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Moments of products of automorphic L-functions ☆

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ABSTRACT

Assuming the generalized Riemann hypothesis, we prove upper bounds for moments of arbitrary products of automorphic L-functions and for Dedekind zeta-functions of Galois number fields on the critical line. As an application, we use these bounds to estimate the variance of the coefficients of these zeta- and L-functions in short intervals. We also prove upper bounds for moments of products of central values of automorphic L-functions twisted by quadratic Dirichlet characters and averaged over fundamental discriminants.

1. Introduction

An important problem in analytic number theory is to understand the behavior of L-functions on the critical line and at the central point. The Langlands program predicts that the most general L-functions are attached to automorphic representations of $\mathrm{GL}(n)$ over a number field and further conjectures that these L-functions should be expressible as products of the Riemann zeta-function and L-functions attached to cuspidal automorphic representations of $\mathrm{GL}(m)$ over the rationals. In this paper, we investigate

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the moments of such products on the critical line. We also prove estimates for moments of Dedekind zeta-functions, $\zeta_K(s)$, of Galois extensions K over $\mathbb{Q}$. In general, unless $\mathrm{Gal}(K/\mathbb{Q})$ is solvable, it is not known whether $\zeta_K(s)$ can be written as a product of automorphic L-functions (though the Langlands reciprocity conjecture predicts that this is the case).

An L-function is called *primitive* if it does not factor as a product of L-functions of smaller degree. Given a primitive L-function, $L(s,\pi)$, normalized so that $\Re(s) = 1/2$ is the critical line, it has been conjectured that there exist constants $C(k,\pi)$ such that

$$\int_0^T \left| L\left(\frac{1}{2} + it, \pi\right)\right|^{2k} dt \sim C(k,\pi) T (\log T)^{k^2} \tag{1.1}$$

for any $k > 0$ as $T \to \infty$, see [11]. The case $L(s,\pi) = \zeta(s)$, the Riemann zeta-function, has received the most attention. In addition to [11], see [12,15,23]. The conjecture in (1.1) has only been established in a few cases and only for small values of k. For degree one L-functions, the Riemann zeta-function and Dirichlet L-functions, the conjecture is known to hold when k is 1 or 2. For degree two L-functions, many cases of the conjecture have been established when $k = 1$. See, for instance, results of Good [16] and Zhang [47,48]. For higher degree L-functions, and for higher values of k, the conjecture seems to be beyond the scope of current techniques.

It is expected that the values of distinct primitive L-functions on the critical line are uncorrelated. Therefore, given r distinct primitive L-functions, $L(s,\pi_1), \ldots, L(s,\pi_r)$, normalized so that $\Re(s) = 1/2$ is the critical line, one might conjecture that for any $k_1, \ldots, k_r > 0$ we have

$$\int_0^T \left| L\left(\frac{1}{2} + it, \pi_1\right)\right|^{2k_1} \cdots \left| L\left(\frac{1}{2} + it, \pi_r\right)\right|^{2k_r} dt \sim C(\vec{\mathbf{k}}, \vec{\pi}) T (\log T)^{k_1^2 + \cdots + k_r^2} \tag{1.2}$$

for some constant $C(\vec{\mathbf{k}}, \vec{\pi})$ as $T \to \infty$ where $\vec{\mathbf{k}} = (k_1, \ldots, k_r)$ and $\vec{\pi} = (\pi_1, \ldots, \pi_r)$. In the case where $k_1, \ldots, k_r$ are natural numbers, Heap [18] has recently modified the approaches in [15] and [11] and made a precise conjecture for the constants $C(\vec{\mathbf{k}}, \vec{\pi})$. Using classical methods, the asymptotic formula in (1.2) can be established for products of two Dirichlet L-functions in the case when $k_1 = k_2 = 1$, $L(s,\pi_1) = L(s,\chi_1)$, and $L(s,\pi_2) = L(s,\chi_2)$ where χ_1 and χ_2 are distinct primitive Dirichlet characters. It seems that there are no other cases where the asymptotic formula in (1.2) has been established. The conjectural order of magnitude of the moments in (1.2) is consistent with the observation that the logarithms of distinct primitive L-functions on the critical line, $\log L(\frac{1}{2} + it, \pi_1)$ and $\log L(\frac{1}{2} + it, \pi_2)$, are (essentially) statistically independent if $\pi_1 \not\cong \pi_2$ as t varies under the assumption of Selberg's orthogonality conjectures[1] for the Dirichlet series

[1] For automorphic L-functions, we state Selberg's orthogonality conjectures in Section 2.

coefficients of $L(s,\pi_1)$ and $L(s,\pi_2)$. This statistical independence can be made precise; see, for instance, the work of Bombieri and Hejhal [4] and Selberg [42].

1.1. Moments of automorphic L-functions

In this paper, in support of the conjecture in (1.2), we prove the following mean-value estimate for arbitrary products of primitive automorphic L-functions.

Theorem 1.1. *Let $L(s,\pi_1),\dots,L(s,\pi_r)$ be L-functions attached to distinct irreducible cuspidal automorphic representations, π_j, of $\mathrm{GL}(m_j)$ over $\mathbb{Q}$ each with unitary central character, and assume that these L-functions satisfy the generalized Riemann hypothesis. Then, if $\max_{1\leqslant j\leqslant r} m_j \leqslant 4$, we have*

$$\int_0^T \left|L\left(\frac{1}{2}+it,\pi_1\right)\right|^{2k_1}\cdots\left|L\left(\frac{1}{2}+it,\pi_r\right)\right|^{2k_r} dt \ll T(\log T)^{k_1^2+\cdots+k_r^2+\varepsilon} \tag{1.3}$$

for any $k_1,\dots,k_r>0$ and every $\varepsilon>0$ when T is sufficiently large. The implied constant in (1.3) depends on $\pi_1,\dots,\pi_r$, $k_1,\dots,k_r$, and ε. If $\max_{1\leqslant j\leqslant r} m_j \geqslant 5$, then the inequality in (1.3) holds under the additional assumption of Hypothesis H described in Section 2.

Some of the standard properties of the L-functions described in Theorem 1.1 are reviewed in Section 2. Observe that the upper bound in Theorem 1.1 is nearly as sharp as the conjectured asymptotic formula in (1.2). Moreover, note that we do not assume that the L-functions in Theorem 1.1 satisfy the Ramanujan–Petersson conjecture. Instead, we assume Hypothesis H of Rudnick and Sarnak [40]. This mild (but unproven) conjecture is implied by the Ramanujan–Petersson conjecture and is known to hold for L-functions attached to irreducible cuspidal automorphic representations on $\mathrm{GL}(m)$ over $\mathbb{Q}$ if $m\leqslant 4$.

Our proof of Theorem 1.1 builds upon techniques of Soundararajan [43] and is inspired by the work of Chandee [8]. Corollary A of [43] states that for the Riemann zeta-function the inequality

$$T(\log T)^{k^2} \ll_k \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll_{k,\varepsilon} T(\log T)^{k^2+\varepsilon}$$

holds for any $k>0$ and every $\varepsilon>0$ assuming the Riemann hypothesis. The upper bound is due to Soundararajan, and the lower bound is due to Ramachandra [38]. In the case $r=1$, combining the result of Theorem 1.1 with the work of Pi [37], we deduce that

$$T(\log T)^{k^2} \ll_{k,\pi} \int_0^T \left|L\left(\frac{1}{2}+it,\pi\right)\right|^{2k} dt \ll_{\pi,k,\varepsilon} T(\log T)^{k^2+\varepsilon} \tag{1.4}$$

for any $k > 0$ and every $\varepsilon > 0$ where π is a self-contragredient irreducible cuspidal automorphic representations of $\mathrm{GL}(m)$ over $\mathbb{Q}$ under the assumptions of the generalized Riemann hypothesis and the Ramanujan–Petersson conjecture for $L(s,\pi)$. As mentioned above, the upper bound holds under weaker assumptions and for more general L-functions. We may let $L(s,\pi_1) = \zeta(s)$ in the proof of Theorem 1.1, so our theorem generalizes Soundararajan's result. As is the case in [43], it is possible to replace the ε in Theorem 1.1 by a quantity which is $O(1/\log\log\log T)$; see Ivić [19]. Moreover, an analogue of Theorem 1.1 for products of derivatives of L-functions can be proved using the techniques in [31] or [32].

There are a couple of aspects which make the proof of Theorem 1.1 different than the proof of the analogous result for the Riemann zeta-function. First of all, we need to understand the correlations of the coefficients of distinct L-functions averaged over the primes. Secondly, we need to handle the contribution of these coefficients at the prime powers. In [43], assuming the Riemann hypothesis, an inequality for the real part of the logarithm of the Riemann zeta-function is derived which depends only on the primes. In the case of $\zeta(s)$, one can handle the contribution of the primes powers relatively easily. If we were willing to assume the Ramanujan–Petersson conjecture and the generalized Riemann hypothesis for the symmetric square L-functions, then we could similarly derive an inequality involving only the primes for the real part of the logarithms of the L-functions in Theorem 1.1. In order to circumvent these additional assumptions, we must estimate the contribution from the prime powers in a different way. To this end, we use a partial result toward the Ramanujan–Petersson conjecture for automorphic L-functions due to Luo, Rudnick, and Sarnak [30] and also Hypothesis H (mentioned above) which is known to hold for automorphic L-functions of small degree. Ideas similar to these were used for degree two L-functions in [33].

Finally we remark that, assuming the generalized Riemann hypothesis and the Ramanujan–Petersson conjecture, Pi [37] has shown that the integral in (1.4) is $\ll T(\log T)^{k^2}$ if π is self-contragredient for any fixed k satisfying $0 < k < \frac{2}{m}$. Moreover, lower bounds for the integral in (1.4) which are $\gg T(\log T)^{k^2}$ for any positive rational number k have been established by Akbary and Fodden [1] assuming unproven bounds toward the Ramanujan–Petersson conjecture but without assuming the generalized Riemann hypothesis. The results in [1] are unconditional in the case $m = 2$.

After proving our main results, we learned that Harper [17] had refined Soundararajan's techniques. Assuming the Riemann hypothesis, he has shown that

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll_k T(\log T)^{k^2}.$$

We note that Harper uses Soundararajan's upper bounds for moments of $\zeta(s)$ in [43] to prove this result. By combining the ideas and results of the present paper with Harper's techniques, it may be possible to prove a version of Theorem 1.1 with $\varepsilon = 0$. We are investigating this possibility.

1.2. Moments of Dedekind zeta-functions

Let K be an algebraic number field, and let $\mathcal{O}_K$ denote its ring of integers. The Dedekind zeta-function, $\zeta_K(s)$, is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq 0} \frac{1}{(N\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{(N\mathfrak{p})^s}\right)^{-1}, \quad \Re(s) > 1, \tag{1.5}$$

where the sum runs over the nonzero ideals $\mathfrak{a}$ of $\mathcal{O}_K$, the product runs over the prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$, and $N = N_{K/\mathbb{Q}}$ denotes the absolute norm on K. It is known that the Dedekind zeta-function factors as a product of Artin L-functions. For instance, if K is a Galois extension of $\mathbb{Q}$ then

$$\zeta_K(s) = \prod_{\chi} L(s, \chi)^{\chi(1)} \tag{1.6}$$

where the product is over the irreducible characters χ of $\mathrm{Gal}(K/\mathbb{Q})$ and

$$\sum_{\chi} \chi(1)^2 = \left|\mathrm{Gal}(K/\mathbb{Q})\right| = [K : \mathbb{Q}]. \tag{1.7}$$

The Langlands reciprocity conjecture implies that each $L(s, \chi) = L(s, \pi)$ for an irreducible cuspidal automorphic representation π of $\mathrm{GL}(m)$ over $\mathbb{Q}$ where $\chi(1) = m$. By (1.2), (1.6), and (1.7), for Galois extensions K over $\mathbb{Q}$, this leads to the conjecture that

$$\int_0^T \left|\zeta_K\left(\frac{1}{2} + it\right)\right|^{2k} dt \sim C(k, K) T (\log T)^{[K:\mathbb{Q}]k^2} \tag{1.8}$$

for any $k > 0$ as $T \to \infty$. Here $C(k, K)$ is a constant depending on k and the number field K. The recent work of Heap [18] discusses this conjecture in more detail.

The conjectural asymptotic formula in (1.8) is known to hold when $k = 1$ for the Dedekind zeta-functions of quadratic extensions of $\mathbb{Q}$. Let d be a fundamental discriminant, and let $K = \mathbb{Q}[\sqrt{d}]$. Then Motohashi [35] has shown that

$$\int_0^T \left|\zeta_K\left(\frac{1}{2} + it\right)\right|^2 dt \sim \frac{6}{\pi^2} L(1, \chi_d)^2 \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} T \log^2 T$$

as $T \to \infty$ using the factorization $\zeta_K(s) = \zeta(s) L(s, \chi_d)$, where $L(s, \chi_d)$ is the Dirichlet L-function associated to χ_d, the Kronecker symbol of d. Also in support of (1.8), for finite Galois extensions K over $\mathbb{Q}$, Akbary and Fodden [1] have shown that the inequality

$$\int_0^T \left|\zeta_K\left(\frac{1}{2}+it\right)\right|^{2k} dt \gg T(\log T)^{[K:\mathbb{Q}]k^2}$$

holds for any rational number $k > 0$ as $T \to \infty$.

Using results of Arthur and Clozel [2], the following mean-value estimate for Dedekind zeta-functions is a consequence of Theorem 1.1.

Corollary 1.2. *Let K be a finite solvable Galois extension of $\mathbb{Q}$, and let $\zeta_K(s)$ be the associated Dedekind zeta-function. Then, assuming the generalized Riemann hypothesis for $\zeta_K(s)$, we have*

$$\int_0^T \left|\zeta_K\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll_{K,k,\varepsilon} T(\log T)^{[K:\mathbb{Q}]k^2+\varepsilon}$$

for any $k, \varepsilon > 0$ when T is sufficiently large.

Proof. If K is a finite solvable Galois extension of $\mathbb{Q}$, then Arthur and Clozel have shown that

$$\zeta_K(s) = \prod_{j=1}^{r} L(s,\pi_j)^{k_j} \tag{1.9}$$

where the π_j are irreducible cuspidal automorphic representations of the appropriate degree over $\mathbb{Q}$ and the exponents k_j are natural numbers satisfying $k_1^2+\cdots+k_r^2 = [K:\mathbb{Q}]$. See the concluding example in Chapter 3 of [2]. Moreover, since $\zeta_K(s)$ satisfies the Ramanujan–Petersson conjecture, Murty [36] observed that each factor $L(s,\pi_j)$ satisfies this conjecture, as well. Hence, Hypothesis H holds for each L-function in the product (1.9), and thus Theorem 1.1 implies that

$$\int_0^T \left|\prod_{j=1}^{r} L\left(\frac{1}{2}+it,\pi_j\right)^{k_j}\right|^{2k} dt \ll T(\log T)^{(k_1^2+\cdots+k_r^2)k^2+\varepsilon} = T(\log T)^{[K:\mathbb{Q}]k^2+\varepsilon}. \tag{1.10}$$

The corollary now follows from (1.9) and (1.10). □

The condition that $\mathrm{Gal}(K/\mathbb{Q})$ be a solvable group can be removed with a little more work. In Section 5, we sketch how to modify the proof of Theorem 1.1 to prove the following mean-value estimate.

Theorem 1.3. *Let K be a finite Galois extension of $\mathbb{Q}$. Then, assuming the generalized Riemann hypothesis for $\zeta_K(s)$, we have*

$$\int_0^T \left|\zeta_K\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll_{K,k,\varepsilon} T(\log T)^{[K:\mathbb{Q}]k^2+\varepsilon}$$

for any $k, \varepsilon > 0$ when T is sufficiently large.

Unlike the proof of Corollary 1.2, our proof of Theorem 1.3 does not rely on a factorization of $\zeta_K(s)$ into automorphic L-functions.

1.3. Coefficients of zeta- and L-functions in short intervals

Let K be a number field and let $r_K(n)$ denote the number of ideals in K of norm n. Then, by (1.5), we see that

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{r_K(n)}{n^s}, \quad \Re(s) > 1.$$

When K is a Galois extension of $\mathbb{Q}$, we can use Theorem 1.3 and a technique of Selberg [41] to study the distribution of $r_K(n)$ in short intervals assuming the generalized Riemann hypothesis for $\zeta_K(s)$. In order to state our result, recall that

$$\operatorname*{Res}_{s=1} \zeta_K(s) = \lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{D}} \tag{1.11}$$

where r_1 is the number of real embeddings of K, r_2 is the number of pairs of complex embeddings, h is the class number of K, R is the regulator, w is the number of roots of unity in K, and $D = |d_K|$ is the absolute value of the discriminant. Landau's classical mean-value estimate for the arithmetic function $r_K(n)$ is

$$\sum_{n\leqslant x} r_K(n) = \frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{D}} x + O\big(x^{1-2/([K:\mathbb{Q}]+1)}\big).$$

We prove the following conditional estimate for the variance of the arithmetic function $r_K(n)$ in short intervals.

Theorem 1.4. *Let K be a finite Galois extension of $\mathbb{Q}$. Let $y = y(x)$ be a positive and increasing function such that $y \to \infty$ and $y/x \to 0$ as $x \to \infty$. Then, assuming the generalized Riemann hypothesis for $\zeta_K(s)$, we have*

$$\frac{1}{X}\int_X^{2X} \left|\sum_{x<n\leqslant x+y} r_K(n) - \frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{D}} y\right|^2 dx \ll y(\log X)^{[K:\mathbb{Q}]+\varepsilon}$$

for $\varepsilon > 0$ when X is sufficiently large. Here the implied constant depends on K and ε.

Assuming the generalized Riemann hypothesis for $\zeta_K(s)$, it follows from Theorem 1.4 that

$$\sum_{x<n\leqslant x+y} r_K(n) \sim \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{D}}y$$

for *almost all* x if we choose y to be a function of x satisfying $y/(\log x)^{[K:\mathbb{Q}]+\varepsilon} \to \infty$ but $y/x \to 0$ as $x \to \infty$.

Using Theorem 1.1, we can similarly study the behavior of coefficients of products of automorphic L-functions in short intervals. To state the results in this situation, we first introduce some notation. For $k \geqslant 0$ an integer and $k_1, \ldots, k_r \in \mathbb{N}$, let

$$L(s) = \zeta(s)^k \prod_{j=1}^{r} L(s,\pi_j)^{k_j}$$

be an (automorphic) L-function. Here we are assuming that the L-functions $L(s,\pi_1), \ldots, L(s,\pi_r)$ are as in Theorem 1.1 and that $L(s,\pi_j) \neq \zeta(s)$ for all $1 \leqslant j \leqslant r$. We distinguish between the case $k=0$, where $L(s)$ is entire, and the case $k \geqslant 1$, where $L(s)$ has a pole of order k at $s=1$. For $\Re(s) > 1$, we set

$$L(s) = \begin{cases} \sum_{n=1}^{\infty} \frac{a_L(n)}{n^s}, & \text{if } k=0, \\ \sum_{n=1}^{\infty} \frac{b_L(n)}{n^s}, & \text{if } k \in \mathbb{N}. \end{cases}$$

As is to be expected, the behavior of $a_L(n)$ and $b_L(n)$ in short intervals differs due to the presence of the pole of the generating function when $k \geqslant 1$. For $x > 0$, we define

$$R_L(x) = \operatorname*{Res}_{s=1}\left(L(s)\frac{x^s}{s}\right).$$

Note that $R_L(x) = 0$ if $k=0$, that

$$R_L(x) = x\prod_{j=1}^{r} L(1,\pi_j)^{k_j}$$

if $k=1$, and that

$$R_L(x) = \frac{x(\log x)^{k-1}}{(k-1)!}\prod_{j=1}^{r} L(1,\pi_j)^{k_j} + O\big(x(\log x)^{k-2}\big)$$

if $k \geqslant 2$. With this set-up, assuming the conditions of Theorem 1.1, the proof of Theorem 1.4 can be modified to show that

$$\frac{1}{X}\int_X^{2X}\Bigg|\sum_{x<n\leqslant x+y} a_L(n)\Bigg|^2 dx \ll y(\log X)^{k_1^2+\cdots+k_r^2+\varepsilon}$$

and

$$\frac{1}{X}\int_{X}^{2X}\Bigg|\sum_{x<n\leqslant x+y} b_L(n) - \big(R_L(x+y) - R_L(x)\big)\Bigg|^2 dx \ll y(\log X)^{k^2+k_1^2+\cdots+k_r^2+\varepsilon}$$

for $\varepsilon > 0$ when X is sufficiently large. Here y is any function satisfying the conditions in Theorem 1.4, and the implied constants depend on $\pi_1, \ldots, \pi_r$, k, $k_1, \ldots, k_r$, and ε. The details are left to the interested reader.

1.4. Quadratic twists of automorphic L-functions

One can also use the methods of Soundararajan in [43] to study the moments of central values of quadratic twists of automorphic L-functions. In this case, the conjecture for the size of moments depends on the symmetry type of the family of these twists. Let $L(s,\pi)$ be an L-function attached to a self-contragredient irreducible cuspidal automorphic representation π on $\mathrm{GL}(m)$ over $\mathbb{Q}$. (We assume the L-function is self-dual so that the central value is real.) Then Katz and Sarnak [22] and Rubinstein [39] have conjectured that the family of quadratic twists of $L(s,\pi)$ has either symplectic or orthogonal symmetry corresponding to whether or not the symmetric square L-function $L(s,\pi,\wedge^2)$ has a pole at $s=1$.

Following the notation in [39], we set $\delta(\pi) = 1$ if $L(s,\pi,\wedge^2)$ does not have a pole at $s=1$ and set $\delta(\pi) = -1$ if $L(s,\pi,\wedge^2)$ has a pole at $s=1$. Then for each $k > 0$ it has been conjectured (see [24,10]) that there are constants $C^\flat(k,\pi) > 0$ such that

$$\sum_{|d|\leqslant X}^{\flat} L\left(\frac{1}{2}, \pi\otimes\chi_d\right)^k \sim C^\flat(k,\pi) X(\log X)^{k(k-\delta(\pi))/2}$$

as $X \to \infty$. Here the superscript $\flat$ indicates that the sums run over fundamental discriminants d, χ_d denotes the corresponding primitive quadratic Dirichlet character, and (as before) we have normalized so that $s = 1/2$ is the central point. In the case of quadratic Dirichlet L-functions and L-functions of quadratic twists of a fixed elliptic curve E over $\mathbb{Q}$, Soundararajan [43] proved that

$$\sum_{|d|\leqslant X}^{\flat} L\left(\frac{1}{2}, \chi_d\right)^k \ll X(\log X)^{k(k+1)/2+\varepsilon} \tag{1.12}$$

and

$$\sum_{|d|\leqslant X}^{\flat} L\left(\frac{1}{2}, E\otimes\chi_d\right)^k \ll X(\log X)^{k(k-1)/2+\varepsilon} \tag{1.13}$$

for every $k > 0$ and any $\varepsilon > 0$ assuming the generalized Riemann hypothesis for the relevant L-functions. (Note that in the first example the L-functions have $\delta(\pi) = -1$, and in the second case the L-functions have $\delta(\pi) = 1$.) We generalize these results and, in analogy with our Theorem 1.1, we prove the following result for central values of quadratic twists of arbitrary products of automorphic L-functions.

Theorem 1.5. *Let d denote a fundamental discriminant, and let χ_d be a primitive quadratic Dirichlet character of conductor $|d|$. Let $L(s,\pi_1),\dots,L(s,\pi_r)$ be L-functions attached to distinct self-contragredient irreducible cuspidal automorphic representations, π_j, of $\mathrm{GL}(m_j)$ over $\mathbb{Q}$ each with unitary central character, and assume that the twisted L-functions $L(s,\pi_1\otimes\chi_d),\dots,L(s,\pi_r\otimes\chi_d)$ satisfy the generalized Riemann hypothesis. Then, if $\max_{1\leqslant j\leqslant r} m_j \leqslant 2$, we have*

$$\sum_{|d|\leqslant X}^{\flat} L\left(\frac{1}{2},\pi_1\otimes\chi_d\right)^{k_1}\cdots L\left(\frac{1}{2},\pi_r\otimes\chi_d\right)^{k_r} \ll X(\log X)^{k_1(k_1-\delta(\pi_1))/2+\cdots+k_r(k_r-\delta(\pi_r))/2+\varepsilon}, \tag{1.14}$$

for any $k_1,\dots,k_r > 0$ and every $\varepsilon > 0$ when X is sufficiently large. Here the superscript $\flat$ indicates that the sum is restricted to fundamental discriminants, and the implied constant depends on $\pi_1,\dots,\pi_r$, $k_1,\dots,k_r$, and ε. If $\max_{1\leqslant j\leqslant r} m_j \geqslant 3$, then the inequality in (1.14) holds under the additional assumptions of Hypothesis H and Hypothesis E described in Section 2.

We now give two examples which are consequences of Theorem 1.5 and generalize Soundararajan's results in (1.12) and (1.13) to biquadratic extensions of $\mathbb{Q}$. Let d_1 and d_2 be coprime fundamental discriminants, and let $K_{d_1,d_2} = \mathbb{Q}[\sqrt{d_1},\sqrt{d_2}]$ be the corresponding biquadratic number field. Then the Dedekind zeta-function of K_{d_1,d_2} factors as

$$\zeta_{K_{d_1,d_2}}(s) = \zeta(s)L(s,\chi_{d_1})L(s,\chi_{d_2})L(s,\chi_{d_1d_2}),$$

and similarly, given an elliptic curve E over $\mathbb{Q}$, the Hasse–Weil L-function $L(s,E/K_{d_1,d_2})$ of E over K_{d_1,d_2} factors as

$$L(s,E/K_{d_1,d_2}) = L(s,E)L(s,E\otimes\chi_{d_1})L(s,E\otimes\chi_{d_2})L(s,E\otimes\chi_{d_1d_2}).$$

Using Theorem 1.5, we can estimate moments of $\zeta_{K_{d_1,d_2}}(\frac{1}{2})$ and $L(\frac{1}{2},E/K_{d_1,d_2})$ by averaging over two sets of fundamental discriminants. (We note that under the assumption of the generalized Riemann hypothesis for these zeta- and L-functions, these central values are non-negative real numbers.) In particular, we have

$$\sum_{\substack{|d_1 d_2| \leqslant X \\ (d_1,d_2)=1}}^{\flat} \zeta_{K_{d_1,d_2}}\left(\frac{1}{2}\right)^k \ll X(\log X)^{3k(k+1)/2+1+\varepsilon} \tag{1.15}$$

and

$$\sum_{\substack{|d_1 d_2| \leqslant X \\ (d_1,d_2)=1}}^{\flat} L\left(\frac{1}{2}, E/K_{d_1,d_2}\right)^k \ll X(\log X)^{3k(k-1)/2+1+\varepsilon} \tag{1.16}$$

for any $\varepsilon > 0$. Here the superscript $\flat$ indicates that the sum runs over two sets fundamental discriminants, d_1 and d_2. When $k = 1$, the conditional estimate in (1.15) is consistent with a result of Chinta [9] who proved that, as $X \to \infty$,

$$\sum_{d_1,d_2 \text{ odd}}^{\flat} a(d_1,d_2)\zeta_{K_{d_1,d_2}}\left(\frac{1}{2}\right)F\left(\frac{d_1 d_2}{X}\right) \sim cX \log^4 X$$

for a constant $c > 0$, where F is a smooth compactly supported test function satisfying $\int_0^\infty F(x)\,dx = 1$ and $a(d_1,d_2)$ is a weighting factor satisfying $a(d_1,d_2) = 1$ if $(d_1,d_2) = 1$ and is (on average) small otherwise.

Since the condition $(d_1,d_2) = 1$ implies that $\chi_{d_1 d_2} = \chi_{d_1}\chi_{d_2}$, and $\delta(\pi) = -1$ for any degree one L-function, under the conditions of Theorem 1.5 we have

$$\begin{aligned}
\sum_{\substack{|d_1 d_2| \leqslant X \\ (d_1,d_2)=1}}^{\flat} \zeta_{K_{d_1,d_2}}\left(\frac{1}{2}\right)^k &= \zeta\left(\frac{1}{2}\right)^k \sum_{|d_1| \leqslant X}^{\flat} L\left(\frac{1}{2}, \chi_{d_1}\right)^k \\
&\quad \times \sum_{\substack{|d_2| \leqslant X/|d_1| \\ (d_1,d_2)=1}}^{\flat} L\left(\frac{1}{2}, \chi_{d_2}\right)^k L\left(\frac{1}{2}, \chi_{d_1 d_2}\right)^k \\
&\ll X(\log X)^{k(k+1)+\varepsilon} \sum_{|d_1| \leqslant X}^{\flat} \frac{L(\frac{1}{2}, \chi_{d_1})^k}{|d_1|} \\
&\ll X(\log X)^{3k(k+1)/2+1+\varepsilon}
\end{aligned}$$

by two applications of (1.14) and summation by parts. This proves that the estimate in (1.15) follows from Theorem 1.5.

To prove (1.16), we observe that the modularity theorems of Wiles [46], Wiles and Taylor [45], and Breuil, Conrad, Diamond, and Taylor [5] imply that $L(s,E)$ and its quadratic twists correspond to L-functions attached to irreducible cuspidal automorphic representations of $\mathrm{GL}(2)$ over $\mathbb{Q}$. Moreover, we have $\delta(\pi) = 1$ for each of these L-functions. Therefore, under the conditions of Theorem 1.5, we similarly have

$$\begin{aligned}
\sideset{}{^\flat}\sum_{\substack{|d_1 d_2|\leqslant X\\(d_1,d_2)=1}} L\left(\frac{1}{2}, E/K_{d_1,d_2}\right)^k &= L\left(\frac{1}{2}, E\right)^k \sideset{}{^\flat}\sum_{|d_1|\leqslant X} L\left(\frac{1}{2}, E\otimes\chi_{d_1}\right)^k \\
&\quad\times \sideset{}{^\flat}\sum_{\substack{|d_2|\leqslant X/|d_1|\\(d_1,d_2)=1}} L\left(\frac{1}{2}, E\otimes\chi_{d_2}\right)^k L\left(\frac{1}{2}, E\otimes\chi_{d_1 d_2}\right)^k \\
&\ll X(\log X)^{k(k-1)+\varepsilon} \sideset{}{^\flat}\sum_{|d_1|\leqslant X} \frac{L(\frac{1}{2}, E\otimes\chi_{d_1})^k}{|d_1|} \\
&\ll X(\log X)^{3k(k-1)/2+1+\varepsilon}
\end{aligned}$$

by two more applications of (1.14) and summation by parts. This shows that the estimate in (1.16) also follows from Theorem 1.5.

1.5. Notation and conventions

Throughout the remainder of this article, we use ε to denote a small positive quantity which may vary from line to line. The letter p is always used to denote a prime. The superscript $\flat$ is used to denote that a sum is restricted to fundamental discriminants. Unless otherwise indicated, all implied constants are allowed to depend on the cuspidal automorphic representations π_j, the non-negative real numbers k_j, and ε.

2. Properties of automorphic L-functions

In this section, we review standard properties of automorphic L-functions on $\mathrm{GL}(m)$ over $\mathbb{Q}$ and their twists by Dirichlet characters. Some of this section overlaps with Section 2 of Rudnick and Sarnak [40] and Section 3.6 of Rubinstein [39] (see also [20]). Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}(m)$ over $\mathbb{Q}$ with unitary central character. For $\Re(s) > 1$, we let

$$L(s,\pi) = \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s} = \prod_p \prod_{j=1}^{m} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \tag{2.1}$$

be the global L-function attached to π (as defined by Godement and Jacquet in [13] and Jacquet and Shalika in [21]). Then $L(s,\pi)$ is either the Riemann zeta-function or $L(s,\pi)$ analytically continues to an entire function of order 1 satisfying a functional equation of the form

$$\begin{aligned}
\Phi(s,\pi) &:= N^{s/2}\gamma(s,\pi)L(s,\pi) \\
&= \epsilon_\pi \overline{\Phi}(1-s,\pi),
\end{aligned}$$

where N is a natural number, $|\epsilon_\pi| = 1$, $\overline{\Phi}(s,\pi) = \overline{\Phi(\overline{s},\pi)}$, and the gamma factor

$$\gamma(s,\pi)=\prod_{j=1}^{m}\Gamma_{\mathbb{R}}(s+\mu_j).$$

Here $\Gamma_{\mathbb{R}}(s)=\pi^{s/2}\Gamma(s/2)$, and the μ_j are complex numbers. Logarithmically differentiating the Euler product, we define

$$-\frac{L'}{L}(s,\pi):=-\frac{d}{ds}\log L(s,\pi)=\sum_{p^\ell,\,\ell\geqslant 1}\frac{(\alpha_1^\ell(p)+\cdots+\alpha_m^\ell(p))\log p}{p^{\ell s}}=\sum_{n=1}^{\infty}\frac{\Lambda_\pi(n)}{n^s}$$

for $\Re(s)>1$. We note that $\Lambda_\pi(p)=a_\pi(p)\log p$.

Let χ be a primitive Dirichlet character modulo q satisfying $(q,N)=1$, and define

$$L(s,\pi\otimes\chi):=\sum_{n=1}^{\infty}\frac{a_\pi(n)\chi(n)}{n^s}=\prod_p\prod_{j=1}^{m}\left(1-\frac{\alpha_j(p)\chi(p)}{p^s}\right)^{-1}$$

for $\Re(s)>1$. Then

$$-\frac{L'}{L}(s,\pi\otimes\chi):=-\frac{d}{ds}\log L(s,\pi\otimes\chi)=\sum_{n=1}^{\infty}\frac{\Lambda_\pi(n)\chi(n)}{n^s},$$

when $\Re(s)>1$. For $q>1$, the function $L(s,\pi\otimes\chi)$ continues to an entire function of order 1 and satisfies a functional equation of the form

$$\begin{aligned}\Phi(s,\pi\otimes\chi)&:=\left(q^mN\right)^{s/2}\gamma_\chi(s,\pi)L(s,\pi\otimes\chi)\\&=\epsilon_{\pi,\chi}\overline{\Phi}(1-s,\pi\otimes\chi),\end{aligned}$$

where $|\epsilon_{\pi,\chi}|=1$, $\overline{\Phi}(s,\pi\otimes\chi)=\overline{\Phi(\bar{s},\pi\otimes\chi)}$, and the gamma factor

$$\gamma_\chi(s,\pi)=\prod_{j=1}^{m}\Gamma_{\mathbb{R}}(s+\mu_{j,\chi})$$

for complex numbers $\mu_{j,\chi}$.

The generalized Riemann hypothesis states that all the zeros of the completed L-functions, $\Phi(s,f)$ and $\Phi(s,f\otimes\chi)$, are on the critical line $\Re(s)=1/2$. We always indicate the L-functions for which we are assuming this hypothesis. The Ramanujan–Petersson conjecture states that the Euler coefficients $\alpha_j(p)$ in (2.1) satisfy $|\alpha_j(p)|=1$ for all but a finite number of primes p. In general, this conjecture is open. Towards the Ramanujan–Petersson conjecture, Luo, Rudnick, and Sarnak [30] have shown that

$$\left|\alpha_j(p)\right|\leqslant p^{1/2-1/(m^2+1)}$$

for all p. It follows that

$$\left|\Lambda_\pi(n)\right| < m\Lambda(n)n^{1/2-1/(m^2+1)} \tag{2.2}$$

where $\Lambda(n)$ is the von Mangoldt function, defined by $\Lambda(n) = \log p$ if $n = p^j, j \geqslant 1$, and $\Lambda(n) = 0$ otherwise. The bound in (2.2) is crucial to our proofs of Theorem 1.1 and Theorem 1.5. Our proofs also assume Hypothesis H of Rudnick and Sarnak [40].

Hypothesis H. Let $j \geqslant 2$ be fixed, and let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}(m)$ over $\mathbb{Q}$. Then we have

$$\sum_p \frac{|\Lambda_\pi(p^j)|^2}{p^j} < \infty.$$

Hypothesis H is known to hold for automorphic L-functions of small degree.

Theorem 2.1. *Hypothesis H is true for $m \leqslant 4$.*

Proof. The case $m = 1$ is trivial, the case $m = 2$ follows from the work of Kim and Sarnak [26], the case $m = 3$ is due to Rudnick and Sarnak [40], and the case $m = 4$ is due to Kim [25]. □

Given distinct automorphic L-functions $L(s,\pi)$ and $L(s,\pi')$, we need to understand the correlation of their Dirichlet series coefficients averaged over the primes. Selberg [42] has made the following conjecture (in a different context).

Selberg's Orthogonality Conjectures. *Let π and π' be two irreducible unitary cuspidal automorphic representations of $\mathrm{GL}(m)$ and $\mathrm{GL}(m')$ over $\mathbb{Q}$, respectively, and let $x \geqslant 3$. Then*

$$\sum_{p\leqslant x} \frac{a_\pi(p)\overline{a_{\pi'}(p)}}{p} = \sum_{p\leqslant x} \frac{\Lambda_\pi(p)\overline{\Lambda_{\pi'}(p)}}{p\log^2 p} = \begin{cases} \log\log x + O(1), & \text{if } \pi \cong \pi', \\ O(1), & \text{if } \pi \ncong \pi'. \end{cases}$$

The following result allows us to use Selberg's orthogonality conjectures in the proofs of Theorem 1.1 and Theorem 1.5.

Theorem 2.2. *Let π and π' be two irreducible unitary cuspidal automorphic representations of $\mathrm{GL}(m)$ and $\mathrm{GL}(m')$ over $\mathbb{Q}$, respectively. If $L(s,\pi)$ and $L(s,\pi')$ satisfy Hypothesis H, then the coefficients of these L-functions satisfy Selberg's orthogonality conjectures. In particular, Selberg's orthogonality conjectures hold if $\max(m,m') \leqslant 4$.*

Proof. This was proved in the special case where at least one of π or π' is self-contragredient in [27,28], and in full generality by Liu and Ye in [29]. See also Avdispahić and Smajlović [3]. □

In order to prove Theorem 1.5, we need to understand the behavior of the Dirichlet series coefficients of automorphic L-functions averaged over the squares of primes. Let $L(s,\pi)$ be an L-function attached to a self-contragredient irreducible cuspidal automorphic representation π of $\mathrm{GL}(m)$ over $\mathbb{Q}$ (i.e. $\pi = \widetilde{\pi}$). The Rankin–Selberg L-function $L(s,\pi\otimes\widetilde{\pi}) = L(s,\pi\otimes\pi)$ factors as the product of the symmetric and exterior square L-functions

$$L(s,\pi\otimes\widetilde{\pi}) = L\bigl(s,\pi,\vee^2\bigr)\cdot L\bigl(s,\pi,\wedge^2\bigr)$$

and has a simple pole at $s=1$, see [6]. This pole must be carried by one of the factors on the right-hand side. Following [39], we denote the order of the pole of $L(s,\pi,\wedge^2)$ as $(1+\delta(\pi))/2$. Then it is known that

$$\sum_{p\leqslant x}\Lambda_\pi\bigl(p^2\bigr) \sim -\delta(\pi)x \tag{2.3}$$

as $x\to\infty$. We use this estimate, in a different form, in Section 6.

The proof of Theorem 1.5 also requires an assumption on the coefficients of the L-functions which is stronger than Hypothesis H.

Hypothesis E. Let $j\geqslant 2$ be a fixed integer, and let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}(m)$ over $\mathbb{Q}$. Then we have

$$\sum_p \frac{|\Lambda_\pi(p^{2j})|}{p^j} < \infty.$$

Note that Hypothesis E only applies to even powers of primes, and the power of the prime in the denominator differs from the corresponding exponent in Hypothesis H. Hypothesis E, though stronger than Hypothesis H, is still considerably weaker than the Ramanujan–Petersson conjecture. Indeed, it would follow if the Euler product coefficients in (2.1) satisfied a bound of the form $|\alpha_j(p)|\leqslant p^{1/4-\vartheta}$ for some $\vartheta>0$. Such a bound trivially holds when $m=1$ and follows from the work of Kim and Sarnak [26] when $m=2$. Therefore, in the proof of Theorem 1.5, we only need to assume Hypothesis E when $\max_{1\leqslant j\leqslant r} m_j\geqslant 3$.

3. Lemmas

In this section, we state three lemmas that will be used in the proof of Theorem 1.1.

Lemma 3.1. *If $\{b_n\}$ is a sequence of complex numbers such that $\sum|b_n|$ and $\sum n|b_n|^2$ are convergent, then*

$$\int_0^T \left| \sum_{n=1}^{\infty} b_n n^{-it} \right|^2 dt = T \sum_{n=1}^{\infty} |b_n|^2 + O\left(\sum_{n=1}^{\infty} n |b_n|^2 \right)$$

where the implied constant is absolute.

Proof. This is Montgomery and Vaughan's mean-value theorem for Dirichlet polynomials (see Corollary 3 of [34]). □

Lemma 3.2. *Let T be large, $x \geqslant 2$, and let ℓ and j be natural numbers satisfying $x^\ell \leqslant T^j$. Then for any complex numbers $b(p)$ we have*

$$\frac{1}{T} \int_T^{2T} \left| \sum_{p^j \leqslant x} \frac{b(p)}{p^{j(\sigma+it)}} \right|^{2\ell} dt \ll \ell! \left\{ \sum_{p^j \leqslant x} \frac{|b(p)|^2}{p^{2j\sigma}} \right\}^\ell$$

where j is fixed and the sum runs over the primes p.

Proof. This is a consequence of Lemma 3.1. The case $j = 1$ essentially corresponds to Lemma 3 of Soundararajan [43]. For any $s \in \mathbb{C}$, write

$$\left\{ \sum_{p \leqslant y} \frac{b(p)}{p^s} \right\}^\ell = \sum_{n \leqslant y^\ell} \frac{\beta_{y,\ell}(n)}{n^s},$$

where $\beta_{y,\ell}(n) = 0$ unless n is the product of ℓ (not necessarily distinct) primes, all less than or equal to y. Thus, we have

$$\int_T^{2T} \left| \sum_{p \leqslant y} \frac{b(p)}{p^{j(\sigma+it)}} \right|^{2\ell} dt = \int_T^{2T} \left| \sum_{n \leqslant y^\ell} \frac{\beta_{y,\ell}(n)}{n^{j\sigma+jit}} \right|^2 dt = \frac{1}{j} \int_{jT}^{2jT} \left| \sum_{n \leqslant y^\ell} \frac{\beta_{y,\ell}(n)}{n^{j\sigma+iu}} \right|^2 du,$$

where in the last step we have made the variable change $u = jt$. If $y^\ell \leqslant T$, then Lemma 3.1 implies that

$$\int_T^{2T} \left| \sum_{p \leqslant y} \frac{b(p)}{p^{j(\sigma+it)}} \right|^{2\ell} dt \ll \frac{2jT - jT}{j} \sum_{n \leqslant y^\ell} \frac{|\beta_{y,\ell}(n)|^2}{n^{2j\sigma}} \ll T \sum_{n \leqslant y^\ell} \frac{|\beta_{y,\ell}(n)|^2}{n^{2j\sigma}}.$$

By modifying the combinatorial argument appearing in the proof of Lemma 3 of [43] in a straightforward manner, it follows that

$$\sum_{n \leqslant y^\ell} \frac{|\beta_{y,\ell}(n)|^2}{n^{2j\sigma}} \ll \ell! \left\{ \sum_{p \leqslant y} \frac{|b(p)|^2}{p^{2j\sigma}} \right\}^\ell.$$

Combining estimates, the lemma follows. □

Lemma 3.3. *Assume that either $L(s,\pi)$ is the Riemann zeta-function or that $\Phi(s,\pi)$ has no pole or zero at $s=0,1$. Let $\lambda_0 = 0.4912\ldots$ denote the unique positive real number satisfying $e^{-\lambda_0} = \lambda_0 + \lambda_0^2/2$. Then, assuming the generalized Riemann hypothesis for $L(s,\pi)$, for all $\lambda_0 \leqslant \lambda \leqslant \log x/2$ and $\log x \geqslant 2$, we have*

$$\log\left|L\left(\frac{1}{2}+it,\pi\right)\right| \leqslant \Re \sum_{n\leqslant x} \frac{\Lambda_\pi(n)}{n^{\frac{1}{2}+\frac{\lambda}{\log x}+it}\log n}\frac{\log x/n}{\log x} + \frac{(1+\lambda)}{2}\frac{m\log T}{\log x} + O\left(\frac{1}{\log x}\right)$$

for $T \leqslant t \leqslant 2T$ and T sufficiently large, where the implied constant in the error term depends only on π.

Proof. The case where $L(s,\pi)$ corresponds to the Riemann zeta-function is due to Soundararajan [43], and the other cases are a consequence of Theorem 2.1 of Chandee [7]. □

4. Proof of Theorem 1.1

In this section, we state and prove a value distribution result for a linear combination of distinct primitive L-functions and use this to deduce Theorem 1.1. This value distribution result is an analogue of the main theorem in [43]. Let $L(s,\pi_1),\ldots,L(s,\pi_r)$ be r distinct primitive L-functions (as in Theorem 1.1) of degrees $m_1,\ldots,m_r$, respectively, let

$$\Delta = \max\{m_1^2+1,\ldots,m_r^2+1\},$$

and let

$$B = k_1m_1 + \cdots + k_rm_r + 1. \tag{4.1}$$

Define the set

$$\mathcal{A}(T,V) = \left\{t\in[T,2T]:\ k_1\log\left|L\left(\frac{1}{2}+it,\pi_1\right)\right| + \cdots + k_r\log\left|L\left(\frac{1}{2}+it,\pi_r\right)\right| \geqslant V\right\}$$

and the quantity

$$W = \left(k_1^2+\cdots+k_r^2\right)\log\log T.$$

Note that

$$\int_T^{2T}\left|L\left(\frac{1}{2}+it,\pi_1\right)\right|^{2k_1}\cdots\left|L\left(\frac{1}{2}+it,\pi_r\right)\right|^{2k_r}dt$$
$$= -\int_{-\infty}^{\infty}\exp(2V)\,d\,\mathrm{meas}\big(\mathcal{A}(T,V)\big)$$

$$= 2 \int_{-\infty}^{\infty} \exp(2V) \operatorname{meas}\bigl(\mathcal{A}(T,V)\bigr)\, dV. \tag{4.2}$$

To prove Theorem 1.1, it suffices to estimate the measure of $\mathcal{A}(T,V)$ for all $V \geqslant 3$ when T is large. Note that the definitions of $\mathcal{A}(T,V)$ and W depend on our choices of $k_1,\dots,k_r$, which we consider to be fixed throughout the proof of Proposition 4.1 below.

We prove estimates for the size of $\mathcal{A}(T,V)$ using Lemma 3.2 and Lemma 3.3. The contribution to the size of $\mathcal{A}(T,V)$ coming from the primes in the sum on the right-hand side of the inequality in Lemma 3.3 is estimated following the method of Soundararajan in [43], and the contribution from the prime powers p^j with $j > \Delta$ is estimated trivially. More care is necessary to handle the contribution from the prime powers p^j with $2 \leqslant j \leqslant \Delta$, and this is where we appeal to (2.2) and Hypothesis H. This allows us to circumvent using the Ramanujan–Petersson conjecture.

As might be expected, the proof of Theorem 1.1 relies on understanding the correlations between coefficients of distinct automorphic L-functions. The key ingredient to the proof of the proposition below (and hence Theorem 1.1) is the fact that the Selberg orthogonality conjectures imply that

$$\sum_{p\leqslant z} \frac{|k_1\Lambda_{\pi_1}(p) + \cdots + k_r\Lambda_{\pi_r}(p)|^2}{p\log^2 p} = \bigl(k_1^2 + \cdots + k_r^2\bigr)\log\log z + O(1) \tag{4.3}$$

as $z \to \infty$, which can be seen by expanding the square on the left hand side of (4.3).

Proposition 4.1. *Let $L(s,\pi_1),\dots,L(s,\pi_r)$ be L-functions attached to distinct irreducible cuspidal automorphic representations, π_j, of $\mathrm{GL}(m_j)$ over $\mathbb{Q}$ with unitary central character, and assume that these L-functions satisfy the generalized Riemann hypothesis. If $\max_{1\leqslant j\leqslant r} m_j \leqslant 4$ or each of the L-functions satisfies Hypothesis H, then the following inequalities hold. If $\sqrt{W} \leqslant V \leqslant \frac{W}{B^2}$, we have*

$$\operatorname{meas}\bigl(\mathcal{A}(T,V)\bigr) \ll T\frac{V}{\sqrt{W}}\exp\biggl(-\frac{V^2}{W}\biggl(1-\frac{4}{\log W}\biggr)\biggr);$$

if $\frac{W}{B^2} \leqslant V \leqslant \frac{1}{2B^2}W\log W$, we have

$$\operatorname{meas}\bigl(\mathcal{A}(T,V)\bigr) \ll T\frac{V}{\sqrt{W}}\exp\biggl(-\frac{V^2}{W}\biggl(1-\frac{7B^2V}{4W\log W}\biggr)^2\biggr);$$

and if $\frac{1}{2B^2}W\log W \leqslant V$, we have

$$\operatorname{meas}\bigl(\mathcal{A}(T,V)\bigr) \ll T\exp\biggl(-\frac{1}{129B^2}V\log V\biggr)$$

for any $k_1,\dots,k_r > 0$ when T is sufficiently large.

Proof. Our proof is similar to the proof of the main theorem of Soundararajan in [43], and our notation follows that of [43] and Chandee [8]. Let $L(s,\pi)$ be a primitive L-function of degree m. Choosing $x=(\log T)^{1-\varepsilon}$ and $\lambda=\lambda_0<\frac{1}{2}$, it follows from Lemma 3.3 and (2.2) that

$$\begin{aligned}\log\left|L\left(\frac{1}{2}+it,\pi\right)\right| &\leqslant m(\log T)^{1-\varepsilon}+\frac{(1+\lambda_0)m\log T}{2(1-\varepsilon)\log\log T}+O\left(\frac{1}{(1-\varepsilon)\log\log T}\right)\\ &\leqslant \frac{3m}{4}\frac{\log T}{\log\log T}\end{aligned}$$

for sufficiently large T. Therefore, we see that

$$k_1\log\left|L\left(\frac{1}{2}+it,\pi_1\right)\right|+\cdots+k_r\log\left|L\left(\frac{1}{2}+it,\pi_r\right)\right|\leqslant\frac{3(k_1m_1+\cdots+k_rm_r)}{4}\frac{\log T}{\log\log T}$$

when T is large. Recalling the definition of B in (4.1), we may assume that

$$\sqrt{W}\leqslant V\leqslant\frac{3(B-1)}{4}\frac{\log T}{\log\log T}$$

while proving the proposition. Note that $B>1$ (a fact that is useful when deriving the estimates below).

Define a parameter A as

$$A=\begin{cases}\frac{B}{2}\log W, & \text{if } \sqrt{W}\leqslant V\leqslant\frac{W}{B^2},\\ \frac{1}{2BV}W\log W, & \text{if } \frac{W}{B^2}<V\leqslant\frac{1}{2B^2}W\log W,\\ B, & \text{if } V>\frac{1}{2B^2}W\log W,\end{cases}$$

and let $x=T^{A/V}$ and $z=x^{1/\log\log T}$. Choosing $\lambda=1/2$ in Lemma 3.3, we deduce that

$$\begin{aligned}&k_1\log\left|L\left(\frac{1}{2}+it,\pi_1\right)\right|+\cdots+k_r\log\left|L\left(\frac{1}{2}+it,\pi_r\right)\right|\\ &\quad\leqslant\left|S_1(t)\right|+\left|S_1^\star(t)\right|+\sum_{2\leqslant j\leqslant\Delta}\left|S_j(t)\right|+\frac{3(B-1)}{4}\frac{V}{A}+O(1),\end{aligned}\tag{4.4}$$

where

$$\begin{aligned}S_1(t)&=\sum_{p\leqslant z}\frac{(k_1\Lambda_{\pi_1}(p)+\cdots+k_r\Lambda_{\pi_r}(p))}{p^{\frac{1}{2}+\frac{\lambda}{\log x}+it}\log p}\frac{\log(x/p)}{\log x},\\ S_1^\star(t)&=\sum_{z<p\leqslant x}\frac{(k_1\Lambda_{\pi_1}(p)+\cdots+k_r\Lambda_{\pi_r}(p))}{p^{\frac{1}{2}+\frac{\lambda}{\log x}+it}\log p}\frac{\log(x/p)}{\log x},\end{aligned}$$

and

$$S_j(t) = \sum_{p^j \leqslant x} \frac{(k_1 \Lambda_{\pi_1}(p^j) + \cdots + k_r \Lambda_{\pi_r}(p^j))}{p^{j(\frac{1}{2}+\frac{\lambda}{\log x}+it)} \log p^j} \frac{\log(x/p^j)}{\log x}$$

for $2 \leqslant j \leqslant \Delta$. The coefficient bound in (2.2) implies that the error term in (4.4) is $O(1)$ since

$$\sum_{j>\Delta} \left|S_j(t)\right| \ll \sum_{j>\Delta} \sum_{p^j \leqslant x} \frac{|k_1 \Lambda_{\pi_1}(p^j) + \cdots + k_r \Lambda_{\pi_r}(p^j)|}{j p^{j/2} \log p} \ll 1.$$

Let

$$V_1 := V\left(1 - \frac{7(B-1)}{8A}\right), \qquad V_1^\star = V_j := \frac{(B-1)V}{8\Delta A}$$

for $2 \leqslant j \leqslant \Delta$. Note that if $t \in \mathcal{A}(T,V)$, then at least one of the following inequalities holds:

$$\left|S_1^\star(t)\right| \geqslant V_1^\star \quad \text{or} \quad \left|S_j(t)\right| \geqslant V_j$$

for some $j = 1, 2, \ldots, \Delta$. If we define

$$N_j(T, V_j) := \operatorname{meas}\left\{t \in [T, 2T]\colon\ \left|S_j(t)\right| \geqslant V_j\right\}$$

for $j = 1, 2, \ldots, \Delta$ and define $N_1^\star(T, V_1^\star)$ similarly, then we can bound $N_j(T, V_j)$ and $N_1^\star(T, V_1^\star)$ using Lemma 3.2 since Chebyshev's inequality implies that

$$N_j(T, V_j) \leqslant (V_j)^{-2\ell} \int_T^{2T} \left|S_j(t)\right|^{2\ell} dt$$

and

$$N_1^\star(T, V_1^\star) \leqslant \left(V_1^\star\right)^{-2\ell} \int_T^{2T} \left|S_1^\star(t)\right|^{2\ell} dt$$

for every non-negative integer ℓ.

Let us first estimate $N_1(T, V_1)$. Letting ℓ be any natural number such that $z^\ell \leqslant T$, Lemma 3.2 and (4.3) imply that

$$\begin{aligned}
\int_T^{2T} \left|S_1(t)\right|^{2\ell} dt &\ll T\ell! \left(\sum_{p \leqslant z} \frac{|k_1 \Lambda_{\pi_1}(p) + \cdots + k_r \Lambda_{\pi_r}(p)|^2}{p \log^2 p}\right)^\ell \\
&\ll T\ell! \left(\left(k_1^2 + \cdots + k_r^2\right) \log\log z + O(1)\right)^\ell \\
&\ll T\ell! \left(\left(k_1^2 + \cdots + k_r^2\right) \log\log T\right)^\ell
\end{aligned}$$

$$\ll T\sqrt{\ell}\left(\frac{\ell(k_1^2+\cdots+k_r^2)\log\log T}{e}\right)^{\ell}$$
$$\ll T\sqrt{\ell}\left(\frac{\ell W}{e}\right)^{\ell}.$$

Thus we have

$$N_1(T,V_1) \ll T\sqrt{\ell}\left(\frac{\ell W}{eV_1^2}\right)^{\ell}. \tag{4.5}$$

We consider separately the two cases where $V \leqslant \frac{W^2}{B^4}$ and $V > \frac{W^2}{B^4}$. In the first case, we choose $\ell = \lfloor \frac{V_1^2}{W} \rfloor$ in (4.5) and find that

$$N_1(T,V_1) \ll T\frac{V}{\sqrt{W}}\exp\left(-\frac{V_1^2}{W}\right).$$

In the case, where $V > \frac{W^2}{B^4}$, we choose $\ell = \lfloor 10V \rfloor$ in (4.5) and find that

$$N_1(T,V_1) \ll T\exp(-4V\log V).$$

Hence

$$N_1(T,V_1) \ll T\frac{V}{\sqrt{W}}\exp\left(-\frac{V_1^2}{W}\right) + T\exp(-4V\log V) \tag{4.6}$$

for all V.

Next, we find an upper bound for $N_1^\star(T,V_1^\star)$. For any natural number ℓ with $x^\ell \leqslant T$, Lemma 3.2 and (4.3) imply that

$$\int_T^{2T} \left|S_1^\star(t)\right|^{2\ell} dt \ll T\ell!\left(\sum_{z\leqslant p\leqslant x}\frac{|k_1\Lambda_{\pi_1}(p)+\cdots+k_r\Lambda_{\pi_r}(p)|^2}{p\log^2 p}\right)^{\ell}$$
$$\ll T\ell!\,\left(\left(k_1^2+\cdots+k_r^2\right)(\log\log x - \log\log z) + O(1)\right)^{\ell}$$
$$\ll T\left(\ell\left(k_1^2+\cdots+k_r^2\right)\log\log\log T + O(1)\right)^{\ell}$$
$$\ll T\left(2\ell\left(k_1^2+\cdots+k_r^2\right)\log\log\log T\right)^{\ell}$$

when T is large. Choosing $\ell = \lfloor \frac{V}{A} \rfloor$, we have that

$$N_1^\star(T,V_1^\star) \ll T\left(\frac{8\Delta A}{(B-1)V}\right)^{2\ell}\left(2\ell\left(k_1^2+\cdots+k_r^2\right)\log\log\log T\right)^{\ell}$$
$$\ll T\exp\left(-\frac{V\log V}{2A}\right). \tag{4.7}$$

Finally, we find an upper bound for $N_j(T, V_j)$ for each $2 \leqslant j \leqslant \Delta$. For $x^{1/j} \leqslant T$, Lemma 3.2 and Hypothesis H imply that

$$\int_T^{2T} \left|S_j(t)\right|^{2\ell} dt \ll T\ell! \left(\sum_{p^j \leqslant x} \frac{|k_1 \Lambda_{\pi_1}(p^j) + \cdots + k_r \Lambda_{\pi_r}(p^j)|^2}{j^2 p^j \log^2 p} \right)^{\ell}$$
$$\ll T\left(\ell C_j \left(k_1^2 + \cdots + k_r^2\right)\right)^{\ell},$$

for each fixed j, where C_j is a constant (depending on j). Let

$$C_{\max} = \max_{2 \leqslant j \leqslant \Delta} C_j$$

be an absolute constant. Then for every $2 \leqslant j \leqslant \Delta$, we have

$$\int_T^{2T} \left|S_j(t)\right|^{2\ell} dt \ll T\left(\ell C_{\max} \left(k_1^2 + \cdots + k_r^2\right)\right)^{\ell}.$$

Comparing this upper bound to the upper bound for $\int_T^{2T} |S_1^\star(t)|^{2\ell}\, dt$, we conclude that

$$N_j(T, V_j) \ll T \exp\left(-\frac{V \log V}{2A}\right), \tag{4.8}$$

for each $2 \leqslant j \leqslant \Delta$. The proposition now follows by combining the estimates in (4.6), (4.7), and (4.8). $\square$

We now use Proposition 4.1 and (4.2) to prove Theorem 1.1.

Proof of Theorem 1.1. Proposition 4.1 implies that

$$\operatorname{meas}\left(\mathcal{A}(T, V)\right) \ll \begin{cases} T(\log T)^{\varepsilon} \exp(-\frac{V^2}{W}), & \text{if } 3 \leqslant V \leqslant \frac{256W}{B^2}, \\ T(\log T)^{\varepsilon} \exp(-\frac{4V}{B^2}), & \text{if } V > \frac{256W}{B^2}. \end{cases}$$

Inserting these bounds into (4.2) and estimating the range $V < 3$ trivially, we deduce that

$$\int_T^{2T} \left|L\left(\frac{1}{2} + it, \pi_1\right)\right|^{2k_1} \cdots \left|L\left(\frac{1}{2} + it, \pi_r\right)\right|^{2k_r} dt$$
$$\ll T(\log T)^{\varepsilon} e^{W} 256W \ll T(\log T)^{k_1^2 + \cdots k_r^2 + \varepsilon}.$$

Theorem 1.1 now follows by summing this estimate over the dyadic intervals $[\frac{T}{2}, T]$, $[\frac{T}{4}, \frac{T}{2}]$, $[\frac{T}{8}, \frac{T}{4}], \ldots$. $\square$

5. Sketch of the proof of Theorem 1.3

We now sketch how to modify the proof of Theorem 1.1 to deduce Theorem 1.3. Throughout this section, let K be a finite extension of $\mathbb{Q}$, and let $\zeta_K(s)$ be the associated Dedekind zeta-function. As before, our starting point is the observation that

$$\int_T^{2T} \left|\zeta_K\left(\frac{1}{2}+it\right)\right|^{2k} dt = 2\int_{-\infty}^{\infty} \exp(2V)\,\mathrm{meas}\big(\mathcal{A}(T,V)\big)\,dV \tag{5.1}$$

where

$$\mathcal{A}(T,V) = \left\{t \in [T,2T]\colon\ k\log\left|\zeta_K\left(\frac{1}{2}+it\right)\right| \geqslant V\right\}.$$

In order to bound the measure of $\mathcal{A}(T,V)$, we need analogues of Lemma 3.3 and (4.3) for $\zeta_K(s)$. For $\Re(s) > 1$, define

$$\frac{\zeta_K'}{\zeta_K}(s) := \frac{d}{ds}\log\zeta_K(s) = -\sum_{n=1}^{\infty}\frac{\Lambda_K(n)}{n^s}.$$

Since $\zeta_K(s)$ satisfies the Ramanujan–Petersson conjecture, we have

$$\big|\Lambda_K(n)\big| \leqslant [K:\mathbb{Q}]\Lambda(n).$$

Then the following analogue of Lemma 3.3 holds.

Lemma 5.1. *Let $\lambda_0 = 0.4912\ldots$ denote the unique positive real number satisfying $e^{-\lambda_0} = \lambda_0 + \lambda_0^2/2$. Then, assuming the generalized Riemann hypothesis for $\zeta_K(s)$, for all $\lambda_0 \leqslant \lambda \leqslant \log x/2$ and $\log x \geqslant 2$, we have*

$$\log\left|\zeta_K\left(\frac{1}{2}+it\right)\right| \leqslant \Re\sum_{n\leqslant x}\frac{\Lambda_K(n)}{n^{\frac{1}{2}+\frac{\lambda}{\log x}+it}\log n}\frac{\log x/n}{\log x} + \frac{(1+\lambda)}{2}\frac{[K:\mathbb{Q}]\log T}{\log x} + O\left(\frac{1}{\log x}\right)$$

for $T \leqslant t \leqslant 2T$ and T sufficiently large, where the implied constant in the error term depends only on K.

Proof. This is a consequence of Theorem 2.1 of Chandee [7]. □

The analogue of (4.3) follows from the Chebotarev density theorem.

Lemma 5.2. *Let K be a finite Galois extension of $\mathbb{Q}$, and let p denote a rational prime. Then*

$$\sum_{p\leqslant x} r_K(p)^2 \sim [K:\mathbb{Q}]\sum_{p\leqslant x} 1$$

as $x\to\infty$, and in particular

$$\sum_{p\leqslant x} \frac{r_K(p)^2}{p} \sim [K:\mathbb{Q}]\log\log x. \tag{5.2}$$

Proof. Let (p) denote the principal ideal in $\mathcal{O}_K$ generated by p. Then

$$(p)=\mathfrak{P}_1^{e_1}\cdots\mathfrak{P}_r^{e_r},$$

where the $\mathfrak{P}_i$ are the distinct prime ideals in $\mathcal{O}_K$ lying above p with norm p^{f_i}. It follows that

$$\sum_{i=1}^{r} e_i f_i=[K:\mathbb{Q}].$$

If p is unramified in K, then $e_1=\cdots=e_r=1$. Since K is Galois, all the $\mathfrak{P}_i$ lying above p are conjugate. Thus $f_1=\cdots=f_r=f$, say. Therefore, for unramified primes p, we see that $r_K(p)\neq 0$ if and only if $f=1$. In this case, p completely splits, $r=[K:\mathbb{Q}]$, and hence $r_K(p)=[K:\mathbb{Q}]$. That is, for unramified primes p, we have

$$r_K(p)=\begin{cases}[K:\mathbb{Q}], & \text{if and only if } p \text{ splits completely},\\ 0, & \text{otherwise.}\end{cases}$$

Since there are only a finite number of ramified primes, it follows that

$$\sum_{p\leqslant x} r_K(p)^2=\sum_{\substack{p\leqslant x\\ p\text{ unramified}}} r_K(p)^2+O(1)=\sum_{\substack{p\leqslant x\\ p\text{ splits completely}}}[K:\mathbb{Q}]^2+O(1).$$

On the other hand, the Chebotarev density theorem implies that

$$\sum_{\substack{p\leqslant x\\ p\text{ splits completely}}} 1\sim\frac{1}{[K:\mathbb{Q}]}\sum_{p\leqslant x}1,$$

as $x\to\infty$. Thus,

$$\sum_{p\leqslant x} r_K(p)^2 \sim [K:\mathbb{Q}]\sum_{p\leqslant x} 1,$$

proving the first assertion of the lemma. Using this estimate, the prime number theorem and partial summation imply (5.2), completing the proof of the lemma. □

We now indicate how to prove Theorem 1.3. Choosing $W = k^2[K:\mathbb{Q}]\log\log T$, $B = k[K:\mathbb{Q}]+1$, $\Delta = 2$ (since the Ramanujan–Petersson conjecture holds for $\zeta_K(s)$), and A as before, a straightforward modification of the analysis in the previous section implies that

$$\operatorname{meas}\big(\mathcal{A}(T,V)\big) \ll \begin{cases} T(\log T)^{\varepsilon}\exp(-\frac{V^2}{W}), & \text{if } 3 \leqslant V \leqslant \frac{256W}{B^2}, \\ T(\log T)^{\varepsilon}\exp(-\frac{4V}{B^2}), & \text{if } V > \frac{256W}{B^2}. \end{cases}$$

Inserting these bounds into (5.1), we deduce Theorem 1.3.

Remark. In order to prove Theorem 1.3, it is not necessary to derive an asymptotic formula for the sum in (5.2). An upper bound of $[K:\mathbb{Q}]\log\log x + O(1)$ for the sum in (5.2) would be sufficient and is more easily derived. For instance, since $0 \leqslant r_K(p) \leqslant [K:\mathbb{Q}]$, we see that

$$\sum_{p\leqslant x}\frac{r_K(p)^2}{p} \leqslant [K:\mathbb{Q}]\sum_{p\leqslant x}\frac{r_K(p)}{p} \leqslant [K:\mathbb{Q}]\log\log x + O(1)$$

by Landau's prime ideal theorem.

6. Sketch of the proof of Theorem 1.5

We now sketch how to modify the proof of Theorem 1.1 to deduce Theorem 1.5. In this case, the starting point is the observation that

$$\begin{aligned} &\sideset{}{^\flat}\sum_{|d|\leqslant X} L\left(\frac{1}{2},\pi_1\otimes\chi_d\right)^{k_1}\cdots L\left(\frac{1}{2},\pi_r\otimes\chi_d\right)^{k_r} \\ &\quad = \int_{-\infty}^{\infty}\exp\left(V - \frac{k_1\delta(\pi_1)+\cdots+k_r\delta(\pi_r)}{2}\log\log X\right)\mathcal{N}(X,V)\,dV, \end{aligned} \tag{6.1}$$

where $\mathcal{N}(X,V)$ denotes the number of fundamental discriminants d with $|d| \leqslant X$ such that

$$\begin{aligned} &k_1\log\left|L\left(\frac{1}{2},\pi_1\otimes\chi_d\right)\right| + \cdots + k_r\log\left|L\left(\frac{1}{2},\pi_r\otimes\chi_d\right)\right| \\ &\quad \geqslant V - \left(\frac{k_1\delta(\pi_1)+\cdots+k_r\delta(\pi_r)}{2}\right)\log\log X. \end{aligned} \tag{6.2}$$

We can bound $\mathcal{N}(X,V)$ with the following analogues of Lemma 3.2 and Lemma 3.3. (Note that the definition of $\mathcal{N}(X,V)$ takes into account the contribution from the squares of primes in Lemma 6.2, below.)

Lemma 6.1. *Let X and y be real numbers, and let ℓ be a natural number with $y^\ell \leqslant X^{1/2}/\log X$. For any complex numbers $b(p)$ we have*

$$\sideset{}{^\flat}\sum_{|d|\leqslant X}\left|\sum_{2<p\leqslant y}\frac{b(p)\chi_d(p)}{p^{1/2}}\right|^{2\ell} \ll X\frac{(2\ell)!}{\ell!2^\ell}\left(\sum_{p\leqslant y}\frac{|b(p)|^2}{p}\right),$$

where the implied constant is absolute.

Proof. This is Lemma 6.3 of Soundararajan and Young [44]. □

Lemma 6.2. *Let $L(s,\pi)$ be an L-function attached to an irreducible cuspidal automorphic representation π on $\mathrm{GL}(m)$ over $\mathbb{Q}$ and let d be a fundamental discriminant. Let $\lambda_0 = 0.4912\ldots$ denote the unique positive real number satisfying $e^{-\lambda_0} = \lambda_0 + \lambda_0^2/2$. Then, assuming the generalized Riemann hypothesis for $L(s,\pi\otimes\chi_d)$, for all $\lambda_0 \leqslant \lambda \leqslant \log x/2$ and $\log x \geqslant 2$, we have*

$$\log\left|L\left(\frac{1}{2},\pi\otimes\chi_d\right)\right| \leqslant \left|\sum_{n\leqslant x}\frac{\Lambda_\pi(n)\chi_d(n)}{n^{\frac{1}{2}+\frac{\lambda}{\log x}}\log n}\frac{\log x/n}{\log x}\right| + \frac{(1+\lambda)}{2}\frac{m\log|d|}{\log x} + O\left(\frac{1}{\log x}\right),$$

where the implied constant depends only on π.

Proof. This follows from Theorem 2.1 of Chandee [7]. □

We now indicate how to prove Theorem 1.5. The primary difference between the proof of this theorem and the proof of Theorem 1.1 is how we handle the contribution from the prime powers. By (2.3), the contribution from the prime squares to the inequality in Lemma 6.2 is

$$\sum_{p^2\leqslant x}\frac{\Lambda_\pi(p^2)\chi_d(p^2)}{p^{1+\frac{2\lambda}{\log x}}\log p^2} \sim -\frac{\delta(\pi)}{2}\log\log x,$$

giving rise to the extra term on the right-hand side of (6.2) in the definition of $\mathcal{N}(X,V)$. In this way, the squares of primes contribute to our bounds for the size of these moments.

In contrast to the proof of Theorem 1.1, we must handle prime powers p^j with $j>2$ differently depending on whether j is odd or even. When j is odd, $\chi_d(p^j) = \chi_d(p)$, and hence we can average over fundamental discriminants using Lemma 6.1, (2.2), and Hypothesis H in a manner analogous to the analysis in Section 4. If j is even, then $\chi_d(p^j) = 1$ for $p \nmid d$, and therefore we cannot average over discriminants to estimate their contribution. Instead, we use Hypothesis E to show that the contribution of these primes to Lemma 6.2 is $O(1)$.

With these changes, choosing B, Δ, and A as in Section 4 and $W = (k_1^2+\cdots+k_r^2)\log\log x$, a relatively straightforward modification of the proof of Proposition 4.1 gives

$$\mathcal{N}(X,V) \ll \begin{cases} X(\log X)^{\varepsilon}\exp(-\frac{V^2}{2W}), & \text{if } 3 \leqslant V \leqslant \frac{512W}{B^2}, \\ X(\log X)^{\varepsilon}\exp(-\frac{4V}{B^2}), & \text{if } V > \frac{512W}{B^2}. \end{cases}$$

Theorem 1.5 now follows by inserting these bounds into (6.1).

7. Proof of Theorem 1.4

We follow the proof of Theorem 1 of Selberg [41], who studied the distribution of primes in short intervals using upper bounds for moments of the logarithmic derivative of $\zeta(s)$ near the critical line. (See also Section 4 of Goldston, Gonek, and Montgomery [14].) For K a finite Galois extension of $\mathbb{Q}$, let

$$c_K = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{D}}, \qquad S(x) = \sum_{n \leqslant x} r_K(n), \quad \text{and} \quad S_0(x) = \frac{1}{2}\lim_{\varepsilon \to 0}\big(S(x+\varepsilon) + S(x-\varepsilon)\big)$$

so that $S(x) = S_0(x)$ for almost all x. Perron's formula implies that

$$S_0(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \zeta_K(s)\frac{x^s}{s}\,ds.$$

Assuming the generalized Riemann hypothesis (GRH) for $\zeta_K(s)$, we move the contour left from $\Re(s) = 2$ to $\Re(s) = 1/2$ passing over a pole of the integrand at $s = 1$ and no other singularities. Here we are implicitly using the generalized Lindelöf hypothesis for $\zeta_K(s)$ in t-aspect (which follows from GRH) to justify the contour shift. Thus by the residue calculation in (1.11) and a variable change, we have

$$S_0(x) - c_K x = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \zeta_K(s)\frac{x^s}{s}\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta_K\left(\frac{1}{2}+it\right)\left(\frac{x^{\frac{1}{2}+it}}{\frac{1}{2}+it}\right)dt.$$

Applying this formula twice with the values $x = e^{\tau+\kappa}$ and $x = e^{\tau}$, it follows that

$$\frac{S_0(e^{\kappa+\tau}) - S_0(e^{\tau}) - c_K(e^{\kappa}-1)e^{\tau}}{e^{\tau/2}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta_K\left(\frac{1}{2}+it\right)\left(\frac{e^{\kappa(\frac{1}{2}+it)}-1}{\frac{1}{2}+it}\right)e^{i\tau t}\,dt,$$

giving a Fourier transform relation for all $\tau \in \mathbb{R}$. By Plancherel's theorem, since $S_0(x) = S(x)$ almost everywhere, we have

$$\int_{-\infty}^{\infty} \left|S\big(e^{\kappa+\tau}\big) - S\big(e^{\tau}\big) - c_K\big(e^{\kappa}-1\big)e^{\tau}\right|^2 \frac{d\tau}{e^{\tau}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\zeta_K\left(\frac{1}{2}+it\right)\right|^2 \left|\frac{e^{\kappa(\frac{1}{2}+it)}-1}{\frac{1}{2}+it}\right|^2 dt.$$

Observing that the integrand on the left-hand side is even and letting $x = e^{\tau}$, $X \geqslant T \geqslant 2$, and $e^{\kappa} = 1 + 1/T$, we derive that

$$\begin{aligned}\int_X^{2X} \left| S\left(x + \frac{x}{T}\right) - S(x) - c_K \frac{x}{T}\right|^2 \frac{dx}{x^2} &\leqslant \int_0^{\infty} \left| S\left(x + \frac{x}{T}\right) - S(x) - c_K \frac{x}{T}\right|^2 \frac{dx}{x^2} \\ &= \frac{1}{\pi} \int_0^{\infty} \left|\zeta_K\left(\frac{1}{2} + it\right)\right|^2 \left| \frac{e^{\kappa(\frac{1}{2}+it)} - 1}{\frac{1}{2} + it}\right|^2 dt \\ &= \frac{1}{\pi} \sum_{\ell=0}^{\infty} \int_{(2^{\ell}-1)T}^{(2^{\ell+1}-1)T} \left|\zeta_K\left(\frac{1}{2} + it\right)\right|^2 \left| \frac{e^{\kappa(\frac{1}{2}+it)} - 1}{\frac{1}{2} + it}\right|^2 dt \\ &\ll \sum_{\ell=0}^{\infty} \frac{1}{(2^{\ell}T)^2} \int_0^{(2^{\ell+1}-1)T} \left|\zeta_K\left(\frac{1}{2} + it\right)\right|^2 dt.\end{aligned}$$

It follows from this and Theorem 1.3 that

$$\frac{1}{X^2} \int_X^{2X} \left| S\left(x + \frac{x}{T}\right) - S(x) - c_K \frac{x}{T}\right|^2 dx \ll \sum_{\ell=0}^{\infty} \frac{1}{2^{\ell}T} (\log T)^{[K:\mathbb{Q}]+\varepsilon} \ll \frac{(\log T)^{[K:\mathbb{Q}]+\varepsilon}}{T}$$

for any $\varepsilon > 0$. Theorem 1.4 now follows by choosing $y = x/T$.

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