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Restricted 132-Avoiding Permutations

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We study generating functions for the number of permutations on n letters avoiding 132 and an arbitrary permutation τ on k letters, or containing τ exactly once. In several interesting cases the generating function depends only on k and is expressed via Chebyshev polynomials of the second kind. © 2001 Academic Press

1. INTRODUCTION

Let $\alpha \in S_n$ and $\tau \in S_k$ be two permutations. We say that α *contains* τ if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that α *avoids* τ, or is τ*-avoiding*, if such a subsequence does not exist. The set of all τ-avoiding permutations in S_n is denoted $S_n(\tau)$. For an arbitrary finite collection of patterns T, we say that α avoids T if α avoids any $\tau \in T$; the corresponding subset of S_n is denoted $S_n(T)$.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns τ_1, τ_2. This problem was solved completely for $\tau_1, \tau_2 \in S_3$ (see [SS]), for $\tau_1 \in S_3$ and $\tau_2 \in S_4$ (see [W, A]), and for $\tau_1, \tau_2 \in S_4$ (see [B1, Km] and references therein). Several recent papers

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[CW, MV1, Kr, MV2] deal with the case $\tau_1 \in S_3$, $\tau_2 \in S_k$ for various pairs τ_1, τ_2. Another natural question is to study permutations avoiding τ_1 and containing τ_2 exactly t times. Such a problem for certain $\tau_1, \tau_2 \in S_3$ and $t = 1$ was investigated in [R], and for certain $\tau_1 \in S_3$, $\tau_2 \in S_k$ in [RWZ, MV1, Kr]. The tools involved in these papers include continued fractions, Chebyshev polynomials, and Dyck paths.

In this paper we present a general approach to the study of permutations avoiding 132 and avoiding an arbitrary pattern $\tau \in S_k$ (or containing it exactly once). As a consequence, we derive all the previously known results for this kind of problem, as well as many new results.

The paper is organized as follows. The case of permutations avoiding both 132 and τ is treated in Section 2. We derive a simple recursion for the corresponding generating function for general τ. This recursion can be solved explicitly for several interesting cases, including 2-layered and 3-layered patterns (see [B2, MV2]) and wedge patterns defined below. It also allows one to write a Maple program that calculates the generating function for any given τ. This program can be obtained from the authors on request. Observe that if τ itself contains 132, then any 132-avoiding permutation avoids τ as well, so in what follows we always assume that $\tau \in S_k(132)$.

The case of permutations avoiding 132 and containing τ exactly once is treated in Section 3. Here again we start from a general recursion and then solve it for several particular cases.

Most of the explicit solutions obtained in Sections 2 and 3 involve Chebyshev polynomials of the second kind. Several identities used for getting these solutions are presented in Section 4. The authors are grateful to the referee for explaining to us a general approach to such identities.

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2. AVOIDING A PATTERN

Consider an arbitrary pattern $\tau = (\tau_1, \ldots, \tau_k) \in S_k(132)$. Recall that τ_i is said to be a *right-to-left maximum* if $\tau_i > \tau_j$ for any $j > i$. Let $m_0 = k, m_1, \ldots, m_r$ be the right-to-left maxima of τ written from left to right. Then τ can be represented as

$$\tau = (\tau^0, m_0, \tau^1, m_1, \ldots, \tau^r, m_r),$$

where each of τ^i may be possibly empty, and all the entries of τ^i are greater than m_{i+1} and all the entries of τ^{i+1}. This representation is called the *canonical decomposition* of τ. Given the canonical decomposition, we

define the ith *prefix* of τ by $\pi^i = (\tau^0, m_0, \ldots, \tau^i, m_i)$ for $1 \le i \le r$ and $\pi^0 = \tau^0$, $\pi^{-1} = \varnothing$. Furthermore, the ith *suffix* of τ is defined by $\sigma^i = (\tau^i, m_i, \ldots, \tau^r, m_r)$ for $0 \le i \le r$ and $\sigma^{r+1} = \varnothing$. Strictly speaking, prefixes and suffixes themselves are not patterns, since they are not permutations (except for $\pi^r = \sigma^0 = \tau$). However, any prefix or suffix is order-isomorphic to a unique permutation, and in what follows we do not distinguish between a prefix (or suffix) and the corresponding permutation.

Let $f_\tau(n)$ denote the number of permutations in $S_n(132)$ avoiding τ, and let $F_\tau(x) = \sum_{n\ge 0} f_\tau(n)x^n$ be the corresponding generating function. By $f_\tau^\rho(n)$ we denote the number of permutations in $S_n(132)$ avoiding τ and containing ρ. The following proposition is the base of all the other results in this section.

THEOREM 2.1. *For any $\tau \in S_k(132)$, $F_\tau(x)$ is a rational function satisfying the relation*

$$F_\tau(x) = 1 + x\sum_{j=0}^{r}\big(F_{\pi^j}(x) - F_{\pi^{j-1}}(x)\big)F_{\sigma^j}(x).$$

Proof. Let $\alpha \in S_n(132, \tau)$. Choose t so that $\alpha_t = n$, then $\alpha = (\alpha', n, \alpha'')$, and α avoids 132 if and only if α' is a permutation of the numbers $n-t+1, n-t+2, \ldots, n$, α'' is a permutation of the numbers $1, 2, \ldots, n-t$, and both α' and α'' avoid 132. On the other hand, it is easy to see that α contains τ if and only if there exists i, $0 \le i \le r+1$, such that α' contains π^{i-1} and α'' contains σ^i. Therefore, α avoids τ if and only if there exists i, $0 \le i \le r$, such that α' avoids π^i and contains π^{i-1}, while α'' avoids σ^i. We thus get the following relation:

$$f_\tau(n) = \sum_{t=1}^{n}\sum_{j=0}^{r} f_{\pi^j}^{\pi^{j-1}}(t-1)f_{\sigma^j}(n-t).$$

To obtain the recursion for $F_\tau(x)$ it remains to observe that

$$f_{\pi^j}^{\pi^{j-1}}(l) + f_{\pi^{j-1}}(l) = f_{\pi^j}(l)$$

for any l and j, and to pass to generating functions. Rationality of $F_\tau(x)$ follows easily by induction. ∎

Though elementary, the above theorem enables us to derive easily various known and new results for a fixed k.

EXAMPLE 2.1 (see [SS, Proposition 11]). Let us find $F_{321}(x)$. The canonical decomposition of 321 is $(\varnothing, 3, \varnothing, 2, \varnothing, 1)$, so $r = 2$, and hence Theorem 2.1 gives

$$F_{321}(x) = 1 + xF_{32}(x)F_{21}(x) + x\big(F_{321}(x) - F_{32}(x)\big)F_1(x).$$

Since evidently $F_1(x) = 1$ and $F_{21}(x) = F_{32}(x) = (1-x)^{-1}$, we finally get

$$F_{321}(x) = \frac{1 - 2x + 2x^2}{(1-x)^3}.$$

EXAMPLE 2.2 (see [W]). Let us find $F_{3214}(x)$. The canonical decomposition is $(321, 4)$, so $r = 0$, and hence $F_{3214}(x) = 1 + xF_{321}(x)F_{3214}(x)$. Thus finally

$$F_{3124}(x) = \frac{(1-x)^3}{1 - 4x + 5x^2 - 3x^3}.$$

A Maple program that calculates $F_\tau(x)$ for any given τ is available from the authors on request.

The case of varying k is more interesting. As an extension of Example 2.1, let us consider the case $\tau = \langle k \rangle = (k, k-1, \ldots, 1)$. We denote by $\Phi(x, y)$ the generating function $\sum_{k \geq 1} F_{\langle k \rangle}(x) y^k$.

THEOREM 2.2.

$$\Phi(x, y) = \frac{y(1 + x - xy) - y\sqrt{(1 + x - xy)^2 - 4x}}{2x(1-y)}.$$

Proof. Indeed, Theorem 2.1 yields

$$F_{\langle k \rangle}(x) = 1 + x \sum_{j=1}^{k-1} \big(F_{\langle j+1 \rangle}(x) - F_{\langle j \rangle}(x)\big) F_{\langle k-j \rangle}(x) + xF_{\langle 1 \rangle}(x) F_{\langle k-1 \rangle}(x).$$

Multiplication by y^k and summation over $k \geq 2$ gives

$$\Phi(x, y) = \frac{y}{1-y} + x(1-y)\Phi(x, y)\left(\frac{\Phi(x, y)}{y} - 1 - \Phi(x, y)\right) + xy\Phi(x, y),$$

and the result follows. ∎

Observe that as a consequence of Theorem 2.2 we get

$$\lim_{k \to \infty} F_{\langle k \rangle}(x) = \lim_{y \to 1} (1-y)\Phi(x, y) = \frac{1 - \sqrt{1 - 4x}}{2x},$$

in which we recognize the generating function of Catalan numbers. This is a predictable result, since as k tends to ∞ the restriction posed by τ vanishes, and we end up with just 132-avoiding permutations, which are enumerated by Catalans.

Let us consider now a richer class of patterns τ. We say that $\tau \in S_k$ is a *layered* pattern if it can be represented as $\tau = (\tau^0, \tau^1, \ldots, \tau^r)$, where each of τ^i is a nonempty permutation of the form $\tau^i = (m_{i+1}+1, m_{i+1}+2, \ldots, m_i)$ with $k = m_0 > m_1 > \ldots > m_r > m_{r+1} = 0$; in this case we denote τ by $[m_0, \ldots, m_r]$. Observe that our definition differs slightly from the one used in [B2, MV2]: their layered patterns are exactly the complements of our layered patterns. It was revealed in several recent papers (see [CW, MV1, Kr] and especially [MV2]) that layered restrictions are intimately related to Chebyshev polynomials of the second kind $U_p(\cos\theta) = \sin(p+1)\theta/\sin\theta$. Following [MV1], introduce

$$R_p(x) = \frac{U_{p-1}(z)}{\sqrt{x}U_p(z)}, \quad \text{where } z = \frac{1}{2\sqrt{x}}.$$

It was proved by different methods in [CW, MV1, Kr] that $F_{[k]}(x) = R_k(x)$.

Our next result is an easy consequence of Theorem 2.1.

THEOREM 2.3. *Let $r \geq 1$ and $k = m_0 > m_1 > \ldots > m_r$; then*

$$\begin{aligned}\big(1 - xR_{m_0-m_1-1}(x) - xR_{m_r}(x)\big)F_{[m_0,\ldots,m_r]}(x) &= 1 - xR_{m_0-m_1-1}(x)F_{[m_1,\ldots,m_r]}(x) \\ &\quad + x\sum_{j=2}^{r} F_{[m_0-m_j,\ldots,m_{j-1}-m_j]}(x)\big(F_{[m_{j-1},\ldots,m_r]}(x) - F_{[m_j,\ldots,m_r]}(x)\big).\end{aligned}$$

Proof. Evidently, $(\tilde{\tau}^0, m_0, \ldots, \tilde{\tau}^r, m_r)$ with $\tilde{\tau}^i = (m_{i+1}+1, \ldots, m_i - 1)$ is the canonical decomposition of $[m_0, \ldots, m_r]$. Next, $F_{\sigma^i}(x) = F_{[m_i,\ldots,m_r]}(x)$ for $0 \leq i \leq r-1$, and $F_{\pi^i}(x) = F_{[m_0-m_{i+1},\ldots,m_i-m_{i+1}]}(x)$ for $1 \leq i \leq r$. Besides, $F_{\sigma^r}(x) = F_{[m_r]}(x) = R_{m_r}(x)$ and $F_{\pi^0}(x) = F_{[m_0-m_1-1]}(x) = R_{m_0-m_1-1}(x)$. The rest follows from Theorem 2.1 via simple algebraic transformations. ∎

For small r one can find explicit expressions for $F_{[m_0,\ldots,m_r]}(x)$.

For $r = 1$ we get the following generalization of [CW, Theorem 3.1, third case], and [Kr, Theorem 6].

THEOREM 2.4. *For any $k > m > 0$,*

$$F_{[k,m]}(x) = R_k(x).$$

Proof. By Theorem 2.3,

$$\big(1 - xR_{k-m-1}(x) - xR_m(x)\big)F_{[k,m]}(x) = 1 - xR_{k-m-1}(x)F_{[m]}(x),$$

and the result follows immediately from Lemma 4.1(iv,v) for $a = k-m-1$ and $b = m$. ∎

The case $r = 2$ is more complicated.

THEOREM 2.5. *For any* $k > m_1 > m_2 > 0$,

$$F_{[k,m_1,m_2]}(x) = \frac{U_{\alpha+\beta}(z)U_{\alpha+\gamma-1}(z)U_{\beta+\gamma}(z) + U_{\beta-1}(z)U_{\beta}(z)}{\sqrt{x}U_{\alpha+\beta}(z)U_{\alpha+\gamma}(z)U_{\beta+\gamma}(z)},$$

where $\alpha = k - m_1$, $\beta = m_1 - m_2$, $\gamma = m_2$, *and* $z = 1/2\sqrt{x}$.

Proof. Indeed, by Theorem 2.3,

$$\begin{aligned}\big(1 - xR_{k-m_1-1}(x) - xR_{m_2}(x)\big)F_{[k,m_1,m_2]}(x) &= 1 - xR_{k-m_1-1}(x)F_{[m_1,m_2]}(x) \\ &\quad + xF_{[k-m_2,m_1-m_2]}(x)\big(F_{[m_1,m_2]}(x) - F_{[m_2]}(x)\big).\end{aligned}$$

Taking into account Theorem 2.4 one gets

$$F_{[k,m_1,m_2]}(x) = \frac{1 - xR_{k-m_1-1}(x)R_{m_1}(x) + xR_{k-m_2}(x)\big(R_{m_1}(x) - R_{m_2}(x)\big)}{1 - xR_{k-m_1-1}(x) - xR_{m_2}(x)}.$$

By Lemma 4.1(iv) for $a = k - m_1 - 1$ and $b = m_1$, Lemma 4.1(vi) for $a = m_1$ and $b = m_2$, and Lemma 4.1(v) for $a = k - m_1 - 1$ and $b = m_2$, one has

$$1 - xR_{k-m_1-1}(x)R_{m_1}(x) = \frac{U_{k-1}(z)}{U_{k-m_1-1}(z)U_{m_1}(z)},$$

$$xR_{k-m_2}(x)\big(R_{m_1}(x) - R_{m_2}(x)\big) = \frac{U_{k-m_2-1}(z)U_{m_1-m_2-1}(z)}{U_{k-m_2}(z)U_{m_1}(z)U_{m_2}(z)},$$

$$\frac{1}{1 - xR_{k-m_1-1}(x) - xR_{m_2}(x)} = \frac{U_{k-m_1-1}(z)U_{m_2}(z)}{\sqrt{x}U_{k-m_1+m_2}(z)},$$

respectively, and we thus get

$$F_{[k,m_1,m_2]}(x) = \frac{U_{\alpha+\beta}(z)U_{\alpha+\beta+\gamma-1}(z)U_{\gamma}(z) + U_{\alpha-1}(z)U_{\beta-1}(z)U_{\alpha+\beta-1}(z)}{\sqrt{x}U_{\alpha+\beta}(z)U_{\alpha+\gamma}(z)U_{\beta+\gamma}(z)}.$$

Finally, we use Lemma 4.1(i) for $s = \alpha + \gamma - 1$, $t = \beta + \gamma$, and $w = \beta$ and Lemma 4.1(ii) for $s = \beta$, $t = 0$, and $w = \alpha - 1$ to get the desired result. ∎

For a further generalization of Theorem 2.4, consider the following definition. We say that $\tau \in S_k$ is a *wedge* pattern if it can be represented as $\tau = (\tau^1, \rho^1, \ldots, \tau^r, \rho^r)$, so that each of τ^i is nonempty, $(\rho^1, \rho^2, \ldots, \rho^r)$ is a layered permutation of $1, \ldots, s$ for some s, and $(\tau^1, \tau^2, \ldots, \tau^r) = (s+1, s+2, \ldots, k)$. For example, 645783912 and 456378129 are wedge patterns. Evidently, $[k, m]$ is a wedge pattern for any m.

THEOREM 2.6. $F_\tau(x) = R_k(x)$ *for any wedge pattern* $\tau \in S_k(132)$.

Proof. We proceed by induction on r. If $r = 1$ then $\tau = [k, m]$ for some m, and the result is true by Theorem 2.4. For an arbitrary $r > 1$, take the canonical decomposition of τ. Evidently, it looks like $\tau = (\tau', k, \tilde{\rho}^r, m)$, where $(\tilde{\rho}^r, m) = \rho^r$, provided ρ^r is nonempty. Therefore, Theorem 2.1, together with $F_{\rho^r}(x) = F_{[m]}(x) = R_m(x)$, gives

$$F_\tau(x) = \frac{1 - xF_{\tau'}(x)R_m(x)}{1 - xF_{\tau'}(x) - xR_m(x)}. \tag{2.1}$$

If $\tau^r = (k)$, then τ' is itself a wedge pattern on $k - m - 1$ elements, so by induction $F_{\tau'}(x) = R_{k-m-1}(x)$; hence the result follows from (2.1) and Lemma 4.1(iv,v) for $a = k - m - 1$ and $b = m$. Let $\tau^r = (l + 1, \ldots, k)$; then applying Theorem 2.1 repeatedly $k - l - 1$ times, we get

$$F_{\tau'}(x) = \cfrac{1}{1 - \cfrac{x}{\ddots \; 1 - \cfrac{x}{1 - xF_{\tau''}(x)}}},$$

where the height of the fraction equals $k - l - 1$ and $\tau'' = (\tau^1, \rho^1, \ldots, \tau^{r-1}, \rho^{r-1})$ is a wedge pattern on $l - m$ elements. So, by induction, $F_{\tau''}(x) = R_{l-m}(x)$; applying Lemma 4.1(iii) repeatedly $k - l - 1$ times, we again get $F_{\tau'}(x) = R_{k-m-1}(x)$ and proceed exactly as in the previous case. The case $\rho^r = \varnothing$ is treated in a similar way. ∎

Remark. A comparison of Theorem 2.6 with the main result of [MV2] suggests that there should exist a bijection between the sets $S_n(321, [k, m])$ and $S_n(132, \tau)$ for any wedge pattern τ. However, we failed to produce such a bijection, and finding it remains a challenging open question.

3. CONTAINING A PATTERN EXACTLY ONCE

Let $g_\tau(n)$ denote the number of permutations in $S_n(132)$ that contain $\tau \in S_k(132)$ exactly once, and let $g_\tau^\rho(n)$ denote the number of permutations in $S_n(132, \rho)$ that contain $\tau \in S_k(132)$ exactly once. We denote by $G_\tau(x)$ and $G_\tau^\rho(x)$ the corresponding ordinary generating functions.

The following statement is similar to Theorem 2.1.

THEOREM 3.1. *Let $\tau = (\tau^0, m_0, \ldots, \tau^r, m_r)$ be the canonical decomposition of $\tau \in S_k(132)$; then*

$$\big(1 - xF_{\pi^0}(x) - xF_{\sigma^r}(x)\big)G_\tau(x) = x\sum_{j=1}^{r} G_{\pi^{j-1}}^{\pi^j}(x)G_{\sigma^j}^{\sigma^{j-1}}(x)$$

for $r \geq 1$, *and*

$$G_\tau(x) = \frac{xF_\tau(x)G_{\pi^0}(x)}{1 - xF_{\pi^0}(x)}$$

for $r = 0$.

Proof. Let $\alpha \in S_n(132)$ contain τ exactly once, and take the same decomposition $\alpha = (\alpha', n, \alpha'')$ as in the proof of Theorem 2.1. Similarly to this proof, α contains τ exactly once if and only if either α' avoids π^0 and α'' contains σ^0 exactly once, or α' contains π^r exactly once and α'' avoids σ^r, or there exists i, $1 \leq i \leq r$, such that α' avoids π^i and contains π^{i-1} exactly once, while α'' avoids σ^{i-1} and contains σ^i exactly once. We thus get the relation

$$g_\tau(n) = \sum_{t=1}^{n} f_{\pi^0}(t-1)g_\tau(n-t) + \sum_{t=1}^{n} g_\tau(t-1)f_{\sigma^r}(n-t) + \sum_{t=1}^{n}\sum_{j=1}^{r} g_{\pi^{j-1}}^{\pi^j}(t-1)g_{\sigma^j}^{\sigma^{j-1}}(n-t),$$

and the result follows. ∎

Remark. Strictly speaking, Theorem 3.1, unlike Theorem 2.1, is *not* a recursion for $G_\tau(x)$, since it involves functions of type G_τ^ρ (unless $r = 0$). However, for these functions one can write further recursions involving similar objects. For example,

$$\left(1 - xF_{\tau^j}(x) - xF_{\pi^0}(x)\right)G_{\pi^{j-1}}^{\pi^j}(x) = x\sum_{i=1}^{j-1} G_{\pi^{i-1}}^{\pi^i}(x)G_{\sigma_{j-1}^i}^{\sigma_{j-1}^{i-1}}(x),$$

where σ_{j-1}^i is the ith suffix of π^{j-1}. Though we have not succeeded in writing down a complete set of equations in the general case, it is possible to do this in certain particular cases.

EXAMPLE 3.1 (see [MV1, Theorem 3.1]). Let $\tau = [k] = (1, 2, \ldots, k)$. Then $r = 0$, and it follows from Theorem 3.1 that

$$G_{[k]}(x) = \frac{xF_{[k]}(x)G_{[k-1]}(x)}{1 - xF_{[k-1]}(x)}.$$

Since $F_{[k]}(x) = R_k(x)$ and $R_k(x)(1 - xR_{k-1}(x)) = 1$ (see Lemma 4.1(iii) below), we get $G_{[k]}(x) = xR_k^2(x)G_{[k-1]}(x)$, which together with $G_{[0]}(x) = 1$ gives

$$G_{[k]}(x) = \frac{1}{U_k^2(z)}, \quad \text{where } z = \frac{1}{2\sqrt{x}}.$$

Similarly to Section 2, we consider now the case $\tau = \langle k \rangle = (k, k-1, \ldots, 1)$ and denote by $\Psi(x, y)$ the generating function $\sum_{k \geq 1} G_{\langle k \rangle}(x) y^k$. The following statement is a counterpart of Theorem 2.2.

THEOREM 3.2.

$$\Psi(x, y) = \frac{(1-x)(1-xy) - \sqrt{(1-x)^2(1-xy)^2 - 4x^2(1-x)y}}{2x}.$$

Proof. Indeed, Theorem 3.1 yields

$$(1-x) G_{\langle k \rangle}(x) = x G_{\langle 0 \rangle}^{\langle 2 \rangle}(x) G_{\langle k-1 \rangle}^{\langle k \rangle}(x) + x \sum_{j=2}^{k-1} G_{\langle j \rangle}^{\langle j+1 \rangle}(x) G_{\langle k-j \rangle}^{\langle k-j+1 \rangle}(x).$$

Observe that if α contains $\langle j+1 \rangle$ then it contains at least $j+1$ copies of $\langle j \rangle$; hence $G_{\langle j \rangle}^{\langle j+1 \rangle}(x) = G_{\langle j \rangle}(x)$ for any $j \geq 1$. Besides, $G_{\langle 0 \rangle}^{\langle 2 \rangle}(x) = 1$ and $G_{\langle 1 \rangle}(x) = x$. Therefore, multiplication of the above equation by y^k and summation over $k \geq 2$ gives

$$(1-x)\big(\Psi(x, y) - xy\big) = x\Psi^2(x, y) + x(1-x) y \Psi(x, y),$$

and the result follows. ∎

In the case of a layered τ we get the following counterpart of Theorem 2.3.

THEOREM 3.3. *Let $r \geq 1$ and $k = m_0 > m_1 > \ldots > m_r > m_{r+1} = 0$; then*

$$\begin{aligned} &\big(1 - xR_{d_{01}-1}(x) - xR_{m_r}(x)\big) G_{[m_0, \ldots, m_r]}(x) \\ &\quad = x G_{[d_{01}-1]}^{[d_{02}, d_{12}]}(x) G_{[m_1, \ldots, m_r]}^{[m_0, \ldots, m_r]}(x) \\ &\qquad + x \sum_{j=2}^{r} G_{[d_{0j}, \ldots, d_{j-1, j}]}^{[d_{0, j+1}, \ldots, d_{j, j+1}]}(x) G_{[m_j, \ldots, m_r]}^{[m_{j-1}, \ldots, m_r]}(x), \end{aligned}$$

where $d_{ij} = m_i - m_j$.

The proof is similar to that of Theorem 2.3.

For the case $r = 1$ one gets the following counterpart of Theorem 2.4, which is a generalization of [Kr, Theorem 7].

THEOREM 3.4. *For any $k > m > 0$,*

$$G_{[k, m]}(x) = \frac{\sqrt{x}}{U_k(z) U_m(z) U_{k-m-1}(z)},$$

where $z = 1/2\sqrt{x}$.

Proof. Indeed, by Theorem 3.3 we have

$$\bigl(1 - xR_{k-m-1}(x) - xR_m(x)\bigr)G_{[k,m]}(x) = xG^{[k,m]}_{[k-m-1]}(x)G^{[k,m]}_{[m]}(x).$$

Without loss of generality we can assume that $2m \geq k$; otherwise it is enough to replace $[k, m]$ by $[k, m]^{-1} = [k, k-m]$, since $(132)^{-1} = (132)$. Under this restriction, one has $G^{[k,m]}_{[k-m-1]}(x) = G_{[k-m-1]}(x)$, and it remains to find $G^{[k,m]}_{[m]}(x)$. Given a decomposition $\alpha = (\alpha', n, \alpha'') \in S_n(132)$ as before, it is easy to see that α avoids $[k, m]$ and contains $[m]$ exactly once if and only if either α' avoids $[k-m-1]$ while α'' avoids $[k, m]$ and contains $[m]$ exactly once, or α' contains $[m-1]$ exactly once while α'' avoids $[m]$. We thus get

$$g^{[k,m]}_{[m]}(n) = \sum_{t=1}^{n} f_{[k-m-1]}(t-1)g^{[k,m]}_{[m]}(n-t) + \sum_{t=1}^{n} g_{[m-1]}(t-1)f_{[m]}(n-t),$$

which on the level of generating functions means

$$G^{[k,m]}_{[m]}(x) = xF_{[k-m-1]}(x)G^{[k,m]}_{[m]}(x) + G_{[m-1]}(x)F_{[m]}(x).$$

Plugging in the expression for $G_{[m-1]}(x)$ calculated in Example 3.1 and using Lemma 4.1(iii) we get

$$G^{[k,m]}_{[m]}(x) = \frac{U_{k-m-1}(z)}{xU_m(z)U_{m-1}(z)U_{k-m}(z)},$$

where $z = 1/2\sqrt{x}$. This, together with (3.1) and Lemma 4.1(iv,v) for $a = k-m-1$ and $b = m$, yields the desired result. ∎

One can try to obtain results similar to Theorems 2.5 and 2.6, but the expressions involved become extremely cumbersome. So we just consider a simplest wedge pattern, which is not layered.

EXAMPLE 3.2. Let $k > m > p > 0$, and let $\tau = \{k, m, p\} = (p+1, p+2, \ldots, m, 1, 2, \ldots, p, m+1, m+2, \ldots, k)$. To find $G_{\{k,m,p\}}(x)$ we use Theorem 3.1 for $r = 0$ repeatedly $k-m$ times and get

$$G_{\{k,m,p\}}(x) = x^{k-m}R_k^2(x)\cdots R_{m+1}^2(x)G_{[m,p]}(x).$$

Now Theorem 2.4 yields

$$G_{\{k,m,p\}}(x) = \frac{\sqrt{x}U_m(z)}{U_k^2(z)U_{m-p-1}(z)U_p(z)}$$

with $z = 1/2\sqrt{x}$.

4. IDENTITIES INVOLVING CHEBYSHEV POLYNOMIALS

In this section we present several identities involving Chebyshev polynomials of the second kind used in the two previous sections. We do not supply the proofs, since any identity which is rational in z and $U_p(z)$ can be proved routinely by a computer program. Indeed, it is enough to perform the following steps:

(1) Replace z by $\cos\theta$, and $U_p(z)$ by $\sin(p+1)\theta/\sin\theta$

(2) Since $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, and $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$, replace $\cos\theta$ by $(w + w^{-1})/2$, and $U_p(\cos\theta)$ by $(w^{p+1} - w^{-p-1})/(w - w^{-1})$

(3) The obtained identity is rational in w and can be checked by any computer algebra program.

In the following lemma we assume that $R_p(x)$, $p \geq 1$, is defined by

$$R_p(x) = \frac{U_{p-1}(z)}{\sqrt{x}U_p(z)}$$

with $z = 1/2\sqrt{x}$.

LEMMA 4.1. (i) *For any* $s + w - 1 \geq t \geq w \geq 1$,

$$U_s(z)U_t(z) - U_{s+w}(z)U_{t+w}(z) = U_{w-1}(z)U_{s-t+w-1}(z).$$

(ii) *For any* $s, t \geq 0$ *and* $w \geq 1$,

$$U_{s+w}(z)U_{t+w}(z) - U_s(z)U_t(z) = U_{w-1}(z)U_{s+t+w+1}(z).$$

(iii) *For any* $p \geq 1$,

$$R_{p+1}(x) = \frac{1}{1 - xR_p(x)}.$$

(iv) *For any* $a, b \geq 1$,

$$1 - xR_a(x)R_b(x) = \frac{U_{a+b}(z)}{U_a(z)U_b(z)}.$$

(v) *For any* $a, b \geq 1$,

$$1 - xR_a(x) - xR_b(x) = \frac{\sqrt{x}U_{a+b+1}(z)}{U_a(z)U_b(z)}.$$

(vi) *For any* $a \geq b + 1 \geq 2$,

$$R_a(x) - R_b(x) = \frac{U_{a-b-1}(z)}{\sqrt{x}U_a(z)U_b(z)}.$$

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