the cases of $f_n(x)$ when $n = 1$ and $n = 3$.

Lemma 3.1. *The following polynomials have exactly one zero for $x \geqslant 1$:*

(i) $u(x) = x^4 - (p^2 + q)x^2 + (1 + pq)x - p$,
(ii) $v(x) = x^6 - (p^4 + 3p^2q + q^2 + 2p)x^2 + (p^3q + 2pq^2 + p^2 + 2q)x - (p^3 + 2pq + 1)$.

Furthermore, this zero is simple.

Proof.

(i) First observe that $u(1) \leqslant 0$ for all possible p and q as

$$p(p - q) + p + q = p^2 + p(1 - q) + q \geqslant 2,$$

since $p \geqslant \max\{1, q\}$, upon considering cases whether $q > 1$ or $0 \leqslant q \leqslant 1$. Note that $u(1) = 0$ if and only if $p = 1$. Next observe that $u'(1) > 0$ if and only if

$$2p^2 - pq + 2q < 5. \tag{7}$$

Also, since $u''(x) = 12x^2 - 2(p^2 + q)$, we either have

(a) $u''(x) \geqslant 0$ for all $x \geqslant 1$, or
(b) $u''(x) < 0$ for $x \in [1, s)$ for some $s > 1$, with $u''(x) \geqslant 0$ for $x \geqslant s$.

If $p = 1$, then $u'(1) > 0$, by (7), with (a) occurring, which implies $u'(x) > 0$ for all $x \geqslant 1$. Thus, there is exactly one zero on the interval $[1, \infty)$ in this case, namely, $x = 1$. It is a simple zero since $u'(1) > 0$ is non-zero.

So let us assume $p > 1$. Then $u(1) < 0$ and, to complete the proof, we consider cases depending on $u'(1)$. If $u'(1) \leqslant 0$, then $u'(x) \leqslant 0$ on the interval $[1, t]$ for some $t \geqslant 1$ and $u'(x) > 0$ on (t, ∞), when either (a) or (b) occurs. Since $u(1) < 0$ and $\lim_{x\to\infty} u(x) = \infty$, this implies $u(x)$ has exactly one zero r on $[1, \infty)$. This zero is simple as $r > t$ implies $u'(r) > 0$ is non-zero. If $u'(1) > 0$, then by (7),

$$(p^2 + q) + (p - q) + q \leqslant (p^2 + q) + p(p - q) + q < 5,$$

which implies $p^2 + q < 4$. Then $u''(1) = 12 - 2(p^2 + q) > 0$, whence $u''(x) > 0$ for $x \geqslant 1$ since $u'''(x) = 24x > 0$. Then $u'(x) > 0$ for all $x \geqslant 1$, which implies $u(x)$ has one (simple) zero on $[1, \infty)$. Note that this zero occurs at $x = p$ in all cases.

(ii) That $v(1) < 0$ follows from comparing positive and negative terms and the assumption $p \geqslant \max\{1, q\}$. If $v'(1) \leqslant 0$, then proceed as in the proof of (i) above in the comparable case. So assume $v'(1) > 0$, which may be written as

$$(2p - q)(p^3 + 2pq + 1) + (2q - 1)(p^2 + q) + 2p < 6. \tag{8}$$

Since $p \geqslant q \geqslant 0$, inequality (8) gives

$$p(p^3 + 2pq + 1) + (2q - 1)(p^2 + q) + 2p < 6,$$

which further implies

$$(p^4 + 3p^2q + q^2 + 2p) + p^2q - p^2 < 6. \tag{9}$$

By (9), we have $p^4 - p^2 < 6$, i.e., $(p^2 - 3)(p^2 + 2) < 0$. Then $p^2 < 3$, together with (9), gives

$$p^4 + 3p^2q + q^2 + 2p < 6 + p^2 < 6 + 3 < 15,$$

which yields $v''(1) > 0$. Then $v''(x) > 0$ for all $x \geqslant 1$ so that $v'(x) > 0$ for all $x \geqslant 1$ as well. Thus, $v(x)$ also has one (simple) zero on $[1, \infty)$ in this case, which completes the proof. $\square$

Lemma 3.2. *If $n \geqslant 5$ is odd, then the polynomial $f_n(x)$ has exactly one zero on the interval $[1, \infty)$, and it is simple.*

Proof. Let $f(x) = f_n(x)$. For $n \geqslant 5$ odd, first note that

$$f(1) = 1 - a_{n+3} + a_{n+1} + (q - 1)a_{n+2} = 1 - (p - q + 1)a_{n+2} + (1 - q)a_{n+1} - a_n < 0.$$

Next observe that $f'(1) < 0$ for $n \geqslant 5$ odd, i.e.,

$$(2p - q)a_{n+2} + (2q - 1)a_{n+1} + 2a_n - (n + 3) = (pa_{n+2} + 2a_n - (n + 3)) + (p - q)a_{n+2} - (1 - 2q)a_{n+1} > 0,$$

since $pa_{n+2} + 2a_n \geqslant n + 3$ for all $n \geqslant 5$, which can be shown by induction, and since $p \geqslant \max\{1, q\}$ implies $(p - q)a_{n+2} - (1 - 2q)a_{n+1} > 0$.

Now $f''(x) = (n + 2)(n + 3)x^{n+1} - 2a_{n+3}$, which is either positive for all $x \geqslant 1$ or has one sign change, from negative to positive. Since $f'(1) < 0$ and $\lim_{x\to\infty} f'(x) = \infty$, it follows that $f'(x)$ has one sign change, from negative to positive, on the interval $(1, \infty)$. The same then holds for $f(x)$ for the same reason. If r denotes the zero of $f(x)$ on the interval $(1, \infty)$, then we have $f'(r) > 0$, whence the zero is simple. $\square$

Lemma 3.3. *If n is even, then the polynomial $f_n(x)$ has no zeros on the interval $[1, \infty)$.*

Proof. Let $f(x) = f_n(x)$. If n is even, then

$$f_n(x) = -x^{n+3} - a_{n+3}x^2 + (a_{n+1} + qa_{n+2})x - a_{n+2}.$$

Note that

$$f(1) = -1 - a_{n+3} + (q - 1)a_{n+2} + a_{n+1} = -1 - (p - q + 1)a_{n+2} + (1 - q)a_{n+1} - a_n < 0.$$

Since $p \geqslant \max\{1, q\}$, we also have for all $x \geqslant 1$,

$$\begin{aligned} f'(x) &= -(n + 3)x^{n+2} - 2a_{n+3}x + qa_{n+2} + a_{n+1} = -(n + 3)x^{n+2} + (q - 2px)a_{n+2} + (1 - 2qx)a_{n+1} - 2xa_n \\ &= -(n + 3)x^{n+2} + (pq + 1 - 2qx - 2p^2x)a_{n+1} + (q^2 - 2x - 2pqx)a_n + (q - 2px)a_{n-1} < 0, \end{aligned}$$

being the sum of negative terms (except for possibly the last, which can be zero). Thus, $f(x) < 0$ for all $x \geqslant 1$. $\square$

Theorem 3.4.

(i) If n is odd, then the polynomial $Q_n(x)$ has one zero on the interval $(-\infty, -1]$, and it is simple.
(ii) If n is even, then the polynomial $Q_n(x)$ has no zeros on $(-\infty, -1]$.

Proof. First suppose n is even. Then the polynomial $f_n(x)$ has no zeros for $x \geqslant 1$, by Lemma 3.3. Since $f_n(x) = c(-x)Q_n(-x)$, it follows that $Q_n(x)$ has no zeros for $x \leqslant -1$. Note further that $c(-x)$ has no zeros for $x \geqslant 1$.

Now assume n is odd. By Lemma 3.1, the polynomials $f_1(x)$ and $f_3(x)$ have one zero for $x \geqslant 1$, and it is simple. By Lemma 3.2, the same holds true of $f_n(x)$ for all $n \geqslant 5$ odd. By (4) and the fact that $c(-x)$ has no zeros for $x \geqslant 1$, it follows that $Q_n(-x)$ has one zero for $x \geqslant 1$. Thus, $Q_n(x)$ has one (simple) zero for $x \leqslant -1$ when n is odd, which completes the proof. $\square$

By Descartes' rule of signs, the equation $c(x) = 0$ has one positive root, which we will denote by λ.

Theorem 3.5. *Let r_n denote the real zero of $Q_n(x)$, where $n \geqslant 1$ is odd. Then $r_n \to -\lambda$ as $n \to \infty$.*

Proof. In what follows, n will denote an odd integer, except in one place where stated otherwise. Equivalently, we show that $s_n \to \lambda$ as $n \to \infty$, where s_n denotes the real zero of $f_n(x)$. Recall that

$$f_n(x) = x^{n+3} - a_{n+3}x^2 + (a_{n+1} + qa_{n+2})x - a_{n+2}.$$

Let i_n denote the positive inflection point of $f_n(x)$. We will show

$$i_n < s_n < \lambda \tag{10}$$

for all n sufficiently large, and

$$\lim_{n\to\infty} i_n = \lambda. \tag{11}$$

Taken together, (10) and (11) imply $s_n \to \lambda$, as desired.

To do so, first recall that for all non-negative integers n, we have

$$a_n = c_1\lambda^n + c_2\alpha^n + c_3\beta^n \tag{12}$$

for some constants c_1, c_2, c_3, where the equation $c(x) = x^3 - px^2 - qx - 1 = 0$ has distinct roots λ, α, and β (with λ the positive root). Then the roots α and β are either distinct negative real numbers, or a conjugate pair of complex numbers that are not real. In the former case, note that $-1 < \alpha, \beta < 0$ since $c(x)$ has no zeros for $x \leqslant -1$. In the latter case, note that $|\alpha|^2 = |\beta|^2 = \alpha\beta = \frac{1}{\lambda} < 1$ (observe that $\lambda > 1$ since $c(1) = -p - q < 0$), which implies $|\alpha|, |\beta| < 1$. In both cases, we then have $a_n \sim c_1\lambda^n$ for large n. If $c(x) = 0$ has a repeated (necessarily negative) root α, then

$$a_n = b_1\lambda^n + b_2\alpha^n + b_3n\alpha^n, \tag{13}$$

where b_1, b_2, and b_3 are constants and $-1 < \alpha < 0$, which implies $a_n \sim b_1\lambda^n$ for large n.

Taking $n = 0, n = 1$ and $n = 2$ in (12), recalling $a_0 = a_1 = 0$ and $a_2 = 1$ and solving the resulting system gives

$$c_1 = \frac{1}{(\lambda - \alpha)(\lambda - \beta)}, \tag{14}$$

which is always seen to be a positive real number (note that in the case when α and β are not real, c_1 is the product of a non-zero complex number with its conjugate). Performing the same calculation using (13) shows that formula (14) also holds in the case when $\alpha = \beta$.

Next, observe that the positive point of inflection i_n when n is odd is given by

$$i_n = \left(\frac{2a_{n+3}}{(n+2)(n+3)}\right)^{1/(n+1)}.$$

Since $a_{n+3} \sim c_1\lambda^{n+3}$, with c_1 a positive constant, we have $\lim_{n\to\infty}(a_{n+3})^{1/(n+1)} = \lambda$ and (11) follows. Furthermore, from the proof of Lemma 3.2 above, we see that if $n \geqslant 5$, then $i_n < s_n$, since both $f_n(1) < 0$ and $f_n'(1) < 0$ in this case.

To complete the proof, we must show that $s_n < \lambda$ for all n sufficiently large. To do so, we show that $f_n(\lambda) > 0$ for all large n as $f_n(1) < 0$ if $n \geqslant 3$. Note first that

$$f_n(\lambda) = \lambda^{n+3} - a_{n+3}\lambda^2 + (a_{n+1} + qa_{n+2})\lambda - a_{n+2} \sim \lambda^{n+3}\left(1 - c_1\lambda^2 + \frac{c_1}{\lambda} + qc_1 - \frac{c_1}{\lambda}\right) = \lambda^{n+3}(1 + qc_1 - c_1\lambda^2),$$

as the remaining terms in $f_n(\lambda)$ coming from the Binet formulas are bounded by M for some positive constant M for all n. Thus, if it is the case that

$$1 + qc_1 - c_1\lambda^2 > 0, \tag{15}$$

then $f_n(\lambda)$ will be positive for all large n.

Since $-q = \lambda\alpha + \alpha\beta + \beta\lambda$, we see from (14) that (15) is equivalent to

$$1 > c_1(\lambda^2 - q) = \frac{\lambda^2 + (\alpha+\beta)\lambda + \alpha\beta}{\lambda^2 - (\alpha+\beta)\lambda + \alpha\beta} = \frac{\lambda^2 + (\alpha+\beta)\lambda + \frac{1}{\lambda}}{\lambda^2 - (\alpha+\beta)\lambda + \frac{1}{\lambda}}.$$

Note that $c(p) = -pq - 1 < 0$, which implies $\lambda > p$. Then $p = \lambda + \alpha + \beta$ implies $\alpha + \beta < 0$, which gives the last inequality and hence (15). This completes the proof. $\square$

Table 1
Some real zeros of $Q_n(-x)$, where λ is the positive zero of $c(x)$.

$n \setminus (p,q)$	(1,0)	(1,1)	(2,1)	(10,1)
$n = 1$	1	1	2	10
$n = 5$	1.34714	1.59674	2.37536	10.07251
$n = 9$	1.39756	1.69002	2.44325	10.08699
$n = 49$	1.45131	1.80885	2.52594	10.10436
$n = 99$	1.45840	1.82403	2.53637	10.10653
$n = 199$	1.46197	1.83165	2.54159	10.10762
λ	1.46557	1.83928	2.54681	10.10871

In Table 1 above, we illustrate Theorem 3.5 for four cases of (p,q).

4. Further results

We have shown that the polynomial $Q_n(x)$ given by (2) above has one real zero when n is odd and no real zeros when n is even, under the assumption that p and q are non-negative real numbers satisfying $p \geqslant \max\{1,q\}$. While the problem of determining necessary and sufficient conditions on positive numbers p and q so as to assure that $Q_n(x)$ has one real zero when n is odd and no real zeros when n is even seems to be more difficult and may not be possible, we remark here that we did find examples where p and q were both positive with $Q_n(x)$ possessing three real zeros. For example, when $n = 19, p = 5 \cdot 10^{-8}$ and $q = .125$, then $Q_n(x)$ has zeros of approximately $-.491, -.763$, and $-.858$.

In the odd case, we have shown further that the sequence of real zeros r_n of $Q_n(x)$ converges to $-\lambda$, where λ is the positive root of the equation $c(x) = 0$. In this section, we consider the monotonicity of the sequence of real zeros r_n of $Q_n(x)$ when n is odd. This monotonicity is equivalent to the following inequality concerning the sequence a_n.

Proposition 4.1. *Suppose $n \geqslant 3$ is odd. Then $r_n < r_{n-2}$ if and only if*

$$(a_{n+2})^{n+1} - pa_{n+2}(a_{n+1})^{n+1} - a_n(a_{n+1})^{n+1} > 0. \tag{16}$$

Proof. First note that

$$Q_n(x) - x^2 Q_{n-2}(x) = a_{n+2} + xa_{n+1}, \quad n \geqslant 3. \tag{17}$$

Then $r_n < r_{n-2}$ if and only $Q_n(r_{n-2}) > 0$, i.e.,

$$r_{n-2} > -\frac{a_{n+2}}{a_{n+1}}, \quad n \geqslant 3, \tag{18}$$

upon taking $x = r_{n-2}$ in (17). Observe that (18) is equivalent to

$$Q_{n-2}(-b_{n+2}) < 0, \tag{19}$$

where $b_n := \frac{a_n}{a_{n-1}}$. Substituting $x = -b_{n+2}$ into

$$Q_{n-2}(x) = \frac{f_{n-2}(-x)}{c(x)}$$

and noting that $-b_{n+2} < -1$ implies $c(-b_{n+2}) < 0$, we see that (19) is equivalent to

$$f_{n-2}(b_{n+2}) > 0. \tag{20}$$

Now

$$\begin{aligned}(a_{n+1})^{n+1} f_{n-2}(b_{n+2}) &= (a_{n+2})^{n+1} - a_{n+2}^2(a_{n+1})^n + a_{n+2}(a_{n-1} + qa_n)(a_{n+1})^n - a_n(a_{n+1})^{n+1} \\ &= (a_{n+2})^{n+1} - a_{n+2}(a_{n+1})^n(a_{n+2} - a_{n-1} - qa_n) - a_n(a_{n+1})^{n+1} \\ &= (a_{n+2})^{n+1} - pa_{n+2}(a_{n+1})^{n+1} - a_n(a_{n+1})^{n+1},\end{aligned}$$

whence (16) follows from (20). □

Remark: In a manner similar to the proof of Theorem 3.5 above, we are able to show that (16) holds for all odd n sufficiently large. That is, the sequence of zeros r_n at some point is decreasing.

Though we do not have a complete proof of inequality (16) for all p and q, we suspect that it does hold in all cases. We do have a proof in the case $p = q = 1$ corresponding to the Tribonacci coefficient polynomials $q_n(x)$, which we now present. As can be seen, the steps in this proof could be applied to sequences a_n for any given non-negative p and q such that $p \geqslant \max\{1,q\}$, especially for p and q near 1. See [2, Theorem 3.1] for the comparable result concerning the Fibonacci coefficient polynomials $p_n(x)$.

To prove (16) in the case $p = q = 1$, we start by estimating the zeros, which we will denote by θ, μ, and ν, of the characteristic polynomial $c(x) = x^3 - x^2 - x - 1$. By any numerical method, we have

$$\theta \approx 1.839, \quad \mu \approx -0.419 + 0.606i, \quad \nu = \overline{\mu}.$$

Recall the constants c_1, c_2 and c_3 appearing in the proof of Theorem 3.5 above, and note that they are given in this case by

$$c_1 = \frac{1}{(\theta-\mu)(\theta-\nu)} \approx 0.182, \quad c_2 = \frac{1}{(\mu-\theta)(\mu-\nu)}, \quad c_3 = \frac{1}{(\nu-\theta)(\nu-\mu)}.$$

We remark that in the estimates of the roots θ, μ and ν, and of the constant c_1, we have truncated the decimal expansion at three places.

We will need the following estimate of the ratio of two adjacent Tribonacci numbers.

Lemma 4.2. *If $n \geqslant 7$, then*

$$\left|\frac{T_{n+2}}{T_{n+1}} - \theta\right| < \frac{2}{c_1\theta^{n+1}}. \tag{21}$$

Proof. By the Binet formula for T_n, we have for all $n \geqslant 1$,

$$\left|\frac{T_{n+2}}{T_{n+1}} - \theta\right| = \frac{|T_{n+2} - T_{n+1}\theta|}{|T_{n+1}|} = \frac{|c_2(\mu-\theta)\mu^{n+1} + c_3(\nu-\theta)\nu^{n+1}|}{|T_{n+1}|} \leqslant \frac{\left|\frac{\mu^{n+1}}{\mu-\nu}\right| + \left|\frac{\nu^{n+1}}{\nu-\mu}\right|}{|T_{n+1}|} \leqslant \frac{\frac{|\mu|^{n+1}}{|\mathrm{Im}(\mu)|}}{c_1\theta^{n+1} - \left|\frac{\mu^{n+1}}{(\mu-\nu)(\mu-\theta)} + \frac{\nu^{n+1}}{(\nu-\mu)(\nu-\theta)}\right|}.$$

To bound the last expression, note that all for $n \geqslant 1$, we have

$$\frac{|\mu^{n+1}|}{|\mathrm{Im}(\mu)|} < \frac{|\mu|}{|\mathrm{Im}(\mu)|} < \frac{5}{4}$$

and

$$\begin{aligned}\left|\frac{\mu^{n+1}}{(\mu-\nu)(\mu-\theta)} + \frac{\nu^{n+1}}{(\nu-\mu)(\nu-\theta)}\right| &= \left|\frac{\theta(\nu^{n+1} - \mu^{n+1}) - \mu\nu(\nu^n - \mu^n)}{(\mu-\nu)(\theta-\mu)(\theta-\nu)}\right| \leqslant \frac{2|\theta||\mu|^{n+1}}{|(\theta-\mu)(\theta-\nu)| \cdot |\mu-\nu|} + \frac{2|\mu\nu||\mu|^n}{|(\theta-\mu)(\theta-\nu)| \cdot |\mu-\nu|} \\ &< \frac{|\mu|^{n+1}}{|\mathrm{Im}(\mu)|} + \frac{|\nu|^{n+1}}{|\mathrm{Im}(\nu)|} = \frac{2|\mu|^{n+1}}{|\mathrm{Im}(\mu)|} < 2 \cdot \frac{5}{4} = \frac{5}{2}.\end{aligned}$$

Thus, we

$$\left|\frac{T_{n+2}}{T_{n+1}} - \theta\right| < \frac{\frac{5}{4}}{c_1\theta^{n+1} - \frac{5}{2}} < \frac{2}{c_1\theta^{n+1}},$$

which implies (21), provided the second inequality holds for all $n \geqslant 7$. And it is seen to hold since $n \geqslant 7$, $.18 < c_1 < .2$, and $\theta > 1.8$ imply

$$\frac{5c_1\theta^{n+1}}{4} + 5 < \frac{\theta^{n+1}}{4} + 5 < (.36)\theta^{n+1} < 2c_1\theta^{n+1}. \qquad \square$$

We now prove the monotonicity of the real zeros of the Tribonacci coefficient polynomials $q_n(x)$ for n odd.

Theorem 4.3. *If $n \geqslant 2$, then*

$$T_{n+2} + T_n < \left(\frac{T_{n+2}}{T_{n+1}}\right)^{n+1}. \tag{22}$$

Thus, the sequence of real zeros r_n of the polynomials $q_n(x) = \sum_{i=0}^{n} T_{i+2}x^{n-i}$ for n odd is strictly decreasing.

Proof. By Proposition 4.1, we need only show the first statement. One may verify (22) directly for $2 \leqslant n \leqslant 6$. By Lemma 4.2, we have for all $n \geqslant 7$,

$$\frac{T_{n+2}}{T_{n+1}} = \theta + \left(\frac{T_{n+2}}{T_{n+1}} - \theta\right) \geqslant \theta - \left|\frac{T_{n+2}}{T_{n+1}} - \theta\right| > \theta - \frac{2}{c_1\theta^{n+1}},$$

which implies

$$\left(\frac{T_{n+2}}{T_{n+1}}\right)^{n+1} > \left(\theta - \frac{2}{c_1\theta^{n+1}}\right)^{n+1} = \theta^{n+1}\left(1 - \frac{2}{c_1\theta^{n+2}}\right)^{n+1} > \theta^{n+1}\left(1 - \frac{7}{(1.8)^{n+1}}\right)^{n+1}.$$

Note that the function $h(x)$ given by

$$h(x) = \left(1 - \frac{7}{(1.8)^x}\right)^x, \quad x \geqslant 8,$$

is increasing since

$$\frac{d}{dx}\ln(h(x)) = \ln\left(1 - \frac{7}{(1.8)^x}\right) + \frac{7x\ln(1.8)}{(1.8)^x - 7} > -\frac{7}{(1.8)^x} - \frac{49}{(1.8)^{2x}} + \frac{7x\ln(1.8)}{(1.8)^x - 7} > -\frac{7}{(1.8)^x} - \frac{49}{3(1.8)^x} + \frac{7x\ln(1.8)}{(1.8)^x - 7} > 0$$

for all $x \geqslant 8$, where we have used the inequality $\ln(1-x) > -x - x^2$ if $0 < x < \frac{1}{2}$. Thus, we have

$$\left(1 - \frac{7}{(1.8)^{n+1}}\right)^{n+1} > .59, \quad n \geqslant 7,$$

which implies

$$\left(\frac{T_{n+2}}{T_{n+1}}\right)^{n+1} > (.59)\theta^{n+1}, \quad n \geqslant 7. \tag{23}$$

On the other hand, we have for all $n \geqslant 1$,

$$\begin{aligned} T_{n+2} + T_n &= c_1\theta^{n+1}\left(\theta + \frac{1}{\theta}\right) + c_2\mu^{n+2} + c_3\nu^{n+2} + c_2\mu^n + c_3\nu^n < c_1\theta^{n+1}\left(\theta + \frac{1}{\theta}\right) + \frac{|\mu|^n(1+|\mu|^2)}{2|\mathrm{Im}(\mu)|} + \frac{|\nu|^n(1+|\nu|^2)}{2|\mathrm{Im}(\nu)|} \\ &= c_1\theta^{n+1}\left(\theta + \frac{1}{\theta}\right) + \frac{|\mu|^n(1+|\mu|^2)}{|\mathrm{Im}(\mu)|} < (.19)\theta^{n+1}\left(1.9 + \frac{1}{1.8}\right) + 2\cdot\frac{5}{4} < (.47)\theta^{n+1} + \frac{5}{2}. \end{aligned} \tag{24}$$

Combining (23) and (24) yields

$$T_{n+2} + T_n < (.47)\theta^{n+1} + \frac{5}{2} < (.59)\theta^{n+1} < \left(\frac{T_{n+2}}{T_{n+1}}\right)^{n+1}, \quad n \geqslant 7,$$

where the middle inequality follows from $\theta > 1.8$. This gives (22) and completes the proof. □

As can be seen, the steps in the proof above may be performed for any *given* p and q, once one has suitably estimated the zeros of $c(x)$ and the constant c_1, though we do not have a unified proof which applies to all p and q. We remark that for some p and q, one might need to verify all of the cases of (16) up to some fairly large n. But for p and q both near 1 at least, this n should be relatively small, as the zeros of $c(x)$ and the constant c_1 are continuous functions of the parameters p and q.

We remind the reader of the following version of Rouché's Theorem (see, e.g., [4]).

Theorem 4.4 (*Rouché*). *If $p(z)$ and $q(z)$ are analytic interior to a simple closed Jordan curve $\mathcal{C}$, and are continuous on $\mathcal{C}$, with*

$$|p(z)| > |q(z)|, \quad z \in \mathcal{C},$$

then the functions $p(z) - q(z)$ and $p(z)$ have the same number of zeros interior to $\mathcal{C}$.

We conclude with the following general result which concerns all of the zeros of $Q_n(x)$. See [2, Theorem 3.2] for the analogous result for the Fibonacci coefficient polynomials $p_n(x)$.

Theorem 4.5. *Suppose $p \geqslant 1 \geqslant q \geqslant 0$ are given real numbers. Then all of the zeros of the associated polynomials $Q_n(x) = \sum_{i=0}^{n} a_{i+2}x^{n-i}$ converge in modulus as $n \to \infty$ to λ, the real zero of $c(x) = x^3 - px^2 - qx - 1$.*

Proof. We first note that the polynomial $c(x)$ has one real zero, which follows from the assumptions on p and q, and this zero, which we will denote by λ, is greater than one since $c(1) < 0$. The other two zeros of $c(x)$ are complex conjugates of modulus $\frac{1}{\sqrt{\lambda}} < 1$. From the Binet formula, we see that $a_n \approx k\lambda^n$ when n is large for some positive constant k.

First suppose $z \in \mathbb{C}$, where $|z| = a$ and $a > \lambda$ is fixed. We then have

$$|z^{n+3}| = a^{n+3} > a_{n+3}|z|^2 + (a_{n+1} + qa_{n+2})|z| + a_{n+2} \geqslant |a_{n+3}z^2 - (a_{n+1} + qa_{n+2})z + a_{n+2}|$$

for all n sufficiently large. By Rouche's Theorem, it follows for such n that

$$f_n(z) = z^{n+3} - a_{n+3}z^2 + (a_{n+1} + qa_{n+2})z - a_{n+2}$$

has the same number of zeros inside the disk $|z| < a$ as does $g(z) = z^{n+3}$, assuming for now that n is odd. Since $f_n(z) = c(-z)Q_n(-z)$ is a polynomial of degree $n+3$, it follows that all of the zeros of $Q_n(z)$ lie within the disk $|z| < a$ for such n.

Now observe that

$$\lim_{x\to\lambda^-}(\lambda x^2 - (q+1)x - 1) = \lambda^3 - (q+1)\lambda - 1 > \lambda^3 - p\lambda^2 - q\lambda - 1 = 0,$$

which implies $\lambda x^2 - (q+1)x - 1 > 0$ on the interval (ω, λ) for some ω, where $1 < \omega < \lambda$. Note that the constant ω depends only on p and q (through λ).

Next suppose $|z| = b$, with b fixed and $\omega < b < \lambda$. Then we have

$$|a_{n+3}z^2 - (a_{n+1} + qa_{n+2})z + a_{n+2}| > a_{n+3}b^2 - (bq + b + 1)a_{n+2} > b^{n+3} = |z^{n+3}|$$

for all n sufficiently large since $a_n \approx k\lambda^n$ implies

$$a_{n+3}b^2 - (bq + b + 1)a_{n+2} \approx k\lambda^{n+2}(\lambda b^2 - (q+1)b - 1),$$

with $\lambda b^2 - (q+1)b - 1 > 0$ as $\omega < b < \lambda$. By Rouche's Theorem, it follows for such n that $f_n(z)$ has the same number of zeros inside the disk $|z| < b$ as does $h(z) = a_{n+3}z^2 - (a_{n+1} + qa_{n+2})z + a_{n+2}$. From the assumptions on p and q, we have $(a_{n+1} + qa_{n+2})^2 - 4a_{n+3}a_{n+2} < 0$, and thus the zeros of $h(z)$ are not real. If r denotes one of these zeros, then

$$|r|^2 = r\bar{r} = \frac{a_{n+2}}{a_{n+3}} < 1,$$

which implies both zeros of $h(z)$ lie within the disk $|z| < b$ and thus $f_n(z)$ has two zeros within this disk. Since $f_n(z) = c(-z)Q_n(-z)$ and since the two complex zeros of the polynomial $c(-z)$ have modulus less than one and thus lie within $|z| < b$, it follows that $Q_n(z)$ has no zeros within $|z| < b$ for n sufficiently large. Thus, given a and b, where $\omega < b < \lambda < a$, we see that $Q_n(z)$ has all of its zeros within the annulus $b \leqslant |z| < a$ for n large enough. Allowing a and b to approach λ completes the proof in the odd case. The even case follows in a similar manner, upon considering $f_n(-z)$ in place of $f_n(z)$. $\square$

The above result will probably hold for other p and q as well, though we do not have a proof which covers cases when $q > 1$. Taking $p = q = 1$ in the prior theorem gives the following result concerning Tribonacci coefficient polynomials.

Corollary 4.6. *All of the zeros of the polynomials* $q_n(z) = \sum_{i=0}^{n} T_{i+2}x^{n-i}$ *converge in modulus as* $n \to \infty$ *to* $\theta \approx 1.839$, *the real root of the equation* $x^3 - x^2 - x - 1 = 0$.

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References

[1] A.T. Benjamin, J.J. Quinn, Proofs that Really Count: The Art of Combinatorial Proof, Mathematical Association of America, 2003.
[2] D. Garth, D. Mills, P. Mitchell, Polynomials generated by the Fibonacci sequence, J. Integer Seq. 10 (2007) (Art. 07.6.8.).
[3] D.E. Knuth, The Art of Computer Programming: Sorting and Searching, Addison-Wesley, vol. 3, 1973.
[4] M. Marden, Geometry of Polynomials, Mathematical Surveys, second ed., vol. 3, American Mathematical Society, 1966.
[5] F. Mátyás, On the generalization of the Fibonacci-coefficient polynomials, Ann. Math. Inform. 34 (2007) 71–75.
[6] F. Mátyás, Further generalizations of the Fibonacci-coefficient polynomials, Ann. Math. Inform. 35 (2008) 123–128.
[7] F. Mátyás, K. Liptai, J.T. Tóth, F. Filip, Polynomials with special coefficients, Ann. Math. Inform. 37 (2010) 101–106.
[8] F. Mátyás, L. Szalay, A note on Tribonacci-coefficient polynomials, Ann. Math. Inform. 38 (2011) 95–98.
[9] A.G. Shannon, Iterative formulas associated with generalized third order recurrences, SIAM J. Appl. Math. 23 (3) (1972) 364–368.
[10] N.J. Sloane, The on-line encyclopedia integer seq., 2010. <http://oeis.org>.