

## Laguerre moments and generalized functions

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### Abstract

Here we explore the link between the moments of the Laguerre polynomials or *Laguerre moments* and the *generalized functions* (as the Dirac delta-function and its derivatives), presenting several interesting relations. A useful application is related to a procedure for calculating mean values in quantum optics that makes use of the so-called quasi-probabilities. One of them, the *P-distribution*, can be represented by a sum over Laguerre moments when the electromagnetic field is in a photon-number state. Consequently, the P-distribution can be expressed in terms of Dirac delta-function and derivatives. More specifically, we found a direct relation between P-distributions and the *Laguerre factorial moments*.

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### 1. Introduction

The probability finding, as a measurement outcome,  $n$  photons in the field state  $\hat{\rho}$  is given by  $\text{Tr}(\hat{\rho}|n\rangle\langle n|)$  ( $\hat{\rho}$  is a traceclass operator and  $|n\rangle$  is the eigenstate of the photon-number operator). So, in a field prepared in a coherent state  $|\alpha\rangle$  ( $\hat{\rho} = |\alpha\rangle\langle\alpha|$ ), where  $\alpha$  is a complex number and  $|\alpha|^2$  is the intensity of the field, or the mean photon number, then  $\text{Tr}(\hat{\rho}|n\rangle\langle n|) = |\langle n|\alpha\rangle|^2$ .

On the other hand, from a formal point of view the state  $|\alpha\rangle$  is used to map a q-number operator  $\hat{O}(a, a^\dagger)$  ( $a$  and  $a^\dagger$  are destruction and creation operators of photons with respect to the number state  $|n\rangle$ ,  $n = 0, 1, 2, \dots$ ) to a c-number function. The trace operation  $\text{Tr}(\hat{\rho}|\alpha\rangle\langle\alpha|) = \langle\alpha|\hat{\rho}|\alpha\rangle = Q_{\hat{\rho}}(\alpha, \alpha^*)$  defines the Husimi distribution, or Q-distribution, for state  $\hat{\rho}$  (actually this is a map:  $\hat{\rho} \Rightarrow Q_{\hat{\rho}}(\alpha, \alpha^*)$ ,  $a \rightarrow \alpha$ ,  $a^\dagger \rightarrow \alpha^*$ ).

The mean value of an operator  $\hat{O}(a, a^\dagger)$  can be written as

$$\text{Tr}(\hat{\rho}\hat{O}) = \int O(\alpha, \alpha^*) P_{\hat{\rho}}(\alpha, \alpha^*) d^2\alpha \quad (1)$$

where  $O(\alpha, \alpha^*) = \langle \alpha | \hat{O} | \alpha \rangle$  (this is also a map:  $\hat{O} \Rightarrow O(\alpha, \alpha^*)$ ) and  $P_{\hat{\rho}}(\alpha, \alpha^*)$  is the Glauber–Sudarshan or P-distribution, related to  $Q_{\hat{\rho}}(\alpha, \alpha^*)$  through

$$P_{\hat{\rho}}(\alpha, \alpha^*) = \exp\left(-\frac{\partial^2}{\partial \alpha \partial \alpha^*}\right) Q_{\hat{\rho}}(\alpha, \alpha^*).$$

The distributions  $Q_{\hat{\rho}}(\alpha, \alpha^*)$  and  $P_{\hat{\rho}}(\alpha, \alpha^*)$  are quasi-probabilities, the former is always a smooth and well-behaved function of its arguments while the latter, depending on the state  $\hat{\rho}$ , may be a regular function or a *generalized function* (GF) as is the case for  $\hat{\rho} = |n\rangle\langle n|$ .

For a field in state  $\hat{\rho} = |n\rangle\langle n|$  the probability to find  $n$  photons in a coherent state  $|\alpha\rangle\langle\alpha|$  is the same as the Q-distribution for state  $|n\rangle\langle n|$ , being a Poisson distribution in variable  $n$  with mean value  $|\alpha|^2$  [1],

$$Q_n(|\alpha|^2) = |\langle n | \alpha \rangle|^2 = \frac{\exp(-|\alpha|^2) |\alpha|^{2n}}{n!} \quad (2)$$

a smooth and well-behaved function of its argument. However, the P-distribution is a quite singular function, as was originally reported in the classical papers of Glauber [2] and Sudarshan [3] and more recently reviewed by Wünsche [4], who found new relations and representations for the P-distribution. Working on this same problem of representing the P-distribution, we obtained several results which we did not find in the current literature, relating the Laguerre moments and Laguerre factorial moments to GFs. We also derived a direct relation between the P-distribution and the Laguerre factorial moments. These results are reported in this paper.

We begin by reminding, with some examples, how the Dirac delta-function arises in mathematical physics [5–10]:

(I) Certain sequences of functions defined on  $\mathbb{R}$ ,  $f_n(x)$ ,  $n = 1, 2, 3, \dots$  are well behaved (continuous with continuous derivatives to all orders) in a domain  $\mathcal{I}$ ; however, they cease to exist as such when  $n \rightarrow \infty$ , acquiring meaning as a continuous linear functional  $T_f \phi \equiv \langle f, \phi \rangle = \int_{\mathcal{I}} f(x) \phi(x) dx$  that maps each continuous test function  $\phi(x)$  ( $\phi \in \Phi$ ,  $\Phi$  is a linear vector space) onto a complex number. So, the functional denoted as  $T_f$  (or simply  $f$ ) is called the distribution or GF. For instance, the functions

$$\frac{n}{\sqrt{\pi}} e^{-n^2/x^2} \quad \frac{n}{\pi} \frac{1}{1+n^2x^2} \quad \frac{\sin nx}{\pi x} \quad (3)$$

although being continuous with continuous derivatives to all orders for any finite integer  $n$ , no longer exhibit this property in the limit  $n \rightarrow \infty$ , thus no longer belong to the class of regular functions. All the examples in (3) converge to the so-called Dirac delta-function  $\delta$ , in reality a GF, to be referred to as the Dirac distribution (DD)

$$\delta(x) = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{\pi}} e^{-n^2/x^2} = \lim_{n \rightarrow \infty} \frac{n}{\pi} \frac{1}{1+n^2x^2} = \lim_{n \rightarrow \infty} \frac{\sin nx}{\pi x}$$

defined by the functional

$$\langle T_{\delta_n}, \phi \rangle = \lim_{n \rightarrow \infty} \int_{\mathcal{I}} \delta_n(x - x_0) \phi(x) dx = \phi(x_0)$$

where  $\delta_n(x)$  stands for any one of the functions displayed in (3) and  $\phi(x)$  is a test function.

(II) From Sturm–Liouville theory we know that a class of second-order differential equations accept, as solution, orthogonal polynomials  $\mathcal{P}_n(x)$  that form a complete set, meaning that any piecewise smooth and bounded function  $f(x)$  defined on  $\mathcal{I}$  ( $x \in \mathcal{I}$ ) can be expanded in terms of the  $\mathcal{P}_n(x)$  (the weight function and normalization factors are included in it),

$$f(x) = \sum_n c_n \mathcal{P}_n(x)$$

the coefficients are obtained by integration,

$$c_n = \int_{\mathcal{I}} f(x) \mathcal{P}_n(x) dx$$

and

$$\sum_{n=0}^{\infty} \mathcal{P}_n(x) \mathcal{P}_n(x') = \delta(x - x')$$

is the completeness property of the polynomials. If the point 0 is contained in  $\mathcal{I}$ , then setting  $x' = 0$  in the previous equation, the DD becomes equal to an infinite weighted sum of polynomials,

$$\sum_{n=0}^{\infty} \mathcal{P}_n(0) \mathcal{P}_n(x) = \delta(x).$$

Concerning the weighted Laguerre polynomials,  $\mathcal{P}_n(x) = e^{-x/2} L_n(x)$ , which are defined on  $[0, \infty)$  with  $\mathcal{P}_n(0) = 1$ , one notes that the infinite sum expansion

$$\sum_{n=0}^{\infty} L_n(x) = \delta_+(x) \quad (4)$$

is a representation of a GF (we will come back to this point in the next section, with a proper demonstration)<sup>3</sup>. The GF on the right-hand side (RHS) of equation (4) is related to the Dirac distribution

$$\int_0^{\infty} \delta_+(x) \phi(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_0^{\infty} \delta(x - \varepsilon) \phi(x) dx = \phi(0) \quad (5)$$

since GFs are properly defined in open intervals.

The infinite sums of polynomials and moments are useful for a certain class of problems, as we will see in section 4. In what follows, we shall consider the associated Laguerre polynomials  $L_n^\alpha(x)$ , whose generating function (GEF) is

$$G(x, t, \alpha) = e^{-\frac{xt}{1-t}} / (1-t)^{\alpha+1} \quad (6)$$

since

$$G(x, t, \alpha) = \sum_{n=0}^{\infty} L_n^\alpha(x) t^n \quad |t| < 1 \quad (7)$$

where  $t$  is a complex variable in the open disc of radius  $|t|$ ,  $t \in \mathcal{D}_t = \{t \in \mathbb{C} \mid 0 \leq |t| < 1\}$ .

In section 2 we present some lemmas involving the ordinary Laguerre polynomials and the theorem for the Laguerre moments, whereas in section 3 we extend the results to the associated Laguerre polynomials. In section 4 we make use of previous results and obtain an expression for the P-distribution in terms of either the Laguerre factorial moments or the GFs. In section 5 we expose our conclusions.

## 2. The ordinary Laguerre polynomials and moments

Initially, we shall consider the ordinary Laguerre polynomials ( $\alpha = 0$ ),  $L_n^0(x) \equiv L_n(x)$ , where  $L_n(0) = 1$ . If we extend the domain of  $t$  to include the additional point  $t = +1$  in

<sup>3</sup> Here the  $\delta_+(x)$  should not be confused with the distributions  $\delta^\pm = \frac{\delta}{2} \pm \frac{1}{2\pi i} vp \frac{1}{x}$  defined in [6], p 91, where  $vp$  stands for the Cauchy principal value.

equations (6) and (7) then  $\mathcal{D}'_t = \{\mathcal{D}_t, 1\}$ ; one verifies that at this point (since  $t = |t| e^{i\varphi}$ ,  $|t| = 1$  and  $\varphi = 0$ ) the GEF (6) becomes

$$\lim_{t \rightarrow 1^-} G(x, t, 0) = \lim_{\varepsilon \rightarrow 0^+} \frac{e^{-\frac{x}{\varepsilon}}}{\varepsilon} = \begin{cases} 0 & \text{for } x > 0 \\ \infty & \text{for } x = 0 \end{cases}. \quad (8)$$

Thus  $G(x, 1, 0)$  is no longer a regular function in the usual sense, it becomes quite singular at  $x = 0$ . Let us look more closely at GEF (6) and analyse its properties:

**Lemma 1.** For  $x \in \mathbb{R}_+$ ,  $\mathbb{R}_+ \equiv (0, \infty)$ , the  $\alpha = 0$  GEF  $G(x, 1, 0)$  can be represented by the GF, equation (5),

$$G(x, 1, 0) = \delta_+(x). \quad (9)$$

**Proof.** Multiplying  $G(x, t, 0)$  by a piecewise smooth test function  $\phi(x)$ , with  $\phi \in \mathbb{R}$  and  $x \in \mathbb{R}_+$ , integrating

$$\int_0^\infty \frac{e^{-\frac{xt}{1-t}}}{(1-t)} \phi(x) dx$$

performing the change of variable  $x = y(1-t)/t$  and considering the limit  $t \rightarrow 1^-$ , we get

$$\lim_{t \rightarrow 1^-} \int_0^\infty e^{-y} \frac{1}{t} \phi\left(\frac{1-t}{t}y\right) dy = \phi(0)$$

which is the main property of the GF, thus

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^\infty G(x, 1 - \varepsilon, 0) \phi(x) dx = \phi(0) \quad (10)$$

and equation (9) is justified, i.e.,  $\lim_{t \rightarrow 1^-} G(x, t, 0)$  is a representation of the GF.  $\square$

Symbolically, equation (10) can be written as

$$G(x, 1, 0)\phi(x) = \phi(0) \quad \text{or} \quad \delta_+(x)\phi(x) = \phi(0)$$

which is a well-known property of the DD, omnipresent in mathematical physics textbooks [6, 9, 10].

As the summation  $\sum_{n=0}^N L_n(x)$  is a regular function for any finite  $N$ , we may ask: does the infinite summation go to a GF? Or, is the equality  $\sum_{n=0}^\infty L_n(x) = \delta_+(x)$  true? Before answering that question we first recall the following theorem of the Laguerre polynomials (we do not present the demonstration since it can be found in the usual textbooks [11]):

**Theorem 1.** If the real function  $f(x)$ , defined in the interval  $[0, \infty)$ , is piecewise smooth in every subinterval  $[x_1, x_2]$ , where  $0 \leq x_1 < x_2 < \infty$  and if the integral

$$\int_0^\infty e^{-x} x^\alpha [f(x)]^2 dx$$

is finite, then the series

$$\sum_{n=0}^\infty c_{n,\alpha} L_n^\alpha(x)$$

with coefficients

$$c_{n,\alpha} = \frac{n!}{\Gamma(n + \alpha + 1)} \int_0^\infty e^{-x} x^\alpha f(x) L_n^\alpha(x) dx$$

converges to  $f(x)$  at every continuity point of  $f(x)$ . At a discontinuity point  $x_0$  the series converges to

$$\frac{1}{2} \lim_{\varepsilon \rightarrow 0} [f(x_0 + \varepsilon) + f(x_0 - \varepsilon)].$$

For  $\alpha = 0$  we have

$$f(x) = \sum_{n=0}^{\infty} c_n L_n(x) \tag{11}$$

with

$$c_n = \int_0^{\infty} e^{-x} f(x) L_n(x) dx. \tag{12}$$

The Laguerre polynomials are defined such that  $L_n(0) = 1$ , implying that  $f(0) = \sum_{n=0}^{\infty} c_n$ ; so we propose

**Lemma 2.** From equation (12) and for  $x \in \mathbb{R}_+ \equiv (0, \infty)$  we obtain

$$\sum_{n=0}^{\infty} L_n(x) = \delta_+(x). \tag{13}$$

**Proof.** Summing over all  $n$  on both sides of equation (12) and interchanging the order of summation and integration, we get

$$\sum_{n=0}^{\infty} c_n = f(0) = \int_0^{\infty} e^{-x} f(x) \left( \sum_{n=0}^{\infty} L_n(x) \right) dx$$

so we verify equation (13). □

We can also verify equation (13) by using a recurrence relation of the Laguerre polynomials and a property of the DD:

**Corollary 1.** From the recurrence relation of the Laguerre polynomials

$$x d_x L_n(x) = n L_n(x) - n L_{n-1}(x) \quad n = 1, 2, 3, \dots \tag{14}$$

where  $d_x \doteq d/dx$ , follows the known relation of the DD

$$x d_x \delta_+(x) = -\delta_+(x) \tag{15}$$

for  $\sum_{n=0}^{\infty} L_n(x) = \delta_+(x)$ .

**Proof.** Summing both sides of equation (14) over  $n$ , from 1 to  $N$ , we get

$$\sum_{n=1}^N n L_n(x) = x d_x \sum_{n=1}^N L_n(x) + \sum_{n=1}^N n L_{n-1}(x)$$

or

$$\sum_{n=0}^N n L_n(x) = x d_x \sum_{n=0}^N L_n(x) + \sum_{n=0}^N (n+1) L_n(x)$$

then

$$x d_x \left( \sum_{n=0}^N L_n(x) \right) = - \sum_{n=0}^N L_n(x).$$

Considering  $N \rightarrow \infty$  we recognize equation (15) for  $\sum_{n=0}^{\infty} L_n(x) = \delta_+(x)$ . □

Therefore, we can write

$$G(x, 1, 0) = \sum_{n=0}^{\infty} L_n(x) = \delta_+(x). \quad (16)$$

We now give some relations involving the GFs that will be necessary to demonstrate a useful theorem. As a preliminary, we

- (i) write in short  $\delta_+^{(n)}(x) \doteq (d_x)^n \delta_+(x)$  (with  $\delta_+^{(0)}(x) \doteq \delta_+(x)$ );  
 (ii) assume  $\lim_{x \rightarrow 0} (x \sum_{n=0}^{\infty} L_n(x)) \doteq 0$  and  $\lim_{x \rightarrow 0} (x (d_x)^n \sum_{m=0}^{\infty} L_m(x)) \doteq 0$ , thus

$$\lim_{x \rightarrow 0} x \delta_+^{(n)}(x) = \lim_{x \rightarrow 0} x (d_x)^n G(x, 1, 0) = 0; \quad (17)$$

- (iii) introduce the bracketed terms  $[x\delta_+^{(1)}(x)]$ ,  $[x\delta_+^{(2)}(x)]$ ,  $\dots$ ,  $[x\delta_+^{(n)}(x)]$  as GFs;

- (iv) define the functional

$$([x\delta_+^{(n)}(x)], \phi) \doteq \int_0^{\infty} [x\delta_+^{(n)}(x)] \phi(x) dx$$

where  $\phi(x)$  is a regular piecewise and bounded test function in  $\mathbb{R}_+$ .

**Example 1.** For  $n = 1$ , the functional is

$$\begin{aligned} \int_0^{\infty} [x\delta_+^{(1)}(x)] \phi(x) dx &= \int_0^{\infty} \delta_+^{(1)}(x) (\phi(x)x) dx = - \int_0^{\infty} \delta_+(x) [d_x(\phi(x)x)] dx \\ &= - \int_0^{\infty} \delta_+(x) (x d_x \phi(x) + \phi(x)) dx = \int_0^{\infty} (-\delta_+(x)) \phi(x) dx = -\phi(0). \end{aligned}$$

or in symbolic notation

$$[x\delta_+^{(1)}(x)] = -\delta_+(x) \quad (18)$$

the term in the brackets is reduced to the GF multiplied by  $-1$ .

**Example 2.** For  $n = 2$ ,

$$\begin{aligned} \int_0^{\infty} [x\delta_+^{(2)}(x)] \phi(x) dx &= \int_0^{\infty} \delta_+^{(2)}(x) (\phi(x)x) dx = (-1)^2 \int_0^{\infty} \delta_+(x) [(d_x)^2 (\phi(x)x)] dx \\ &= \int_0^{\infty} \delta_+(x) [x(d_x)^2 \phi(x) + 2d_x \phi(x)] dx = 2 \int_0^{\infty} \delta_+(x) [d_x \phi(x)] dx \\ &= \int_0^{\infty} (-2\delta_+^{(1)}(x)) \phi(x) dx = 2\phi'(0) \end{aligned}$$

therefore, in symbolic notation

$$[x\delta_+^{(2)}(x)] = -2\delta_+^{(1)}(x). \quad (19)$$

**Remark 1.** The bracket  $[x\delta_+(x)] = 0$ , since

$$\int_0^{\infty} [x\delta_+(x)] \phi(x) dx = \int_0^{\infty} \delta_+(x) [\phi(x)x] dx = 0.$$

The general term  $[x\delta_+^{(n)}(x)]$  is obtained by induction:

**Lemma 3.** For  $x \in \mathbb{R}_+$ , the factor  $(-x)$  in  $[(-x)\delta_+^{(n)}(x)]$  acts as a first-order derivative on  $\delta_+^{(n)}(x)$ ,

$$[x\delta_+^{(n)}(x)] = -n\delta_+^{(n-1)}(x) \quad n = 1, 2, 3, \dots \quad (20)$$

**Proof.** From examples 1 and 2 we have  $[x\delta_+^{(1)}(x)] = -\delta_+(x)$  and  $[x\delta_+^{(2)}(x)] = -2\delta_+^{(1)}(x)$ . For  $[x\delta_+^{(3)}(x)]$  we use this last equation,

$$[x\delta_+^{(3)}(x)] = d_x [x\delta_+^{(2)}(x)] - \delta_+^{(2)}(x) = d_x (-2\delta_+^{(1)}(x)) - \delta_+^{(2)}(x) = -3\delta_+^{(2)}(x) \tag{21}$$

and so forth for higher order derivatives, so equation (20) stands for any positive integer  $n$ . □

We can generalize this result for higher powers of  $x$  through

**Lemma 4.** For  $x \in \mathbb{R}_+$ , any positive integer  $p$ , and assuming relation (20), it follows that in  $[(-x)^p \delta_+^{(n)}(x)]$  the factor  $(-x)^p$  acts as a  $p$ th-order derivative multiplied by a constant,

$$[x^p \delta_+^{(n)}(x)] = \begin{cases} (-1)^p \frac{n!}{(n-p)!} \delta_+^{(n-p)}(x) & \text{for } n \geq p \\ 0 & \text{for } n < p. \end{cases} \tag{22}$$

**Proof.** Since  $[x\delta_+^{(n)}(x)] = -n\delta_+^{(n-1)}(x)$  then

$$\begin{aligned} [x^2 \delta_+^{(n)}(x)] &= [x(-n\delta_+^{(n-1)}(x))] \\ &= -n[x\delta_+^{(n-1)}(x)] = -n(-n+1)\delta_+^{(n-2)}(x) \quad \text{for } n \geq 2. \end{aligned}$$

However,  $[x^2 \delta_+^{(1)}(x)] = -[x\delta_+(x)] = 0$ , where the second equality follows from remark 1. Repeating this procedure for any positive integer  $p$ , we verify equation (22). □

**Lemma 5.** For the differential operator

$$\Lambda(x) \doteq -((1-x)d_x + x d_x^2)$$

the following equation

$$[\Lambda(x)\delta_+^{(n)}(x)] = (n+1)(\delta_+^{(n+1)}(x) - \delta_+^{(n)}(x)) \tag{23}$$

holds for  $n = 0, 1, 2, \dots$

**Proof.** Setting  $n = 0$  in equation (23) and by using relation (22) we get

$$\begin{aligned} [\Lambda(x)\delta_+(x)] &= -\left([ (1-x)\delta_+^{(1)}(x) ] + [x\delta_+^{(2)}(x)]\right) \\ &= -\left(\delta_+^{(1)}(x) - (-\delta_+(x)) + \left(-\frac{2!}{1!}\delta_+^{(1)}(x)\right)\right) \\ &= \delta_+^{(1)}(x) - \delta_+(x). \end{aligned} \tag{24}$$

Following the same procedure for  $n \geq 1$  we get equation (23). □

**Lemma 6.** For any positive integer  $s$  we have

$$\begin{aligned} [(\Lambda(x))^s \delta_+^{(r)}(x)] &= \left[ \prod_{k=1}^s (r+k) \right] \delta_+^{(s+r)}(x) - \left[ \prod_{k=1}^{s-1} (r+k) \sum_{j=1}^s (r+j) \right] \delta_+^{(s+r-1)}(x) \\ &+ \left[ \prod_{k=1}^{s-2} (r+k) \sum_{j=1}^{s-1} (r+j) \sum_{l=j}^{s-1} (r+l) \right] \delta_+^{(s+r-2)}(x) \\ &- \left[ \prod_{k=1}^{s-3} (r+k) \sum_{j=1}^{s-2} (r+j) \sum_{l=j}^{s-2} (r+l) \sum_{m=l}^{s-2} (r+m) \right] \delta_+^{(s+r-3)}(x) + \dots \\ &+ \left[ (-1)^s \underbrace{\sum_{j=1}^1 (r+j) \sum_{l=j}^1 (r+l) \dots \sum_{n=m}^1 (r+n)}_{\text{product of } s \text{ summation symbols}} \right] \delta_+^{(r)}(x). \end{aligned} \tag{25}$$

**Proof.** Starting with equation (23) and applying  $\Lambda(x)$  successively, by induction we arrive at equation (25).  $\square$

We now define the Laguerre moments

$$\mathcal{M}_s(x, 0) \doteq \sum_{n=0}^{\infty} n^s L_n(x) \tag{26}$$

and propose the following theorem.

**Theorem 2.** For any positive integer  $s$  and  $x \in \mathbb{R}_+$ , the moments  $\mathcal{M}_s(x, 0)$  can be expanded in terms of GFs as

$$\mathcal{M}_s(x, 0) = \sum_{l=0}^s c_{s,l} \delta_+^{(l)}(x) \tag{27}$$

with the coefficients given by

$$c_{s,l} = (-1)^{s-l} l! \underbrace{\sum_{j=1}^{l+1} j \sum_{k=j}^{l+1} k \cdots \sum_{n=m}^{l+1} n}_{\text{product of } s-l \text{ summation symbols}}. \tag{28}$$

**Proof.** The eigenvalue equation for the Laguerre polynomials is

$$\Lambda(x)L_n(x) = nL_n(x). \tag{29}$$

Summing both sides of the equation over  $n$  from 0 to  $\infty$  gives the first moment,

$$\begin{aligned} \mathcal{M}_1(x, 0) &= \sum_{n=0}^{\infty} nL_n(x) \\ &= [\Lambda(x)\delta_+(x)] = \delta_+^{(1)}(x) - \delta_+(x) \end{aligned} \tag{30}$$

where we used equation (24) to obtain the second equality. By applying  $s - 1$  times  $\Lambda(x)$  on both sides of equation (29) we obtain

$$(\Lambda(x))^s L_n(x) = n^s L_n(x)$$

summing over  $n$  we get the  $s$ th-order moment

$$\mathcal{M}_s(x, 0) = \sum_{n=0}^{\infty} n^s L_n(x) = [(\Lambda(x))^s \delta_+(x)].$$

Setting  $r = 0$  in equation (25) we get

$$\begin{aligned} \mathcal{M}_s(x, 0) &= s! \delta_+^{(s)}(x) - (s-1)! \left( \sum_{j=1}^s j \right) \delta_+^{(s-1)}(x) + (s-2)! \left( \sum_{j=1}^{s-1} j \sum_{k=j}^{s-1} k \right) \delta_+^{(s-2)}(x) \\ &\quad - (s-3)! \left( \sum_{j=1}^{s-2} j \sum_{k=j}^{s-2} k \sum_{m=k}^{s-2} m \right) \delta_+^{(s-3)}(x) + \cdots + (-)^s \delta_+(x) \\ &= \sum_{l=0}^s \left[ (-1)^{s-l} l! \underbrace{\sum_{j=1}^{l+1} j \sum_{k=j}^{l+1} k \cdots \sum_{n=m}^{l+1} n}_{\text{product of } s-l \text{ summation symbols}} \right] \delta_+^{(l)}(x) = \sum_{l=0}^s c_{s,l} \delta_+^{(l)}(x). \end{aligned}$$

$\square$



**Example 3.**

$$\begin{aligned}
\mathcal{M}_2(x, 0) &= [\Lambda(x) [\Lambda(x) \delta_+(x)]] = [\Lambda(x) (\delta_+^{(1)}(x) - \delta_+(x))] \\
&= 2 (\delta_+^{(2)}(x) - \delta_+^{(1)}(x)) - (\delta_+^{(1)}(x) - \delta_+(x)) \\
&= 2\delta_+^{(2)}(x) - 3\delta_+^{(1)}(x) + \delta_+(x)
\end{aligned} \tag{31}$$

proceeding analogously we get

$$\mathcal{M}_3(x, 0) = 6\delta_+^{(3)}(x) - 12\delta_+^{(2)}(x) + 7\delta_+^{(1)}(x) - \delta_+(x) \tag{32}$$

and

$$\mathcal{M}_4(x, 0) = 24\delta_+^{(4)}(x) - 60\delta_+^{(3)}(x) + 50\delta_+^{(2)}(x) - 15\delta_+^{(1)}(x) + \delta_+(x). \tag{33}$$

It can be verified that

$$\sum_{l=0}^s c_{s,l} = 0. \tag{34}$$

**2.1. The Laguerre factorial moments**

The Laguerre factorial moment of order  $s$ , associated with the ordinary Laguerre polynomials, is defined as

$$\mathcal{F}_s(x, 0) \doteq \sum_{n=s}^{\infty} n(n-1)(n-2) \cdots (n-s+1) L_n(x) = \sum_{n=s}^{\infty} \frac{n!}{(n-s)!} L_n(x) \tag{35}$$

and it is related to the Laguerre moments through

$$\mathcal{F}_s(x, 0) = \sum_{r=1}^s S_s^{(r)} \mathcal{M}_r(x, 0) \tag{36}$$

where the coefficients  $S_s^{(r)}$  are the Stirling numbers of the first kind [12]<sup>4</sup>. By substituting equation (27) into (35) and after summing over the coefficients we arrive at a simple expression for the factorial moments in terms of ultradistributions,

$$\mathcal{F}_s(x, 0) = s! \sum_{r=0}^s (-1)^{s-r} \binom{s}{r} \delta_+^{(r)}(x). \tag{37}$$

**3. The associated Laguerre polynomial and moments**

**Lemma 7.** For  $\alpha$  a positive integer and  $x \in \mathbb{R}_+$ , the equation

$$\sum_{n=0}^{\infty} L_n^\alpha(x) = (1 - d_x)^\alpha \delta_+(x) \tag{38}$$

holds and for  $t = 1$  the GEF also becomes

$$G(x, 1, \alpha) = \lim_{t \rightarrow 1} \frac{e^{-\frac{x}{1-t}}}{(1-t)^{\alpha+1}} = (1 - d_x)^\alpha \delta_+(x). \tag{39}$$

<sup>4</sup> The Stirling numbers of the first kind have the properties  $S_s^{(0)} = \delta_{s,0}$  and  $\sum_{r=1}^s S_s^{(r)} = 0$ . The inverse relation is  $\mathcal{M}_r(x, 0) = \sum_{l=0}^r S_r^{(l)} \mathcal{F}_l(x, 0)$  where  $S_r^{(l)}$  are the Stirling numbers of the second kind.

**Proof.** Considering the following properties of the Laguerre polynomials, [13, 14]

$$L_n^\alpha(x) = (-d_x)^\alpha L_{n+\alpha}(x) \quad \text{and} \quad (d_x)^\alpha L_{n+\alpha}(x) = (-1 + d_x)^\alpha L_n(x)$$

it follows that

$$L_n^\alpha(x) = (1 - d_x)^\alpha L_n(x) \quad (40)$$

and summing both sides of equation (40) over  $n$  from 0 to  $\infty$  one gets equation (38).

Concerning equation (39), looking at  $G(x, t, \alpha)$ , equation (7), one can easily verify that

$$G(x, t, \alpha) = (1 - \partial_x)^\alpha G(x, t, 0) \quad (41)$$

( $\partial_x \equiv \partial/\partial x$ ) thus

$$\lim_{t \rightarrow 1^-} G(x, t, \alpha) = (1 - d_x)^\alpha \lim_{t \rightarrow 1^-} G(x, t, 0) = (1 - d_x)^\alpha \delta_+(x)$$

(the second equality follows from lemma 1) so verifying equation (39). Therefore, one arrives at

$$G(x, 1, \alpha) = \sum_{n=0}^{\infty} L_n^\alpha(x) = (1 - d_x)^\alpha \delta_+(x) \quad (42)$$

which generalizes equation (9).  $\square$

We now propose the following theorem.

**Theorem 3.** For any positive integers  $\alpha$  and  $s$ , and  $x \in \mathbb{R}_+$ , the moments  $\mathcal{M}_s(x; \alpha)$  are given by the following summation of GFs:

$$\mathcal{M}_s(x; \alpha) = \sum_{n=0}^{\infty} n^s L_n^\alpha(x) = \sum_{l=0}^s c_{s,l} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \delta_+^{(l+j)}(x). \quad (43)$$

**Proof.** Considering equation (40), multiplying both sides by  $n^s$  ( $s$  a non-negative integer) and summing over  $n$  from 0 to  $\infty$ , one gets

$$\begin{aligned} \mathcal{M}_s(x; \alpha) &= (1 - d_x)^\alpha \mathcal{M}_s(x; 0) = (1 - d_x)^\alpha \sum_{l=0}^s c_{s,l} \delta_+^{(l)}(x) \\ &= \sum_{l=0}^s c_{s,l} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \delta_+^{(l+j)}(x). \end{aligned} \quad (44)$$

$\square$

### 3.1. The factorial moments

The associated Laguerre factorial moment of order  $s$  is defined as

$$\mathcal{F}_s(x, \alpha) \doteq \sum_{n=s}^{\infty} \frac{n!}{(n-s)!} L_n^\alpha(x) \quad (45)$$

and using relations (44) we get

$$\mathcal{F}_s(x, \alpha) = (1 - d_x)^\alpha \mathcal{F}_s(x, 0) \quad (46)$$

$$= s! \sum_{r=0}^s (-1)^{s-r} \binom{s}{r} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \delta_+^{(r+j)}(x). \quad (47)$$

#### 4. The P-distribution for the photon-number state

Writing the complex variable  $\alpha$  in the polar form and calling  $|\alpha|^2 = y$  in equation (2) we obtain

$$Q_n(y) = \frac{\exp(-y)y^n}{n!}$$

and

$$P_n(y) = \frac{1}{n!} \exp\left(-\frac{d}{dy}y\frac{d}{dy}\right)y^n \exp(-y). \quad (48)$$

It is convenient to introduce an auxiliary function,

$$R(y, \beta) = \exp\left(-\frac{d}{dy}y\frac{d}{dy}\right)\exp(-y\beta) \quad (49)$$

and since

$$\left(-\frac{d}{dy}y\frac{d}{dy}\right)^n \exp(y) = (-1)^n n! L_n(y) \exp(y)$$

we can write

$$R(y, \beta) = \sum_{n=0}^{\infty} \beta^n L_n(\beta y) \exp(-y\beta). \quad (50)$$

Now we can express equation (48) in terms of equation (50) as

$$\begin{aligned} P_n(y) &= \frac{1}{n!} \lim_{\beta \rightarrow 1} \left(-\frac{\partial}{\partial \beta}\right)^n R(y, \beta) \\ &= \frac{1}{n!} \lim_{\beta \rightarrow 1} \left(-\frac{\partial}{\partial \beta}\right)^n \left[ \exp(-y\beta) \sum_{k=0}^{\infty} \beta^k L_k(\beta y) \right] \end{aligned} \quad (51)$$

thus the auxiliary function  $R(y, \beta)$  stands for the GEF of the P-distributions. Using the definition of GEF (7) we have

$$\exp(-y\beta) \sum_{k=0}^{\infty} \beta^k L_k(\beta y) = G(y, \beta, 0) = \sum_{k=0}^{\infty} \beta^k L_k(y) \quad (52)$$

and substituting this result in equation (51) we obtain

$$P_n(y) = (-1)^n \sum_{k=n}^{\infty} \binom{k}{n} L_k(y). \quad (53)$$

Using equations (35) and (37) we get a direct relation between the P-distributions and the Laguerre factorial moments,

$$P_n(y) = \frac{(-1)^n}{n!} \mathcal{F}_n(y, 0) \quad (54)$$

or in terms of the GFs

$$P_n(y) = \sum_{k=0}^n (-1)^k \binom{n}{k} \delta_+^{(k)}(y) = (1 - d_y)^n \delta_+(y) \quad (55)$$

which simplifies the task of calculating mean values in equation (1) when the state of the field can be written as  $\hat{\rho} = \sum_{n=0}^{\infty} p_n |n\rangle \langle n|$ , where  $p_n$  is the probability associated with the state  $|n\rangle$ . As an illustration, we display the  $n = 0, 1, 2$  P-distributions,

$$P_0(y) = \delta_+(y) \quad P_1(y) = -\delta_+^{(1)}(y) + \delta_+(y) \quad P_2(y) = \delta_+^{(2)}(y) - 2\delta_+^{(1)}(y) + \delta_+(y).$$

## 5. Conclusions

We have obtained an explicit expression for the Laguerre moments and factorial moments, for ordinary and associated Laguerre polynomials, written as sums of generalized functions. We showed that the P-distribution of an electromagnetic field in a photon-number state  $|n\rangle$  is proportional to the Laguerre factorial moment of order  $n$  and that it acquires a simple form when expressed as a sum of generalized functions.

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