RESTRICTED PERMUTATIONS, CONTINUED FRACTIONS, AND CHEBYSHEV POLYNOMIALS

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Abstract. Let $f_n^r(k)$ be the number of 132-avoiding permutations on n letters that contain exactly r occurrences of $12\dots k$, and let $F_r(x;k)$ and $F(x,y;k)$ be the generating functions defined by $F_r(x;k)=\sum_{n\geqslant 0} f_n^r(k)x^n$ and $F(x,y;k)=\sum_{r\geqslant 0} F_r(x;k)y^r$. We find an explicit expression for $F(x,y;k)$ in the form of a continued fraction. This allows us to express $F_r(x;k)$ for $1\leqslant r\leqslant k$ via Chebyshev polynomials of the second kind.

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1. Introduction

Let $[p] = \{1, \dots, p\}$ denote a totally ordered alphabet on p letters, and let $\alpha = (\alpha_1, \dots, \alpha_m) \in [p_1]^m$, $\beta = (\beta_1, \dots, \beta_m) \in [p_2]^m$. We say that α is *order-isomorphic* to β if for all $1 \leqslant i < j \leqslant m$ one has $\alpha_i < \alpha_j$ if and only if $\beta_i < \beta_j$. For two permutations $\pi \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$, an *occurrence* of τ in π is a subsequence $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n$ such that $(\pi_{i_1}, \dots, \pi_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called the *pattern*. We say that π *avoids* τ, or is τ-*avoiding*, if there is no occurrence of τ in π. The set of all τ-avoiding permutations of all possible sizes including the empty permutation is denoted $\mathfrak{S}(\tau)$. Pattern avoidance proved to be a useful language in a variety of seemingly unrelated problems, from stack sorting [5] to singularities of Schubert varieties [6]. A complete study of pattern avoidance for the case $\tau \in \mathfrak{S}_3$ is carried out in [11]. For the case $\tau \in \mathfrak{S}_4$ see [14, 11, 12, 1].

A natural generalization of pattern avoidance is the restricted pattern inclusion, when a prescribed number of occurrences of τ in π is required. Papers [8] and [3] contain simple expressions for the number of permutations containing exactly one 123 and 132 patterns, respectively. The main result of [B2] is that the generating function for the number of permutations containing exactly r 132 patterns is a rational function in variables x and $\sqrt{1-4x}$. This proves a particular case of the general conjecture of Noonan and Zeilberger [9] which is that for any set T of patterns, the sequence of numbers enumerating permutations having a prescribed number of occurrences of patterns in T is P-recursive. Recent paper [10] presents the generating function for the number of 132-avoiding permutations that contain a prescribed number of 123 patterns. The generating function is given in the form of a continued fraction. In the present note we generalize the argument of [10] to get the generating function for the number of 132-avoiding permutations that contain a prescribed number of $12\dots k$ patterns for arbitrary $k \geqslant 3$. The study of the obtained continued fraction allows us to recover and to generalize the result of [4] that relates the number of 132-avoiding permutations that contain no $12\dots k$ patterns to Chebyshev polynomials of the second kind.

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2. Continued fractions

Let $f_n^r(k)$ stand for the number of 132-avoiding permutations on n letters that contain exactly r occurrences of $12\dots k$. We denote by $F(x, y; k)$ the generating function of the sequence $\{f_n^r(k\}$), that is,

$$F(x, y; k) = \sum_{n \geqslant 0} \sum_{r \geqslant 0} f_n^r(k) x^n y^r.$$

Our first result is a natural generalization of the main theorem of [10].

Theorem 2.1. *The generating function $F(x,y;k)$ for $k \geqslant 1$ is given by the continued fraction*

$$F(x,y;k) = \cfrac{1}{1 - \cfrac{xy^{d_1}}{1 - \cfrac{xy^{d_2}}{1 - \cfrac{xy^{d_3}}{\cdots}}}},$$

where $d_i = \binom{i-1}{k-1}$, and $\binom{a}{b}$ is assumed 0 whenever $a < b$ or $b < 0$.

Proof. Following [10] we define $\eta_j(\pi)$, $j \geqslant 1$, as the number of occurrences of $12\dots j$ in π. Define $\eta_0(\pi) = 1$ for any π, which means that the empty pattern occurs exactly once in each permutation. The *weight* of a permutation π is a monomial in k independent variables $q_1, \dots, q_k$ defined by

$$w_k(\pi) = \prod_{j=1}^{k} q_j^{\eta_j(\pi)}.$$

The *total weight* is a polynomial

$$W_k(q_1, \dots, q_k) = \sum_{\pi \in \mathfrak{S}(132)} w_k(\pi).$$

The following proposition is implied immediately by the definitions.

Proposition 2.1. *$F(x,y;k) = W_k(x,1,\dots,1,y)$ for $k \geqslant 2$, and $F(x,y;1) = W_1(xy)$.*

We now find a recurrence relation for the numbers $\eta_j(\pi)$. Let $\pi \in \mathfrak{S}_n$, so that $\pi = (\pi', n, \pi'')$.

Proposition 2.2. *For any $j \geqslant 1$ and any nonempty $\pi \in \mathfrak{S}(132)$*

$$\eta_j(\pi) = \eta_j(\pi') + \eta_j(\pi'') + \eta_{j-1}(\pi').$$

Proof. Let $l = \pi^{-1}(n)$. Since π avoids 132, each number in π' is greater than any of the numbers in π''. Therefore, π' is a 132-avoiding permutation of the numbers $\{n-l+1, n-l+2, \dots, n-1\}$, while π'' is a 132-avoiding permutation of the numbers $\{1, 2, \dots, n-l\}$. On the other hand, if π' is an arbitrary 132-avoiding permutation of the numbers $\{n-l+1, n-l+2, \dots, n-1\}$ and π'' is an arbitrary 132-avoiding permutation of the numbers $\{1, 2, \dots, n-l\}$, then $\pi = (\pi', n, \pi'')$ is 132-avoiding. Finally, if $(i_1, \dots, i_j)$ is an occurrence of $12\dots j$ in π then either $i_j < l$, and so it is also an occurrence of $12\dots j$ in π', or $i_1 > l$, and so it is also an occurrence of $12\dots j$ in π'', or $i_j = l$, and so $(i_1, \dots, i_{j-1})$ is an occurrence of $12\dots j-1$ in π'. The result follows. $\square$

Now we are able to find the recurrence relation for the total weight W. Indeed, by Proposition 2.2,

$$\begin{aligned} W_k(q_1,\dots,q_k) &= 1 + \sum_{\varnothing\neq\pi\in\mathfrak{S}(132)} \prod_{j=1}^{k} q_j^{\eta_j(\pi')+\eta_j(\pi'')+\eta_{j-1}(\pi')} \\ &= 1 + \sum_{\pi'\in\mathfrak{S}(132)} \sum_{\pi''\in\mathfrak{S}(132)} \prod_{j=1}^{k} q_j^{\eta_j(\pi'')} \cdot q_1 \prod_{j=1}^{k-1} (q_j q_{j+1})^{\eta_j(\pi')} \cdot q_k^{\eta_k(\pi')} \\ &= 1 + q_1 W_k(q_1,\dots,q_k) W_k(q_1q_2, q_2q_3,\dots,q_{k-1}q_k, q_k). \qquad (1) \end{aligned}$$

For any $d \geqslant 0$ and $1 \leqslant m \leqslant k$ define

$$\mathbf{q}^{d,m} = \prod_{j=1}^{k} q_j^{\binom{d}{j-m}};$$

recall that $\binom{a}{b} = 0$ if $a < b$ or $b < 0$. The following proposition is implied immediately by the well-known properties of binomial coefficients.

Proposition 2.3. *For any $d \geqslant 0$ and $1 \leqslant m \leqslant k$*

$$\mathbf{q}^{d,m}\mathbf{q}^{d,m+1} = \mathbf{q}^{d+1,m}.$$

Observe now that $W_k(q_1,\dots,q_k) = W_k(\mathbf{q}^{0,1},\dots,\mathbf{q}^{0,k})$ and that by (1) and Proposition 2.3

$$W_k(\mathbf{q}^{d,1},\dots,\mathbf{q}^{d,k}) = 1 + \mathbf{q}^{d,1} W_k(\mathbf{q}^{d,1},\dots,\mathbf{q}^{d,k}) W_k(\mathbf{q}^{d+1,1},\dots,\mathbf{q}^{d+1,k}),$$

therefore

$$W_k(q_1,\dots,q_k) = \cfrac{1}{1 - \cfrac{\mathbf{q}^{0,1}}{1 - \cfrac{\mathbf{q}^{1,1}}{1 - \cfrac{\mathbf{q}^{2,1}}{\dots}}}}.$$

To obtain the continued fraction representation for $F(x,y;k)$ it is enough to use Proposition 2.1 and to observe that

$$\mathbf{q}^{d,1}\Big|_{q_1=x,q_2=\cdots=q_{k-1}=1,q_k=y} = xy^{\binom{d}{k-1}}. \qquad \square$$

Remark. For $k = 1$ one recovers from Theorem 2.1 the well-known generating function for the Catalan numbers, $(1-\sqrt{1-4z})/2z$. This result also follows immediately from Proposition 2.1 and equation (1), which for $k = 1$ is reduced to $W_1(q) = 1 + qW_1^2(q)$.

3. Chebyshev polynomials

Let us denote by $F_r(x;k)$ the generating function of the sequence $\{f_n^r(k)\}$ for a given r, that is,

$$F_r(x;k) = \sum_{n\geqslant 0} f_n^r(k)x^n.$$

Recall that $F(x,y;k) = \sum_{r\geqslant 0} F_r(x;k)y^r$. In this section we find explicit expressions for $F_r(x;k)$ in the case $0 \leqslant r \leqslant k$.

Consider a recurrence relation

$$T_j = \frac{1}{1 - xT_{j-1}}, \quad j \geqslant 1. \tag{2}$$

The solution of (2) with the initial condition $T_0 = 0$ is denoted by $R_j(x)$, and the solution of (2) with the initial condition

$$T_0 = G(x,y;k) = \cfrac{y}{1 - \cfrac{xy^{\binom{k}{1}}}{1 - \cfrac{xy^{\binom{k+1}{2}}}{1 - \cfrac{xy^{\binom{k+2}{3}}}{\cdots}}}}$$

is denoted by $S_j(x,y;k)$, or just S_j when the value of k is clear from the context. Our interest in (2) is stipulated by the following relation, which is an easy consequence of Theorem 2.1:

$$F(x,y;k) = S_k(x,y;k). \tag{3}$$

First of all, we find an explicit formula for the functions $R_j(x)$. Let $U_j(\cos\theta) = \sin(j+1)\theta/\sin\theta$ be the Chebyshev polynomials of the second kind.

Lemma 3.1. *For any $j \geqslant 1$*

$$R_j(x) = \frac{U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_j\left(\frac{1}{2\sqrt{x}}\right)}. \tag{4}$$

Proof. Indeed, it follows immediately from (2) that $R_j(x)$ is the jth approximant for the continued fraction

$$\cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{\cdots}}}}.$$

Hence, by [7, Theorem 2, p. 194], for any $j \geqslant 1$ one has $R_j(x) = A_j(x)/A_{j+1}(x)$, where

$$A_j(x) = \left(\frac{1+\sqrt{1-4x}}{2}\right)^j - \left(\frac{1-\sqrt{1-4x}}{2}\right)^j.$$

Using substitution $x \to 1/4t^2$ one gets $(2t)^j A_j(1/4t^2) = 2\sqrt{t^2-1}U_{j-1}(t)$, which gives $A_j(x) = \sqrt{1/x-4}\,x^{j/2}U_{j-1}(1/2\sqrt{x})$, and the result follows. □

Next, we find an explicit expression for S_j in terms of G and R_j.

Lemma 3.2. *For any $j \geqslant 1$ and any $k \geqslant 1$*

$$S_j(x,y;k) = R_j(x)\frac{1 - xR_{j-1}(x)G(x,y;k)}{1 - xR_j(x)G(x,y;k)}. \tag{5}$$

Proof. Indeed, from (2) and $S_0 = G$ we get $S_1 = 1/(1-xG)$. On the other hand, $R_0 = 0$, $R_1 = 1$, so (5) holds for $j = 1$. Now let $j > 1$, then by induction

$$S_j = \frac{1}{1-xS_{j-1}} = \frac{1}{1-xR_{j-1}} \cdot \frac{1-xR_{j-1}G}{1 - \dfrac{x(1-xR_{j-2})R_{j-1}G}{1-xR_{j-1}}}.$$

Relation (2) for R_j and R_{j-1} yields $(1-xR_{j-2})R_{j-1} = (1-xR_{j-1})R_j = 1$, which together with the above formula gives (5). □

As a corollary from Lemma 3.2 and (3) we get the following expression for the generating function $F(x,y;k)$.

Corollary.

$$F(x,y;k) = R_k(x) + \big(R_k(x) - R_{k-1}(x)\big)\sum_{m \geqslant 1}\big(xR_k(x)G(x,y;k)\big)^m.$$

Now we are ready to express the generating functions $F_r(x;k)$, $0 \leqslant r \leqslant k$, via Chebyshev polynomials.

Theorem 3.1. *For any $k \geqslant 1$, $F_r(x;k)$ is a rational function given by*

$$F_r(x;k) = \frac{x^{\frac{r-1}{2}}U_{k-1}^{r-1}\left(\frac{1}{2\sqrt{x}}\right)}{U_k^{r+1}\left(\frac{1}{2\sqrt{x}}\right)}, \quad 1 \leqslant r \leqslant k,$$

$$F_0(x;k) = \frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)},$$

where U_j is the jth Chebyshev polynomial of the second kind.

Proof. Observe that $G(x,y;k) = y + y^{k+1}P(x,y)$, so from Corollary we get

$$F(x,y;k) = R_k(x) + \big(R_k(x) - R_{k-1}(x)\big)\sum_{m=1}^{k}\big(xR_k(x)\big)^m y^m + y^{k+1}P'(x,y),$$

where $P(x,y)$ and $P'(x,y)$ are formal power series. To complete the proof, it suffices to use (4) together with the identity $U_{n-1}^2(z) - U_n(z)U_{n-2}(z) = 1$, which follows easily from the trigonometric identity $\sin^2 n\theta - \sin^2\theta = \sin(n+1)\theta\sin(n-1)\theta$. □

For the case $r = 0$ this result was proved by a different method in [4].

4. Further results

There are several ways to generalize the results of the previous sections. First, one can try to get exact formulas for $F_r(x;k)$ in the case $r > k$. The method described in Section 3 allows, in principle, to obtain such formulas, though they become more and more complicated. For example, the following theorem gives an explicit expression for $F_r(x;k)$ when $r \leqslant k(k+3)/2$.

Theorem 4.1. *For any $k \geqslant 1$ and $1 \leqslant r \leqslant k(k+3)/2$, $F_r(x;k)$ is a rational function given by*

$$F_r(x;k) = \frac{x^{\frac{r-1}{2}} U_{k-1}^{r-1}\left(\frac{1}{2\sqrt{x}}\right)}{U_k^{r+1}\left(\frac{1}{2\sqrt{x}}\right)} \sum_{j=0}^{\lfloor (r-1)/k\rfloor} \binom{r-kj+j-1}{j} \left(\frac{U_k\left(\frac{1}{2\sqrt{x}}\right)}{x^{\frac{k-2}{2k}} U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}\right)^{kj},$$

where U_j is the jth Chebyshev polynomial of the second kind.

Proof. Indeed, the explicit expression for $G(x,y;k)$ gives

$$G(x,y;k) = y(1 + xy^k + \cdots + x^s y^{ks}) + y^t P(x,y),$$

where $s = \lceil (k+1)/2\rceil$, $t = 1 + k(k+3)/2$, and $P(x,y)$ is a formal power series. Hence, by Corollary,

$$\begin{aligned}
\frac{F(x,y;k) - R_k(x)}{R_k(x) - R_{k-1}(x)} &= \sum_{m\geqslant 1}\big(xR_k(x)\big)^m y^m (1 + xy^k + \cdots + x^s y^{ks})^m + y^t P'(x,y) \\
&= \sum_{m\geqslant 1}\big(xR_k(x)\big)^m y^m \sum_{j=0}^{ms}\binom{m+j-1}{j} x^j y^{kj} + y^t P'(x,y) \\
&= \sum_{r\geqslant 1} y^r \big(xR_k(x)\big)^r \sum_{j=0}^{\lfloor (r-1)/k\rfloor} \frac{\binom{r-kj+j-1}{j} x^j}{\big(xR_k(x)\big)^{kj}} + y^t P''(x,y),
\end{aligned}$$

where $P'(x,y)$ and $P''(x,y)$ are formal power series. The rest of the proof follows the proof of Theorem 3.1. $\square$

Another possibility is to analyze the case of permutations containing exactly one 132 pattern and r $12\dots k$ patterns. Introducing the modified total weight $\Omega_k(q_1,\dots,q_k)$ as the sum of the weights $w_k(\pi)$ over all permutations containing exactly one 132 pattern, we get the following equation:

$$\begin{aligned}\Omega_k(q_1,\dots,q_k) &= q_1W_k(q_1q_2,\dots,q_{k-1}q_k,q_k)\Omega_k(q_1,\dots,q_k)\\ &+ q_1W_k(q_1,\dots,q_k)\Omega_k(q_1q_2,\dots,q_{k-1}q_k,q_k)\\ &+ q_1^2q_2^2W_k(q_1q_2,\dots,q_{k-1}q_k,q_k)\big(W_k(q_1,\dots,q_k)-1\big);\end{aligned}$$

for the case $k=3$ see [10]. By (1) and Proposition 2.3 this is equivalent to

$$\begin{aligned}\Omega_k(\mathbf{q}^{d,1},\dots,\mathbf{q}^{d,k}) &= \mathbf{q}^{d,1}\left(\mathbf{q}^{d,2}\right)^2\left(W_k(\mathbf{q}^{d,1},\dots,\mathbf{q}^{d,k})-1\right)^2\\ &+ \mathbf{q}^{d,1}W_k^2(\mathbf{q}^{d,1},\dots,\mathbf{q}^{d,k})\Omega_k(\mathbf{q}^{d+1,1},\dots,\mathbf{q}^{d+1,k}).\end{aligned} \tag{6}$$

Let now $\varphi_n^r(k)$ be the number of permutations on n letters that contain exactly one 132 pattern and r $12\dots k$ patterns, and $\Phi_r(x;k)$ be the generating function of the sequence $\{\varphi_n^r(k)\}$ for a given r. In general, equation (6) allows us to find explicit expressions for $\Phi_r(x;k)$. However, they are rather cumbersome, so we restrict ourselves to the case $r=0$.

Theorem 4.2. *For any $k \geqslant 3$, $\Phi_0(x;k)$ is a rational function given by*

$$\begin{aligned}\Phi_0(x;k) &= \frac{x}{U_k^2\left(\frac{1}{2\sqrt{x}}\right)}\sum_{j=1}^{k-2}U_j^2\left(\frac{1}{2\sqrt{x}}\right)\\ &= \frac{1}{16\sin^2(k+1)t\cos^2 t}\left(2k-5+4\cos^2 t-\frac{\sin(2k-1)t}{\sin t}\right),\end{aligned}$$

where U_j is the jth Chebyshev polynomial of the second kind and $\cos t = 1/2\sqrt{x}$.

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