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Diego Marques

FIBONOMIAL COEFFICIENTS AT MOST ONE AWAY FROM FIBONACCI NUMBERS

Abstract. Let F_n be the nth Fibonacci number. For $1 \le k \le m$, let

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1} \cdots F_{m-k+1}}{F_1 \cdots F_k}$$

be the corresponding Fibonomial coefficient. It is known that $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_F$ is a Fibonacci number if and only if either $k = 1$ or $m \in \{k, k+1\}$. In this note, we find all solutions of the Diophantine equation $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_F \pm 1 = F_n$.

1. Introduction

Let $(F_n)_{n \ge 0}$ be the *Fibonacci sequence* given by $F_0 = 0$ and $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, for $n \ge 1$. The first few terms of this sequence are $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

The problem of the existence of infinitely many prime numbers in the Fibonacci sequence remains open, however several results on the prime factors of a Fibonacci number are known. For instance, a *primitive divisor* p of F_n is a prime factor of F_n which does not divide $\prod_{j=1}^{n-1} F_j$. It is known that a primitive divisor p of F_n exists whenever $n \ge 13$. The above statement is usually referred to the *Primitive Divisor Theorem* (see [1] for the most general version).

The *Fibonomial coefficient* $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_F$ is defined, for $1 \le k \le m$, by replacing each integer appearing in the numerator and denominator of $\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k(k-1)\cdots 1}$ with its respective Fibonacci number. That is

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1} \cdots F_{m-k+1}}{F_1 \cdots F_k}.$$

It is surprising that this quantity will always take integer values. This

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can be shown by an induction argument and the recursion formula

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = F_{k+1}\begin{bmatrix} m-1 \\ k \end{bmatrix}_F + F_{m-k-1}\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F,$$

which is a consequence of the formula $F_m = F_{k+1}F_{m-k} + F_kF_{m-k-1}$.

As an application of the Primitive Divisor Theorem, it is immediate that if $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_F = F_n$, then $\max\{m, n\} < 13$. Hence, assuming that $m - 1 > k > 1$ a quick computation reveals that there are no solutions for the previous Diophantine equation in that obtained range.

In this paper, we find all Fibonomial coefficients at most one away from a Fibonacci number. Our result is the following

Theorem 1. *The solutions of the Diophantine equation*

$$\begin{bmatrix} m \\ k \end{bmatrix}_F \pm 1 = F_n \tag{1}$$

with $m-1 \geq k > 1$, *are* $(m, k, n) = (3, 2, 4)$ *and* $(m, k, n) = (3, 2, 1)$, $(3, 2, 2)$, $(4, 2, 5)$, $(4, 3, 3)$ *according to whether the sign is* $+$ *or* $-$, *respectively.*

2. The proof of Theorem

2.1. Auxiliary results. The sequence of the *Lucas numbers* is defined by $L_{n+1} = L_n + L_{n-1}$, with $L_0 = 2$ and $L_1 = 1$. Let us state some interesting and helpful facts which will be essential ingredients in what follows.

Let $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. For all $n \geq 1$, we have

(L1) $F_{2n} = F_nL_n$;

(L2) (Binet's formulae) $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$;

(L3) $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$.

We may note that the Fibonacci and Lucas sequences can be extrapolated backwards using $F_n = F_{n+2} - F_{n+1}$ and $L_n = L_{n+2} - L_{n+1}$. Thus, for example, $F_{-1} = 1, F_{-2} = -1$, and so on. Furthermore, Binet's formulae (L2) remain valid for Fibonacci and Lucas numbers with negative indices, and they allow us to show that

Lemma 1. *For any integers* a, b, *we have*

$$F_aL_b = F_{a+b} + (-1)^bF_{a-b}.$$

Proof. Since $\alpha = (-\beta)^{-1}$ and thus $\beta = (-\alpha)^{-1}$, we have

$$F_aL_b = \frac{\alpha^a - \beta^a}{\alpha - \beta}(\alpha^b + \beta^b) = F_{a+b} + \frac{\alpha^a\beta^b - \beta^a\alpha^b}{\alpha - \beta} = F_{a+b} + (-1)^bF_{a-b}. \blacksquare$$

As a consequence of the previous lemma, a straight calculation gives a different factorization for $F_n \pm 1$ depending on the class of n modulo 4:

$$(2)\qquad \begin{aligned} F_{4\ell}+1 &= F_{2\ell-1}L_{2\ell+1} \ ; & F_{4\ell}-1 &= F_{2\ell+1}L_{2\ell-1}\\ F_{4\ell+1}+1 &= F_{2\ell+1}L_{2\ell} \ ; & F_{4\ell+1}-1 &= F_{2\ell}L_{2\ell+1}\\ F_{4\ell+2}+1 &= F_{2\ell+2}L_{2\ell} \ ; & F_{4\ell+2}-1 &= F_{2\ell}L_{2\ell+2}\\ F_{4\ell+3}+1 &= F_{2\ell+1}L_{2\ell+2} \ ; & F_{4\ell+3}-1 &= F_{2\ell+2}L_{2\ell+1}. \end{aligned}$$

Now, we are ready to deal with the proof of the theorem.

2.2. The proof. For $1 \leq n \leq 8$, a quick computation reveals that the only solutions are those in the statement of the theorem. So, let us assume that $n > 8$.

The equation (1) can be rewritten as $\begin{bmatrix} m \\ k \end{bmatrix}_F = F_n \mp 1$. By the relations in (2), we have eight possibilities for this Diophantine equation (again depending on the class of n modulo 4): For the $(+)$ case

$$(3)\qquad \begin{aligned} \begin{bmatrix} m \\ k \end{bmatrix}_F &= F_{2\ell+1}L_{2\ell-1} \ ; & \begin{bmatrix} m \\ k \end{bmatrix}_F &= F_{2\ell}L_{2\ell+1}\\ \begin{bmatrix} m \\ k \end{bmatrix}_F &= F_{2\ell}L_{2\ell+2} \ ; & \begin{bmatrix} m \\ k \end{bmatrix}_F &= F_{2\ell+2}L_{2\ell+1} \end{aligned}$$

and the (-) case

$$(4)\qquad \begin{aligned} \begin{bmatrix} m \\ k \end{bmatrix}_F &= F_{2\ell-1}L_{2\ell+1} \ ; & \begin{bmatrix} m \\ k \end{bmatrix}_F &= F_{2\ell+1}L_{2\ell}\\ \begin{bmatrix} m \\ k \end{bmatrix}_F &= F_{2\ell+2}L_{2\ell} \ ; & \begin{bmatrix} m \\ k \end{bmatrix}_F &= F_{2\ell+1}L_{2\ell+2}. \end{aligned}$$

We shall work only on the first equation in the left-hand side of (3) (the other ones can be handled in much the same way). Let us assume that $m \geq \max\{14, k+1\}$. Thus, we have

$$(5)\qquad F_m \cdots F_{m-k+1} = F_{2\ell+1}L_{2\ell-1}F_1 \cdots F_k.$$

Since $L_{2\ell-1} = F_{4\ell-2}/F_{2\ell-1}$ (see (L1)), we get

$$(6)\qquad F_m \cdots F_{m-k+1}F_{2\ell-1} = F_{2\ell+1}F_{4\ell-2}F_1 \cdots F_k.$$

However $4\ell-2 > 2\ell+1$, since $\ell = \lfloor n/4 \rfloor > 2$, and then the Primitive Divisor Theorem yields $m = 4\ell-2$. Thus, the identity (6) becomes

$$(7)\qquad F_{m-1} \cdots F_{m-k+1}F_{2\ell-1} = F_{2\ell+1}F_1 \cdots F_k.$$

Since $m-1 \geq 13$, we can use again the Primitive Divisor Theorem to get $m-1 = \max\{2\ell+1, k\}$. However $m-1 = 4\ell-3 > 2\ell+1$ and therefore $m-1 = k$ and (7) becomes $F_{2\ell-1} = F_{2\ell+1}$ which is an absurd.

So, we only need to consider the range $2 \leq k \leq 10$ and $k+2 \leq m \leq 12$. By using (L3) we get

$$\left(\frac{F_m}{F_1}\right) < \alpha^{m-1} \text{ and } \left(\frac{F_{m-t}}{F_{t+1}}\right) < \alpha^{m-2t}, \text{ for } 1 \leq t \leq k-1.$$

Therefore, we have

$$\begin{bmatrix} m \\ k \end{bmatrix}_F \leq \alpha^{m-1+m-2+\cdots+m-2(k-1)} = \alpha^{m-1+(m-k)(k-1)} \leq \alpha^{43} < 9.7 \times 10^8 - 1. \tag{8}$$

Thus $F_n \leq \left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_F + 1 < 9.7 \times 10^8$ and then $n < 40$.

We have written a simple program in Mathematica to see that in the obtained range $2 \leq k \leq m - 1 \leq 10$ and $9 \leq n \leq 39$ there is no further solution. Thus we have our desired result. Explicitly, $\left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right]_F + 1 = F_4$, $\left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right]_F - 1 = F_1 = F_2$, $\left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]_F - 1 = F_5$ and $\left[\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}\right]_F - 1 = F_3$. ∎

We finish by pointing out that the same method can be applied to provide all solutions of $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_F + 1 = F_n^2$. In fact, if $n > 2$, we have $\left[\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}\right]_F + 1 = F_3^2$, $\left[\begin{smallmatrix} 6 \\ 5 \end{smallmatrix}\right]_F + 1 = F_4^2$, as the only such solutions. The useful fact here is that $F_n^2 - 1$ can be factored as $(F_n - 1)(F_n + 1)$ and thus we can use the formulas in (2).

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References

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRASILIA
BRASILIA, P.O. BOX 70910-900, BRAZIL
E-mail: D.M.Ferreira@mat.unb.br

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