



# Non-classical orthogonality relations for big and little $q$ -Jacobi polynomials

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## Abstract

Big  $q$ -Jacobi polynomials  $\{P_n(\cdot; a, b, c; q)\}_{n=0}^{\infty}$  are classically defined for  $0 < a < q^{-1}$ ,  $0 < b < q^{-1}$  and  $c < 0$ . For the family of little  $q$ -Jacobi polynomials  $\{p_n(\cdot; a, b|q)\}_{n=0}^{\infty}$ , classical considerations restrict the parameters imposing  $0 < a < q^{-1}$  and  $b < q^{-1}$ . In this work we extend both families in such a way that wider sets of parameters are allowed, and we establish orthogonality conditions for those cases for which Favard's theorem does not work. As a by-product, we obtain similar results for the families of big and little  $q$ -Laguerre polynomials.

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*The watchmaker had been examining the watch closely for over two minutes. This he had done with great care and sense of purpose. Then, his features hardened somewhat; He inhaled in quite a conspicuous way and suddenly blew the air over the watch. He immediately handed it over to its owner, who had been watching over the whole process with apprehension. – Good God, the watch works! How puzzling, just a puff and the thing is fixed...! – said the man hardly concealing his excitement. Then he came down to more earthly matters. – How much do I owe you? And I must insist on payment... – said the man. – Twenty dollars – replied the watchmaker. – Excuse me –, said the man; – Twenty dollars for just blowing on the watch...? – complained the owner. – Of course not, – said the watchmaker – the blowing is for free; What costs twenty dollars is knowing where to blow.*

*We want to dedicate this work to the kind, generous and anonymous referees of our previous paper [Samuel G. Moreno, E.M. García-Caballero Linear interpolation and Sobolev orthogonality, J. Approx. Theory (2008) doi:10.1016/j.jat.2008.08.005] who showed us "where to blow": Their suggestions have been our inspiration for further and ongoing investigations.*

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**1. Introduction**

*1.1. Preliminaries*

We shall denote by  $\mathbb{N}$  and  $\mathbb{N}_0$ , respectively, the set of positive integers and the set of nonnegative integers. The set of real numbers will be denoted by  $\mathbb{R}$ . All polynomials considered will be real-valued in one real variable, and  $\mathbb{P}$  will stand for the set of all such polynomials. For each  $n \in \mathbb{N}_0$ , the subset of  $\mathbb{P}$  of all polynomials of degree not greater than  $n$  will be denoted by  $\mathbb{P}_n$ . By a system of *monic* polynomials we will mean a sequence  $\{P_n\}_{n=0}^\infty$  of polynomials fulfilling  $P_n^{(n)} = n!$  for each  $n \in \mathbb{N}_0$ . Observe that for a sequence of monic polynomials we have  $P_0 = 1$ . For notational convenience we will use  $P_{-1}$  to denote the null polynomial.

For  $n \in \mathbb{N}$ , a square matrix of order  $n$ , with real entries  $a_{ij}$ , will be denoted by  $A = (a_{ij})_{i,j=0}^{n-1}$ , and  $(a_i)_{i=0}^{n-1} \in \mathbb{R}^n$  will stand for  $(a_0, a_1, \dots, a_{n-1})$ . The transpose of a matrix will be denoted by using the superscript  $t$ . As usual, we will identify the only element of a matrix of order 1 with the matrix itself. We shall denote by  $\delta_{ij}$  the Kronecker delta.

With respect to the  $q$ -calculus, the conventions adopted in this paper will be the usual ones (see [12,15]). For our purpose, it will suffice to consider (and that is what we shall always assume)  $0 < q < 1$ . Roughly speaking, the  $q$ -calculus is a scenery in which mathematical objects suffer a distortion, and the parameter  $q$  is the measure of such deformation; in the limit as  $q$  tends to one both objects, the classical one and its  $q$ -analogue, will coalesce.

The symbol  $(\cdot; q)_n$  will denote the so-called  $q$ -shifted factorial defined, for  $x \in \mathbb{R}$ , by

$$(x; q)_0 = 1, \quad (x; q)_{n+1} = \prod_{k=0}^n (1 - xq^k), \quad n \in \mathbb{N}_0, \quad (x; q)_\infty = \prod_{k=0}^\infty (1 - xq^k).$$

For notational convenience we also define, for  $m \in \mathbb{N}$ ,  $x_1, x_2, \dots, x_m \in \mathbb{R}$ , and  $n \in \mathbb{N}_0 \cup \{\infty\}$ ,

$$(x_1, x_2, \dots, x_m; q)_n = \prod_{k=1}^m (x_k; q)_n.$$

The  $q$ -binomial coefficient, for real numbers  $a, b$  is

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{\Gamma_q(a+1)}{\Gamma_q(b+1)\Gamma_q(a-b+1)}, \quad -a, -b, -(a-b) \notin \mathbb{N},$$

where the  $q$ -gamma function  $\Gamma_q$  is defined by  $\Gamma_q(x) = ((q; q)_\infty / (q^x; q)_\infty) (1-q)^{1-x}$  for  $x \in \mathbb{R}$ . For  $m, n \in \mathbb{N}_0$ , the  $q$ -hypergeometric (also, basic hypergeometric) series  ${}_m\phi_n$  is defined by

$${}_m\phi_n \left( \begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_n \end{matrix} \middle| q; x \right) = \sum_{k=0}^\infty \frac{(a_1; q)_k \cdots (a_m; q)_k}{(b_1; q)_k \cdots (b_n; q)_k} \frac{(-1)^{(n-m+1)k} x^k}{q^{(m-n-1)k(k-1)/2} (q; q)_k}, \quad (1.1)$$

where  $b_1, \dots, b_n \notin \{q^{-k}\}_{k \in \mathbb{N}_0}$ . When  $m = 0$  ( $n = 0$ ), the numerator (denominator) of the coefficient  $(a_1; q)_k \cdots (a_m; q)_k / (b_1; q)_k \cdots (b_n; q)_k$  becomes 1. In the case that one of the numerator parameters  $a_j$  equals  $q^{-k}$  for a nonnegative integer  $k$ , the  $q$ -hypergeometric series is a polynomial. Basic hypergeometric series are used to define  $q$ -extensions of elementary and special functions. For example, one of the  $q$ -analogues of the exponential function reads

$$e_q(x) = {}_1\phi_0 \left( \begin{matrix} 0 \\ - \end{matrix} \middle| q; x \right) = \sum_{k=0}^\infty \frac{x^k}{(q; q)_k},$$

and clearly  $\lim_{q \uparrow 1} e_q((1 - q)x) = e^x$ . As a consequence of the  $q$ -binomial theorem

$${}_1\phi_0 \left( \begin{matrix} a \\ - \end{matrix} \middle| q; x \right) = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} x^k = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}}, \quad |x| < 1,$$

it follows that  $e_q(x) = 1/(x; q)_{\infty}$  for  $|x| < 1$ . (In Section 5 we will use that  $e_q(q) = 1/(q; q)_{\infty}$ .)

In concluding this preliminary part about notations, conventions and terminologies, let us recall the definitions of the  $q$ -analogues of the derivative and the definite integral operators. As well known, the  $q$ -derivative operator  $D_q : \mathbb{P} \rightarrow \mathbb{P}$  is defined, for each polynomial  $p$ , by

$$D_q p(x) = \begin{cases} \frac{p(x) - p(qx)}{(1 - q)x}, & x \neq 0, \\ p'(0), & x = 0. \end{cases}$$

Further, the  $n$ th iteration of the  $q$ -derivative operator is recursively defined by means of  $D_q^0 = I$  ( $I$  is the identity operator) and  $D_q^{n+1} = D_q \circ D_q^n$  for  $n \in \mathbb{N}_0$ . For real numbers  $a, b$ , and for each polynomial  $p$ , the following standard definitions for the  $q$ -integrals will be used:

$$\int_0^a p(x) d_q x = a(1 - q) \sum_{k=0}^{\infty} p(aq^k) q^k,$$

$$\int_a^b p(x) d_q x = \int_0^b p(x) d_q x - \int_0^a p(x) d_q x.$$

### 1.2. Non-standard orthogonality

By a *non-standard orthogonality* result we will mean an orthogonality statement for a system of monic polynomials  $\{P_n^{(\lambda_1, \dots, \lambda_m)}\}_{n=0}^{\infty}$ , fulfilling the three term recurrence relation

$$x P_n^{(\lambda_1, \dots, \lambda_m)}(x) = P_{n+1}^{(\lambda_1, \dots, \lambda_m)}(x) + a_n P_n^{(\lambda_1, \dots, \lambda_m)}(x) + b_n P_{n-1}^{(\lambda_1, \dots, \lambda_m)}(x), \quad n \in \mathbb{N}_0,$$

(where  $a_n = a_n^{(\lambda_1, \dots, \lambda_m)} \in \mathbb{R}$ ,  $b_n = b_n^{(\lambda_1, \dots, \lambda_m)} \in \mathbb{R}$ ), for those values of the parameters  $\lambda_1, \lambda_2, \dots, \lambda_m$  for which  $b_n$  vanishes for some  $n \geq 1$ ; that is to say, non-standard orthogonality will be understood as the orthogonality beyond Favard’s theorem. This topic has been the subject of an increasing number of papers in the last decade. Moreover, all these papers reveal the crucial role that a kind of Sobolev orthogonality plays when stating orthogonality conditions for classical families of polynomials with non-classical parameters. We refer the reader to [16,21], for the case of non-classical Laguerre families; [17,3,2,1] for non-standard orthogonality concerning Jacobi polynomials; [4,10] for the case of Meixner polynomials with non-standard parameters; [7,8], for the case of symmetric Meixner–Pollaczek polynomials with parameters out of classical considerations; [11], for (not necessarily symmetric) generalized Meixner–Pollaczek polynomials with null parameter  $\lambda$ . Recently, we have given in [19] non-standard orthogonality results for another discrete extension of the Laguerre polynomials: Concretely, and after extending the classical family of Meixner–Pollaczek polynomials  $\{P_n^{(\lambda)}(\cdot; \phi)\}_{n=0}^{\infty}$  to arbitrary complex values of the parameter  $\lambda$ , we have introduced a non-standard discrete–continuous inner product that fills up the “Favard’s gap” in the orthogonality scenery. Finally, we refer the reader to [10] for the case of suitable extensions of the classical families of polynomials which satisfy

a discrete orthogonality with a finite number of masses (i.e., the Hahn, Racah, Dual Hahn and Krawtchouk polynomials).

As far as we know, there are no results concerning non-standard orthogonality results for  $q$ -analogues of classical polynomials. For a kind of a standard orthogonality result (when Favard’s characterization theorem is fulfilled), in a non-standard case (that is, for parameters out of classical considerations), we refer to [9]. Our objective is to accomplish a kind of “ $q$ -Sobolev orthogonality” for the (generalized) big and little  $q$ -Jacobi polynomials ( $\{P_n(\cdot; a, b, c; q)\}_{n=0}^\infty$  and  $\{p_n(\cdot; a, b|q)\}_{n=0}^\infty$ , respectively), when their first parameter  $a$  is an arbitrary real number, and when the remainder parameters are much less restricted than in the classical setting. These results will provide, as a by-product, analogous ones for the cases of big and little  $q$ -Laguerre polynomials.

### 1.3. Structure of the paper

The structure of the paper is the following. In Section 2 we recall some basic facts of classical monic  $q$ -Jacobi polynomials  $\{P_n(\cdot; a, b, c; q)\}_{n=0}^\infty$ , where  $0 < a, b < q^{-1}$  and  $c < 0$ , and we give an extension of this system by allowing the parameters  $a$  and  $c$  to be arbitrary real numbers, and with the restriction on  $b$  in the form  $ab \notin \{q^{-2}, q^{-3}, \dots\}$ . In this section we will give some preparatory results to state some of the main results. In Section 3, by means of a bilinear form involving a discrete part and also a part with a  $q$  integral (both terms with the presence of the  $D_q$  operator), we define a non-standard inner product which provides the orthogonality of the generalized family of monic big  $q$ -Jacobi polynomials when its first parameter takes values for which the hypothesis of Favard’s theorem does not hold. Section 4 is devoted to the same study as in Section 2, but now on the monic little  $q$ -Jacobi polynomials  $\{p_n(\cdot; a, b|q)\}_{n=0}^\infty$ , where the classical values  $0 < a, b < q^{-1}$  are now extended to  $a \in \mathbb{R}$  and  $ab \notin \{q^{-2}, q^{-3}, \dots\}$ . Similarly to Section 3, we introduce in Section 5 a discrete bilinear form involving  $q$ -derivatives with respect to which the monic little  $q$ -Jacobi polynomials (with first parameter out of the range of application of Favard’s theorem) become orthogonal. When the second parameter of big and little monic  $q$ -Jacobi polynomials vanishes, we recover (respectively) big and little monic  $q$ -Laguerre polynomials: The purpose of Section 6 is to distinguish these specific cases that led us to particular results of those previously given in this paper, and that will concern the systems of big and little monic  $q$ -Laguerre polynomials. In the Appendix, we include some interesting results, closely related with those appeared in the previous sections, but that are not essential in establishing the main results in Sections 3 and 5.

## 2. Generalized big $q$ -Jacobi polynomials

There exist several  $q$ -analogues of Jacobi polynomials living in the  $q$ -world. One of them, the so-called big  $q$ -Jacobi polynomials (which carries three free real parameters, other than  $q$ ), was hinted at by Hahn [13] and explicitly introduced and studied by Andrews and Askey (see [6, pp. 166-167]). Monic big  $q$ -Jacobi polynomials can be defined in terms of the  $q$ -hypergeometric series  ${}_3\phi_2$  by means of [15, 3.5.1, 3.5.4]

$$P_n(x; a, b, c; q) = \frac{(aq; q)_n (cq; q)_n}{(abq^{n+1}; q)_n} {}_3\phi_2 \left( \begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \middle| q; q \right), \quad n \in \mathbb{N}_0, \tag{2.2}$$

where  $a, c \notin \{q^{-n}\}_{n \in \mathbb{N}}$  and  $ab \notin \{q^{-n-1}\}_{n \in \mathbb{N}}$ , in order to be  $P_n(\cdot; a, b, c; q)$  a well-defined  $n$ th degree polynomial. Classical considerations (much more restrictive than the previous ones)

assume that  $0 < a, b < q^{-1}$  and  $c < 0$ . With such restrictions, monic big  $q$ -Jacobi polynomials can be alternatively defined as the system of polynomials fulfilling the orthogonality relation (see, for example, [15, 3.5.2, 3.5.4])

$$\begin{aligned}
 & \int_{cq}^{aq} \frac{(a^{-1}x, c^{-1}x; q)_\infty}{(x, bc^{-1}x; q)_\infty} P_m(x; a, b, c; q) P_n(x; a, b, c; q) d_q x \\
 &= a(1-q) \sum_{k=0}^\infty \frac{(q^{k+1}, (a/c)q^{k+1}; q)_\infty}{(aq^{k+1}, (ab/c)q^{k+1}; q)_\infty} \\
 & \quad \times P_m(aq^{k+1}; a, b, c; q) P_n(aq^{k+1}; a, b, c; q) q^{k+1} \\
 & - c(1-q) \sum_{k=0}^\infty \frac{((c/a)q^{k+1}, q^{k+1}; q)_\infty}{(cq^{k+1}, bq^{k+1}; q)_\infty} \\
 & \quad \times P_m(cq^{k+1}; a, b, c; q) P_n(cq^{k+1}; a, b, c; q) q^{k+1} \\
 &= a^{n+1} (-c)^n q^{(n+2)(n+1)/2} (1-q) \frac{(q; q)_n}{(abq^{n+1}; q)_n} \\
 & \quad \times \frac{(q, c/a, (a/c)q, abq^{2n+2}; q)_\infty}{(aq^{n+1}, bq^{n+1}, cq^{n+1}, (ab/c)q^{n+1}; q)_\infty} \delta_{mn}, \quad m, n \in \mathbb{N}_0.
 \end{aligned} \tag{2.3}$$

We can easily generalize the definition of monic big  $q$ -Jacobi polynomials in such a way that all real values of the first and third parameters  $a$  and  $c$  are allowed, and maintaining the restriction on the second parameter  $b$  in the form mentioned above, i.e.,  $ab \notin \{q^{-2}, q^{-3}, \dots\}$ . Using the basic hypergeometric representation (2.2), with the restrictions  $a, c \notin \{q^{-n}\}_{n \in \mathbb{N}}$  and  $ab \notin \{q^{-n-1}\}_{n \in \mathbb{N}}$ , and taking into account that  $(q^{-n}; q)_k$  vanishes for  $k > n$ , we have

$$\begin{aligned}
 P_n(x; a, b, c; q) &= \frac{(aq; q)_n (cq; q)_n}{(abq^{n+1}; q)_n} \sum_{k=0}^\infty \frac{(q^{-n}; q)_k (abq^{n+1}; q)_k (x; q)_k}{(aq; q)_k (cq; q)_k} \frac{q^k}{(q; q)_k} \\
 &= \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} q^k \frac{(aq^{k+1}; q)_{n-k} (cq^{k+1}; q)_{n-k}}{(abq^{n+k+1}; q)_{n-k}} (x; q)_k.
 \end{aligned}$$

Noting that the last representation is meaningful for all real values of  $a$  and  $c$ , and using that

$$\frac{(q^{-n}; q)_k}{(q; q)_k} = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-k(2n-k+1)/2}, \quad 0 \leq k \leq n, \tag{2.4}$$

(see [15, 0.3.3]), we obtain:

**Definition 2.1.** Let  $a, c$  be arbitrary real numbers and let  $b$  be a real number such that  $ab \notin \{q^{-2}, q^{-3}, \dots\}$ . For each  $n \in \mathbb{N}_0$  we define the  $n$ th degree generalized monic big  $q$ -Jacobi polynomial  $P_n(\cdot; a, b, c; q)$  by

$$\begin{aligned}
 P_n(x; a, b, c; q) &= \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(aq^{k+1}; q)_{n-k} (cq^{k+1}; q)_{n-k}}{(abq^{n+k+1}; q)_{n-k}} \\
 & \quad \times q^{k(k+1-2n)/2} (x; q)_k.
 \end{aligned} \tag{2.5}$$

By using the representation (2.5) it can be easily verified that generalized monic big  $q$ -Jacobi polynomials satisfy the same three term recurrence relation as the classical monic big  $q$ -Jacobi ones [15, 3.5.4, 3.5.3], namely:

**Proposition 2.1.** *Let  $a, c$  be arbitrary real numbers and let  $b$  be a real number such that  $ab \notin \{q^{-2}, q^{-3}, \dots\}$ . For each  $n \geq 0$ , the generalized monic big  $q$ -Jacobi polynomials fulfill*

$$P_{n+1}(x; a, b, c; q) = \left(x - A_n^{(a,b,c;q)}\right) P_n(x; a, b, c; q) - B_n^{(a,b,c;q)} P_{n-1}(x; a, b, c; q),$$

where

$$A_n^{(a,b,c;q)} = 1 - \frac{(1 - aq^{n+1})(1 - abq^{n+1})(1 - cq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})} + aq^{n+1} \frac{(1 - q^n)(1 - bq^n)(c - abq^n)}{(1 - abq^{2n})(1 - abq^{2n+1})},$$

$$B_n^{(a,b,c;q)} = -aq^{n+1} \frac{(1 - q^n)(1 - aq^n)(1 - bq^n)(1 - cq^n)(1 - abq^n)(c - abq^n)}{(1 - abq^{2n-1})(1 - abq^{2n})^2(1 - abq^{2n+1})}.$$

We remark that the restriction  $ab \neq q^{-1}$  is not necessary in the previous result, because in the expression of  $B_1^{(a,b,c;q)}$  the fifth factor in the numerator simplifies with the first factor in the denominator.

When  $a = 0$ ,  $B_n^{(0,b,c;q)}$  vanishes for all nonnegative integers  $n$ . In this case, generalized monic big  $q$ -Jacobi polynomials are defined, when  $c \neq 0$ , by  $P_n(x; 0, b, c; q) = (-c)^n q^{n(n+1)/2} (q^{-n}x/c; q)_n$  for  $n \geq 0$  (observe that there is no dependence with the parameter  $b$ ). When  $a = c = 0$  we have  $P_n(x; 0, b, 0; q) = x^n$  for  $n \geq 0$ .

In the case that for some positive integer  $N$ ,  $a = q^{-N}$  and  $b \notin \{q^{N-2}, q^{N-3}, \dots\}$ , no orthogonality results can be deduced from Favard’s theorem for the generalized monic big  $q$ -Jacobi polynomials, due to the fact that  $B_N^{(q^{-N},b,c;q)} = 0$ . One of the main results in this paper consists precisely in an orthogonality statement for the generalized monic big  $q$ -Jacobi polynomials with these outstanding values of the parameter  $a$ . To achieve this aim we will need some results, some concerning the action of the  $q$ -derivative operator on the generalized monic big  $q$ -Jacobi polynomials, and the other ones concerning a factorization property of these polynomials.

Using (2.5), and noting that

$$D_q(x; q)_k = \begin{cases} -\frac{1 - q^k}{1 - q} (qx; q)_{k-1}, & k \geq 1, \\ 0, & k = 0, \end{cases} \tag{2.6}$$

it is an easy matter to show that the  $q$ -derivative operator  $D_q$  acts on the generalized monic big  $q$ -Jacobi polynomials in the same way that it acts on the classical monic big  $q$ -Jacobi ones [15, 3.5.7, 3.5.4].

**Proposition 2.2.** *For arbitrary real parameters  $a$  and  $c$  and for a real parameter  $b$  such that  $ab \notin \{q^{-n-1}\}_{n \in \mathbb{N}}$ , the generalized monic big  $q$ -Jacobi polynomials verify the forward shift relation*

$$D_q P_n(x; a, b, c; q) = \frac{(1 - q^n)}{(1 - q)} q^{1-n} P_{n-1}(qx; aq, bq, cq; q), \quad n \geq 0. \tag{2.7}$$

Iterating (2.7) we readily obtain:

**Corollary 2.1.** *Let  $a, c \in \mathbb{R}$  and let  $b$  be a real number such that  $ab \notin \{q^{-n-1}\}_{n \in \mathbb{N}}$ . For each nonnegative integer  $n$ , and for each  $k \in \{0, 1, \dots, n + 1\}$ ,*

$$D_q^k P_n(x; a, b, c; q) = \frac{(q^{n-k+1}; q)_k}{(1 - q)^k} q^{k(k-n)} P_{n-k}(q^k x; a q^k, b q^k, c q^k; q).$$

We will show now that for  $0 \leq k \leq N - 1$ , the points  $x_k = q^{-k}$  are roots of the polynomials  $P_{N+n}(\cdot; q^{-N}, b, c; q)$  (and also, as we will state in the [Appendix](#), of  $P_{N+n}(\cdot; a, b, q^{-N}; q)$ ).

**Proposition 2.3.** *For a fixed  $N \in \mathbb{N}$  we have, for each  $c \in \mathbb{R}$ , each  $b \in \mathbb{R} \setminus \{q^{N-2}, q^{N-3}, \dots\}$ , and each  $n \geq N$ ,*

$$P_n(x; q^{-N}, b, c; q) = (-1)^N q^{N(N+1-2n)/2} (x; q)_N P_{n-N}(q^N x; q^N, b, c q^N; q).$$

**Proof.** For a fixed positive integer  $N$ , (2.5) yields, for each  $n \in \mathbb{N}_0$ ,

$$P_n(x; q^{-N}, b, c; q) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{k+1-N}; q)_{n-k} (c q^{k+1}; q)_{n-k}}{(b q^{n-N+k+1}; q)_{n-k}} \times q^{k(k+1-2n)/2} (x; q)_k.$$

For a fixed  $n \geq N$ , it is clear that  $(q^{k+1-N}; q)_{n-k} = 0$  for each  $k \leq N - 1$ . Therefore, for  $n \geq N$ ,

$$\begin{aligned} P_n(x; q^{-N}, b, c; q) &= \sum_{k=N}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{k+1-N}; q)_{n-k} (c q^{k+1}; q)_{n-k}}{(b q^{n-N+k+1}; q)_{n-k}} \\ &\quad \times q^{k(k+1-2n)/2} (x; q)_k \\ &= \sum_{k=0}^{n-N} (-1)^{N+k} \begin{bmatrix} n \\ N+k \end{bmatrix}_q \frac{(q^{k+1}; q)_{n-N-k} (c q^{N+k+1}; q)_{n-N-k}}{(b q^{n+k+1}; q)_{n-N-k}} \\ &\quad \times q^{(N+k)(N+k+1-2n)/2} (x; q)_{N+k}. \end{aligned}$$

Using that

$$\begin{bmatrix} n \\ N+k \end{bmatrix}_q (q^{k+1}; q)_{n-N-k} = \begin{bmatrix} n-N \\ k \end{bmatrix}_q (q^{k+1+N}; q)_{n-N-k}, \tag{2.8}$$

and also that  $(x; q)_{N+k} = (x; q)_N (q^N x; q)_k$ , we simplify the above equation to

$$\begin{aligned} P_n(x; q^{-N}, b, c; q) &= ((-1)^N q^{N(N+1-2n)/2} (x; q)_N) \\ &\quad \times \left( \sum_{k=0}^{n-N} (-1)^k \begin{bmatrix} n-N \\ k \end{bmatrix}_q \frac{(q^N q^{k+1}; q)_{n-N-k} (c q^N q^{k+1}; q)_{n-N-k}}{(q^N b q^{n-N+k+1}; q)_{n-N-k}} \right. \\ &\quad \left. \times q^{k(k+1-2(n-N))/2} (q^N x; q)_k \right) \\ &= (-1)^N q^{N(N+1-2n)/2} (x; q)_N P_{n-N}(q^N x; q^N, b, c q^N; q). \quad \square \end{aligned}$$

In the light of the above factorization, and using that for fixed  $N \in \mathbb{N}$  the  $q$ -shifted factorial  $(q^{-j}; q)_N$  vanishes for  $0 \leq j \leq N - 1$ , we deduce:

**Corollary 2.2.** *Let  $N$  be a fixed positive integer, let  $c \in \mathbb{R}$  and let  $b \in \mathbb{R} \setminus \{q^{N-2}, q^{N-3}, \dots\}$ . For  $n \geq N$*

$$P_n(q^{-j}; q^{-N}, b, c; q) = 0, \quad 0 \leq j \leq N - 1. \tag{2.9}$$

The previous result can be generalized in the form:

**Corollary 2.3.** *Let  $N \in \mathbb{N}$ ,  $c \in \mathbb{R}$  and  $b \in \mathbb{R} \setminus \{q^{N-2}, q^{N-3}, \dots\}$ . We have, for  $n \geq N$ ,*

$$D_q^k P_n(q^{-j}; q^{-N}, b, c; q) = 0, \quad 0 \leq k \leq j \leq N - 1. \tag{2.10}$$

**Proof.** For a fixed  $N \in \mathbb{N}$ , and after iterating expression (2.6), we get

$$D_q^k(x; q)_N = \begin{cases} \frac{(-1)^k (q^{N-k+1}; q)_k}{(1-q)^k} q^{k(k-1)/2} (q^k x; q)_{N-k}, & 0 \leq k \leq N, \\ 0, & k \geq N + 1. \end{cases} \tag{2.11}$$

Therefore, we readily obtain that

$$(D_q^k(\cdot; q)_N)(q^{-j}) = 0, \quad 0 \leq k \leq N - 1, k \leq j \leq N - 1. \tag{2.12}$$

Using the well-known  $q$ -analogue of the Leibniz rule

$$D_q^k(f(x)g(x)) = \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix}_q (D_q^j f)(x)(D_q^{k-j} g)(q^j x), \quad k \in \mathbb{N}_0,$$

on the factorizations in Proposition 2.3 (with the choice  $f(x) = (-1)^N q^{N(N+1-2n)/2} (x; q)_N$ ), and taking into account (2.12), we get the desired conclusion.  $\square$

### 3. $q$ -Sobolev orthogonality of big $q$ -Jacobi polynomials

As mentioned in the previous section, the hypothesis of Favard’s theorem does not hold for the family of generalized monic big  $q$ -Jacobi polynomials  $\{P_n(\cdot; q^{-N}, b, c; q)\}_{n=0}^\infty$  when  $N$  is a positive integer,  $b \notin \{q^{N-2}, q^{N-3}, \dots\}$  and  $c \in \mathbb{R}$  (observe that we are focusing our attention on the first parameter). We will use a suitable modification of our previous result [18, Theorem 3], changing the derivative and integral operators by the  $q$ -derivative and  $q$ -integral operators, to establish two (not essentially different) non-standard orthogonality results for the family of generalized monic big  $q$ -Jacobi polynomials.

**Theorem 3.1.** *For each positive integer  $N$  there exists a symmetric and positive definite matrix  $A$ , of order  $N$ , such that the family of generalized monic big  $q$ -Jacobi polynomials  $\{P_n(\cdot; q^{-N}, b, c; q)\}_{n=0}^\infty$ , with  $0 < b < q^{-N-1}$ ,  $b \notin \{q^{N-k}\}_{k=2}^{2N}$  and  $c < 0$ , is orthogonal with respect to the inner product  $(\cdot, \cdot)_{(b,c;q)}^{(N;A)}$  defined by*

$$\begin{aligned} (p_1, p_2)_{(b,c;q)}^{(N;A)} &= \left( \left( p_1(q^{-k}) \right)_{k=0}^{N-1} \right) A \left( \left( p_2(q^{-k}) \right)_{k=0}^{N-1} \right)^t \\ &+ \int_{cq^{N+1}}^q \frac{(c^{-1}q^{-N}x; q)_\infty}{(bc^{-1}x; q)_\infty} (D_q^N p_1(q^{-N}x))(D_q^N p_2(q^{-N}x)) d_q x, \quad p_1, p_2 \in \mathbb{P}. \end{aligned} \tag{3.13}$$



**Proof.** Let  $\{l_j(\cdot; q)\}_{j=0}^{N-1} \subset \mathbb{P}_{N-1}$  be the set of Lagrange interpolating polynomials at the nodes  $\{q^{-k}\}_{k=0}^{N-1}$ , explicitly defined by

$$l_j(x; q) = \prod_{\substack{i=0 \\ i \neq j}}^{N-1} \frac{x - q^{-i}}{q^{-j} - q^{-i}}, \quad 0 \leq j \leq N - 1.$$

Due to the fact that

$$P_j(x; q^{-N}, b, c; q) = \sum_{k=0}^{N-1} P_j(q^{-k}; q^{-N}, b, c; q)l_k(x; q), \quad 0 \leq j \leq N - 1,$$

and since both  $\{l_j(\cdot; q)\}_{j=0}^{N-1}$  and  $\{P_j(\cdot; q^{-N}, b, c; q)\}_{j=0}^{N-1}$  are bases of  $\mathbb{P}_{N-1}$ , we can ensure that the matrix  $C = (P_j(q^{-k}; q^{-N}, b, c; q))_{j,k=0}^{N-1}$  is nonsingular. Also, if  $D = (\kappa_j \delta_{jk})_{j,k=0}^{N-1}$  is a nonsingular diagonal matrix of order  $N$  ( $\kappa_j \in \mathbb{R} \setminus \{0\}$ ), then the symmetric matrix  $A = C^{-1}D^2(C^{-1})^t = (C^{-1}D)(C^{-1}D)^t$  is positive definite.

In order to state the orthogonality we will consider three cases:

(i) First suppose that  $0 \leq m, n \leq N - 1$ . As a consequence that  $\deg(D_q p) = \deg(p) - 1$  for each  $p \in \mathbb{P} \setminus \mathbb{P}_0$ , we have  $D_q^N P_m(x; q^{-N}, b, c; q) = D_q^N P_n(x; q^{-N}, b, c; q) = 0$  for all reals  $x$ . Therefore

$$\begin{aligned} & (P_m(\cdot; q^{-N}, b, c; q), P_n(\cdot; q^{-N}, b, c; q))_{(b,c;q)}^{(N;A)} \\ &= \left( \left( P_m(q^{-k}; q^{-N}, b, c; q) \right)_{k=0}^{N-1} C^{-1} \right) D^2 \left( \left( P_n(q^{-k}; q^{-N}, b, c; q) \right)_{k=0}^{N-1} C^{-1} \right)^t \\ &= (\delta_{mk})_{k=0}^{N-1} D^2 ((\delta_{nk})_{k=0}^{N-1})^t = \kappa_n^2 \delta_{mn}. \end{aligned}$$

(ii) If  $0 \leq m \leq N - 1$  and  $n \geq N$ , then  $D_q^N P_m(x; q^{-N}, b, c; q) = 0$  and also  $P_n(q^{-k}; q^{-N}, b, c; q) = 0$  for  $0 \leq k \leq N - 1$  (see Corollary 2.2); so clearly we have

$$(P_m(\cdot; q^{-N}, b, c; q), P_n(\cdot; q^{-N}, b, c; q))_{(b,c;q)}^{(N;A)} = 0.$$

(iii) Now consider  $m, n \geq N$ . In this case  $P_m(q^{-k}; q^{-N}, b, c; q) = P_n(q^{-k}; q^{-N}, b, c; q) = 0$  for  $0 \leq k \leq N - 1$ . Using Corollary 2.1 we get, for all  $n \geq N$ ,

$$D_q^N P_n(x; q^{-N}, b, c; q) = \frac{(q^{n-N+1}; q)_N}{(1-q)^N} q^{N(N-n)} P_{n-N}(q^N x; 1, bq^N, cq^N; q).$$

Therefore, we deduce

$$\begin{aligned} & (P_m(\cdot; q^{-N}, b, c; q), P_n(\cdot; q^{-N}, b, c; q))_{(b,c;q)}^{(N;A)} \\ &= \int_{cq^{N+1}}^q \frac{(c^{-1}q^{-N}x; q)_\infty}{(bc^{-1}x; q)_\infty} \left( (D_q^N P_m(\cdot; q^{-N}, b, c; q))(q^{-N}x) \right) \\ & \quad \times \left( (D_q^N P_n(\cdot; q^{-N}, b, c; q))(q^{-N}x) \right) d_q x \\ &= h_m h_n \int_{(cq^N)_q}^q \frac{(x, (cq^N)^{-1}x; q)_\infty}{(x, (bq^N)(cq^N)^{-1}x; q)_\infty} \\ & \quad \times P_{m-N}(x; 1, bq^N, cq^N; q) P_{n-N}(x; 1, bq^N, cq^N; q) d_q x, \end{aligned}$$

where we have defined, for  $n \geq N$ ,

$$h_n = \frac{(q^{n-N+1}; q)_N}{(1 - q)^N} q^{N(N-n)}.$$

With the aid of the classical orthogonality condition for the generalized monic big  $q$ -Jacobi polynomials (2.3), replacing  $a$  by 1,  $b$  by  $bq^N$  and  $c$  by  $cq^N$ , we can finally state

$$\begin{aligned} & (P_m(\cdot; q^{-N}, b, c; q), P_n(\cdot; q^{-N}, b, c; q))_{(b,c;q)}^{(N;A)} \\ &= h_n^2 (-cq^N)^{n-N} q^{(n-N+2)(n-N+1)/2} (1 - q) \frac{(q; q)_{n-N}}{(bq^{n+1}; q)_{n-N}} \\ & \quad \times \frac{(q, cq^N, (cq^N)^{-1}q, bq^{2n-N+2}; q)_\infty}{(q^{n-N+1}, bq^{n+1}, cq^{n+1}, (b/c)q^{n-N+1}; q)_\infty} \delta_{mn} \\ &= \frac{(q; q)_n}{(1 - q)^{2N}} (-c)^{n-N} q^{d_n} (1 - q) \\ & \quad \times \frac{(q, cq^N, c^{-1}q^{1-N}, bq^{2n-N+2}; q)_\infty}{(q^{n+1}, bq^{n+1}, cq^{n+1}, (b/c)q^{n-N+1}; q)_\infty (bq^{n+1}; q)_{n-N}} \delta_{mn} \\ &= \left( \frac{(q; q)_n}{(1 - q)^n} \right)^2 (-c)^{n-N} q^{d_n} (1 - q)^{2(n-N)+1} \\ & \quad \times \frac{(cq^N, c^{-1}q^{1-N}, bq^{2n-N+2}; q)_\infty}{(bq^{n+1}, cq^{n+1}, (b/c)q^{n-N+1}; q)_\infty (bq^{n+1}; q)_{n-N}} \delta_{mn}, \end{aligned}$$

where

$$d_n = \frac{(n - N)^2 + (n - N)(3 - 2N) + 2}{2}. \quad \square$$

In the above theorem, the first part of the discrete inner product, namely, the term

$$\left( \left( p_1(q^{-k}) \right)_{k=0}^{N-1} \right) A \left( \left( p_2(q^{-k}) \right)_{k=0}^{N-1} \right)^t,$$

is designed to exploit the fact that for  $n \geq N$  we have  $P_n(q^{-k}; q^{-N}, b, c; q) = 0$  for  $0 \leq k \leq N - 1$ . The matrix  $A$  is defined in terms of an arbitrary nonsingular diagonal matrix  $D$  of order  $N$ , and in terms of a matrix  $C$  in which entries are the numbers  $P_j(q^{-k}; q^{-N}, b, c; q)$ , with  $0 \leq j, k \leq N - 1$ . By considering the condition  $D_q^k P_n(q^{-k}; q^{-N}, b, c; q) = 0$  for  $n \geq N$  and  $0 \leq k \leq N - 1$ , obtained by choosing  $j = k$  in (2.10), we can reformulate Theorem 3.1 in such a way that both terms of the inner product defined explicitly depend on the  $q$ -derivative operator. Using the same notation as in Theorem 3.1 to define the inner product, we have:

**Theorem 3.2.** For each positive integer  $N$  there exists a symmetric and positive definite matrix  $A$ , of order  $N$ , such that the family of generalized monic big  $q$ -Jacobi polynomials  $\{P_n(\cdot; q^{-N}, b, c; q)\}_{n=0}^\infty$ , with  $0 < b < q^{-N-1}$ ,  $b \notin \{q^{N-k}\}_{k=2}^{2N}$  and  $c < 0$ , is orthogonal with respect to the inner product  $(\cdot, \cdot)_{(b,c;q)}^{(N;A)}$  defined by

$$(p_1, p_2)_{(b,c;q)}^{(N;A)} = \left( \left( D_q^k p_1(q^{-k}) \right)_{k=0}^{N-1} \right) A \left( \left( D_q^k p_2(q^{-k}) \right)_{k=0}^{N-1} \right)^t$$

$$+ \int_{cq^{N+1}}^q \frac{(c^{-1}q^{-N}x; q)_\infty}{(bc^{-1}x; q)_\infty} (D_q^N p_1(q^{-N}x))(D_q^N p_2(q^{-N}x)) d_q x, \quad p_1, p_2 \in \mathbb{P}. \quad (3.14)$$

**Proof.** It is easily seen (see (2.11)) that the polynomials  $l_j(\cdot; q)$  defined by

$$l_j(x; q) = (-1)^j \frac{(1-q)^j}{(q; q)_j} \frac{1}{q^{j(j-1)/2}} (x; q)_j, \quad j = 0, 1, \dots, N-1,$$

verify that  $D_q^k l_j(q^{-k}; q) = \delta_{jk}$  for  $0 \leq j, k \leq N-1$ . Therefore

$$P_j(x; q^{-N}, b, c; q) = \sum_{k=0}^{N-1} D_q^k P_j(q^{-k}; q^{-N}, b, c; q) l_k(x; q), \quad 0 \leq j \leq N-1;$$

so the matrix  $C = (D_q^k P_j(q^{-k}; q^{-N}, b, c; q))_{j,k=0}^{N-1}$  is nonsingular. Fix  $D$ , a nonsingular diagonal matrix of order  $N$ , and define  $A = C^{-1} D^2 (C^{-1})^t$ . The rest of the proof follows as in Theorem 3.1.  $\square$

#### 4. Generalized little $q$ -Jacobi polynomials

As in the case of the big  $q$ -Jacobi polynomials, Hahn introduced in [13] other  $q$ -analogues of the Jacobi polynomials, later studied in detail by Andrews and Askey in [5,6], who named them as little  $q$ -Jacobi polynomials. For real parameters  $a, b$  such that  $0 < a, b < q^{-1}$ , monic little  $q$ -Jacobi polynomials  $p_n(\cdot; a, b|q)$  are the ones fulfilling the orthogonality condition [15, 3.12.2, 3.12.4]

$$\begin{aligned} & \frac{(bq; q)_\infty}{(q; q)_\infty} \int_0^1 \frac{(qx; q)_\infty}{(bqx; q)_\infty} x^{\log_q a} p_m(x; a, b|q) p_n(x; a, b|q) d_q x \\ &= \sum_{k=0}^\infty \frac{(bq; q)_k}{(q; q)_k} (aq)^k p_m(q^k; a, b|q) p_n(q^k; a, b|q) \\ &= a^n q^{n^2} \frac{(abq^{2n+2}; q)_\infty}{(aq^{n+1}; q)_\infty} \frac{(q; q)_n (bq; q)_n}{(abq^{n+1}; q)_n} \delta_{mn}, \quad m, n \in \mathbb{N}_0. \end{aligned} \quad (4.15)$$

For each  $n \in \mathbb{N}_0$ , these polynomials can be defined in terms of the  $q$ -hypergeometric series  ${}_2\phi_1$  by means of [15, 3.12.1, 3.12.4]

$$p_n(x; a, b|q) = (-1)^n q^{n(n-1)/2} \frac{(aq; q)_n}{(abq^{n+1}; q)_n} {}_2\phi_1 \left( \begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \middle| q; qx \right). \quad (4.16)$$

Observe that the above representation works perfectly for all real values of the parameters  $a$  and  $b$ , except when  $a \in \{q^{-1}, q^{-2}, \dots\}$  or  $ab \in \{q^{-2}, q^{-3}, \dots\}$ . Our intention is to accomplish the extension of monic little  $q$ -Jacobi polynomials for all real values of the first parameter  $a$ .

Starting with (4.16), and using that  $(q^{-n}; q)_k$  vanishes for  $k > n$ , we get

$$\begin{aligned} p_n(x; a, b|q) &= (-1)^n q^{n(n-1)/2} \frac{(aq; q)_n}{(abq^{n+1}; q)_n} \sum_{k=0}^\infty \frac{(q^{-n}; q)_k (abq^{n+1}; q)_k}{(aq; q)_k} \frac{(qx)^k}{(q; q)_k} \\ &= (-1)^n q^{n(n-1)/2} \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{(aq^{k+1}; q)_{n-k}}{(abq^{n+k+1}; q)_{n-k}} (qx)^k. \end{aligned}$$

Using (2.4), we obtain:

**Definition 4.1.** Let  $a$  be an arbitrary real number and let  $b$  be a real number such that  $ab \notin \{q^{-2}, q^{-3}, \dots\}$ . For each  $n \in \mathbb{N}_0$  we define the  $n$ th degree generalized monic little  $q$ -Jacobi polynomial  $p_n(\cdot; a, b|q)$  by

$$p_n(x; a, b|q) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(aq^{k+1}; q)_{n-k}}{(abq^{n+k+1}; q)_{n-k}} q^{(n-k)(n-k-1)/2} (-x)^k. \tag{4.17}$$

Working directly with the representation (4.17), one can verify that generalized monic little  $q$ -Jacobi polynomials satisfy the same three term recurrence relation as the classical monic little  $q$ -Jacobi polynomials [15, 3.12.4, 3.12.3]. Concretely, we obtain the following proposition.

**Proposition 4.1.** Let  $a$  be an arbitrary real number and let  $b$  be a real number such that  $ab \notin \{q^{-2}, q^{-3}, \dots\}$ . For each  $n \in \mathbb{N}_0$ , the generalized monic little  $q$ -Jacobi polynomials fulfill the three term recurrence relation

$$p_{n+1}(x; a, b|q) = \left(x - A_n^{(a,b;q)}\right) p_n(x; a, b|q) - B_n^{(a,b;q)} p_{n-1}(x; a, b|q),$$

where

$$A_n^{(a,b;q)} = \frac{q^n}{(1 - abq^{2n+1})} \left( \frac{(1 - aq^{n+1})(1 - abq^{n+1})}{(1 - abq^{2n+2})} + \frac{a(1 - q^n)(1 - bq^n)}{(1 - abq^{2n})} \right),$$

$$B_n^{(a,b;q)} = aq^{2n-1} \frac{(1 - q^n)(1 - aq^n)(1 - bq^n)(1 - abq^n)}{(1 - abq^{2n-1})(1 - abq^{2n})^2(1 - abq^{2n+1})}.$$

Again, the restriction  $ab \neq q^{-1}$  is not necessary because in  $B_1^{(a,b;q)}$ , the last factor in the numerator simplifies with the first factor in the denominator.

If  $a = 0$ , then  $B_n^{(0,b;q)}$  vanishes for all nonnegative integers  $n$ , and the corresponding generalized monic little  $q$ -Jacobi polynomials are defined, for each  $n \geq 0$ , by  $p_n(x; 0, b|q) = (-1)^n q^{n(n-1)/2} (q^{1-n}x; q)_n$  (observe that there is no dependence with the parameter  $b$ ).

For  $N \in \mathbb{N}$  and  $b \in \mathbb{R} \setminus \{q^{N-2}, q^{N-3}, \dots\}$ ,  $B_N^{(q^{-N}, b;q)}$  vanishes; so no orthogonality results can be deduced from Favard’s theorem. Our objective is to give an orthogonality statement for the generalized monic little  $q$ -Jacobi polynomials for these outstanding values of the first parameter  $a$  (i.e., for  $a \in \{q^{-1}, q^{-2}, \dots\}$ ). We will need two key tools (the corollaries below) for this purpose.

Directly, from (4.17), it is an easy matter to verify that the  $q$ -derivative operator  $D_q$  acts on the generalized monic little  $q$ -Jacobi polynomials as it acts on the classical monic little  $q$ -Jacobi ones [15, 3.12.7, 3.12.4].

**Proposition 4.2.** For  $a \in \mathbb{R}$  and  $ab \in \mathbb{R} \setminus \{q^{-2}, q^{-3}, \dots\}$ , monic generalized little  $q$ -Jacobi polynomials verify the forward shift relation

$$D_q p_n(x; a, b|q) = \frac{(1 - q^n)}{(1 - q)} p_{n-1}(x; aq, bq|q), \quad n \geq 0. \tag{4.18}$$

Iterating expression (4.18) it follows:

**Corollary 4.1.** *Let  $a$  be an arbitrary real number and let  $b$  be a real number such that  $ab \notin \{q^{-2}, q^{-3}, \dots\}$ . For each nonnegative integer  $n$ , and for each  $k \in \{0, 1, \dots, n + 1\}$ ,*

$$D_q^k p_n(x; a, b|q) = \frac{(q^{n-k+1}; q)_k}{(1 - q)^k} p_{n-k}(x; aq^k, bq^k|q).$$

We will show now that the point  $x = 0$  is a zero of precise order  $N$  of the polynomials  $p_{N+n}(\cdot; q^{-N}, b|q)$ , where  $b \in \mathbb{R} \setminus \{q^{N-2}, q^{N-3}, \dots\}$ .

**Proposition 4.3.** *For a fixed  $N \in \mathbb{N}$  we have, for  $b \in \mathbb{R} \setminus \{q^{N-2}, q^{N-3}, \dots\}$  and  $n \geq N$ ,*

$$p_n(x; q^{-N}, b|q) = x^N p_{n-N}(x; q^N, b|q).$$

**Proof.** For a fixed positive integer  $N$ , (4.17) yields, for each  $n \in \mathbb{N}_0$ ,

$$p_n(x; q^{-N}, b|q) = \sum_{k=0}^n (-1)^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{k+1-N}; q)_{n-k}}{(bq^{n+k+1-N}; q)_{n-k}} q^{(n-k)(n-k-1)/2} (-x)^k.$$

For all  $n \geq N$ ,  $(q^{k+1-N}; q)_{n-k} = 0$  for each  $k \leq N - 1$ . Therefore, for  $n \geq N$

$$\begin{aligned} p_n(x; q^{-N}, b|q) &= \sum_{k=N}^n (-1)^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{k+1-N}; q)_{n-k}}{(bq^{n+k+1-N}; q)_{n-k}} q^{(n-k)(n-k-1)/2} (-x)^k \\ &= x^N \sum_{k=0}^{n-N} (-1)^{n-N} \begin{bmatrix} n \\ N+k \end{bmatrix}_q \frac{(q^{k+1}; q)_{n-N-k}}{(bq^{n+k+1}; q)_{n-N-k}} q^{(n-N-k)(n-N-k-1)/2} (-x)^k. \end{aligned}$$

Using (2.8), we simplify the above equation to

$$\begin{aligned} p_n(x; q^{-N}, b|q) &= x^N \sum_{k=0}^{n-N} (-1)^{n-N} \begin{bmatrix} n-N \\ k \end{bmatrix}_q \frac{(q^N q^{k+1}; q)_{n-N-k}}{(q^N bq^{n-N+k+1}; q)_{n-N-k}} \\ &\quad \times q^{(n-N-k)(n-N-k-1)/2} (-x)^k \\ &= x^N p_{n-N}(x; q^N, b|q). \quad \square \end{aligned}$$

The previous result implies that  $p_n^{(k)}(0; q^{-N}, b|q) = 0$  for  $0 \leq k \leq N - 1$  and  $n \geq N$ . Taking into account that

$$D_q^k f(0) = \frac{(q; q)_k}{(1 - q)^k} \frac{f^{(k)}(0)}{k!}, \quad k \geq 0, \tag{4.19}$$

for a function  $f$  analytic in a neighborhood of 0 (see [14]), we can also establish that  $x = 0$  is a “ $q$ -zero” of precise order  $N$  of the polynomials  $p_n(\cdot; q^{-N}, b|q)$ .

**Corollary 4.2.** *Let  $N$  be a fixed positive integer. For  $b \in \mathbb{R} \setminus \{q^{N-2}, q^{N-3}, \dots\}$  and  $n \geq N$ ,*

$$D_q^k p_n(0; q^{-N}, b|q) = 0, \quad 0 \leq k \leq N - 1. \tag{4.20}$$

### 5. $q$ -Sobolev orthogonality of little $q$ -Jacobi polynomials

In Section 4 we have shown that no orthogonality results can be deduced from Favard’s theorem for the family of generalized monic little  $q$ -Jacobi polynomials  $\{p_n(\cdot; q^{-N}, b|q)\}_{n=0}^\infty$  when  $N$  is a positive integer and  $b \in \mathbb{R} \setminus \{q^{N-2}, q^{N-3}, \dots\}$ . Again, as in Section 3, we will consider a suitable modification of our previous result [18, Theorem 3] to establish a non-standard orthogonality condition for the system of generalized monic little  $q$ -Jacobi polynomials.

**Theorem 5.1.** *For each positive integer  $N$ , there exists a symmetric and positive definite matrix  $A$ , of order  $N$ , such that the family of generalized monic little  $q$ -Jacobi polynomials  $\{p_n(\cdot; q^{-N}, b|q)\}_{n=0}^\infty$ , with  $b < q^{-N-1}$  and  $b \notin \{q^{N-2}, q^{N-3}, \dots\}$ , is orthogonal with respect to the inner product  $(\cdot, \cdot)_{(b; q)}^{(N; A)}$  defined by*

$$\begin{aligned}
 (p_1, p_2)_{(b; q)}^{(N; A)} &= \left( p_1(0), D_q p_1(0), \dots, D_q^{N-1} p_1(0) \right) A \left( p_2(0), D_q p_2(0), \dots, D_q^{N-1} p_2(0) \right)^t \\
 &\quad + \sum_{k=0}^\infty \frac{(bq^{N+1}; q)_k}{(q; q)_k} q^k \left( D_q^N p_1(q^k) \right) \left( D_q^N p_2(q^k) \right), \quad p_1, p_2 \in \mathbb{P}. \quad (5.21)
 \end{aligned}$$

**Proof.** Let  $\{l_j(\cdot; q)\}_{j=0}^{N-1} \subset \mathbb{P}_{N-1}$  be the set of polynomials defined by

$$l_j(x; q) = \frac{(1 - q)^j}{(q; q)_j} x^j, \quad 0 \leq j \leq N - 1.$$

Since both  $\{l_j(\cdot; q)\}_{j=0}^{N-1}$  and  $\{p_j(\cdot; q^{-N}, b|q)\}_{j=0}^{N-1}$  are bases of  $\mathbb{P}_{N-1}$ , and taking into account that  $D_q^k l_j(0; q) = \delta_{kj}$  (see (4.19)), we have

$$p_j(x; q^{-N}, b|q) = \sum_{k=0}^{N-1} D_q^k p_j(0; q^{-N}, b|q) l_k(x; q), \quad 0 \leq j \leq N - 1.$$

Therefore, we can ensure that the matrix  $C = (D_q^k p_j(0; q^{-N}, b|q))_{j,k=0}^{N-1}$  is nonsingular. Also, if  $D = (\kappa_j \delta_{jk})_{j,k=0}^{N-1}$  is a nonsingular diagonal matrix of order  $N$  ( $\kappa_j \in \mathbb{R} \setminus \{0\}$ ), then the symmetric matrix  $A = C^{-1} D^2 (C^{-1})^t = (C^{-1} D)(C^{-1} D)^t$  is positive definite.

In order to state the orthogonality we will consider three cases:

(i) If  $0 \leq m, n \leq N - 1$ , then  $D_q^N p_m(x; q^{-N}, b|q) = D_q^N p_n(x; q^{-N}, b|q) = 0$ . Therefore

$$\begin{aligned}
 (p_m(\cdot; q^{-N}, b|q), p_n(\cdot; q^{-N}, b|q))_{(b; q)}^{(N; A)} &= \left( \left( D_q^k p_m(0; q^{-N}, b|q) \right)_{k=0}^{N-1} C^{-1} \right) D^2 \left( \left( D_q^k p_n(0; q^{-N}, b|q) \right)_{k=0}^{N-1} C^{-1} \right)^t \\
 &= (\delta_{mk})_{k=0}^{N-1} D^2 ((\delta_{nk})_{k=0}^{N-1})^t = \kappa_n^2 \delta_{mn}.
 \end{aligned}$$

(ii) If  $0 \leq m \leq N - 1$  and  $n \geq N$ , then  $D_q^N p_m(x; q^{-N}, b|q) = 0$  and also (Corollary 4.2)  $D_q^k p_n(0; q^{-N}, b|q) = 0$  for  $0 \leq k \leq N - 1$ . Thus

$$(p_m(\cdot; q^{-N}, b|q), p_n(\cdot; q^{-N}, b|q))_{(b; q)}^{(N; A)} = 0.$$

(iii) Finally, if  $m, n \geq N$ , then  $D_q^k p_m(0; q^{-N}, b|q) = D_q^k p_n(0; q^{-N}, b|q) = 0$  for  $0 \leq k \leq N - 1$ . Using Corollary 4.1 and the orthogonality condition for the classical monic little  $q$ -Jacobi polynomials, replacing  $a$  by 1 and  $b$  by  $bq^N$  in (4.15), it follows

$$\begin{aligned} & (p_m(\cdot; q^{-N}, b|q), p_n(\cdot; q^{-N}, b|q))_{(b; q)}^{(N; A)} \\ &= \sum_{k=0}^{\infty} \frac{(bq^{N+1}; q)_k}{(q; q)_k} q^k \left( D_q^N p_m(q^k; q^{-N}, b|q) \right) \left( D_q^N p_n(q^k; q^{-N}, b|q) \right) \\ &= \frac{(q^{m-N+1}; q)_N}{(1-q)^N} \frac{(q^{n-N+1}; q)_N}{(1-q)^N} \sum_{k=0}^{\infty} \frac{(bq^{N+1}; q)_k}{(q; q)_k} q^k \\ &\quad \times p_{m-N}(q^k; 1, bq^N|q) p_{n-N}(q^k; 1, bq^N|q) \\ &= \left( \frac{(q^{n-N+1}; q)_N}{(1-q)^N} \right)^2 q^{(n-N)^2} \frac{(bq^{2n-N+2}; q)_{\infty}}{(q^{n-N+1}; q)_{\infty}} (q; q)_{n-N} \frac{(bq^{N+1}; q)_{n-N}}{(bq^{n+1}; q)_{n-N}} \delta_{mn} \\ &= \left( \frac{(q; q)_n}{(1-q)^n} \right)^2 q^{(n-N)^2} (1-q)^{2(n-N)} e_q(q) \frac{(bq^{2n-N+2}; q)_{\infty} (bq^{N+1}; q)_{n-N}}{(bq^{n+1}; q)_{n-N}} \delta_{mn}. \quad \square \end{aligned}$$

### 6. Particular cases: Big and little $q$ -Laguerre polynomials

When the second parameter of the families of big and little  $q$ -Jacobi polynomials vanishes, we get (respectively) the families of big and little  $q$ -Laguerre polynomials. With this in mind, all the above relations and results can be given for the the families of big and little  $q$ -Laguerre polynomials. Neither new arguments, nor new computations are needed to describe the previous  $q$ -Sobolev orthogonality history, developed in the scenery of the  $q$ -Jacobi families, into the new scenery of the  $q$ -Laguerre families.

#### 6.1. The generalized big $q$ -Laguerre polynomials

Monic big  $q$ -Laguerre polynomials  $P_n(\cdot; a, b; q)$  are monic big  $q$ -Jacobi polynomials  $P_n(\cdot; a, c, b; q)$  with  $c = 0$ . Therefore, first setting  $b = 0$  and then replacing  $c$  by  $b$  in all the results of Sections 2 and 3 (which concern big  $q$ -Jacobi polynomials), we will obtain similar results for the family of monic big  $q$ -Laguerre polynomials. We will briefly summarize all these results.

(i) *Definition as a basic series (2.2)*

Let  $a, b \in \mathbb{R} \setminus \{q^{-1}, q^{-2}, \dots\}$ . For each  $n \in \mathbb{N}_0$  we define the  $n$ th degree monic big  $q$ -Laguerre polynomial  $P_n(\cdot; a, b; q)$  by

$$P_n(x; a, b; q) = (aq; q)_n (bq; q)_{n3} \phi_2 \left( \begin{matrix} q^{-n}, 0, x \\ aq, bq \end{matrix} \middle| q; q \right), \quad x \in \mathbb{R}.$$

(ii) *Orthogonality relation (2.3)*

For  $0 < a < q^{-1}$  and  $b < 0$ ,

$$\begin{aligned} & \int_{bq}^{aq} \frac{(a^{-1}x, b^{-1}x; q)_{\infty}}{(x; q)_{\infty}} P_m(x; a, b; q) P_n(x; a, b; q) d_q x \\ &= a(1-q) \sum_{k=0}^{\infty} \frac{(q^{k+1}, (a/b)q^{k+1}; q)_{\infty}}{(aq^{k+1}; q)_{\infty}} P_m(aq^{k+1}; a, b; q) P_n(aq^{k+1}; a, b; q) q^{k+1} \end{aligned}$$

$$\begin{aligned}
 & -b(1 - q) \sum_{k=0}^{\infty} \frac{(q^{k+1}, (b/a)q^{k+1}; q)_{\infty}}{(bq^{k+1}; q)_{\infty}} P_m(bq^{k+1}; a, b; q) P_n(bq^{k+1}; a, b; q) q^{k+1} \\
 & = a^{n+1} (-b)^n q^{(n+2)(n+1)/2} (1 - q)(q; q)_n \frac{(q, b/a, (a/b)q; q)_{\infty}}{(aq^{n+1}, bq^{n+1}; q)_{\infty}} \delta_{mn}, \quad m, n \in \mathbb{N}_0.
 \end{aligned}$$

(iii) *Generalized monic big  $q$ -Laguerre polynomials (Definition 2.1)*

Let  $a, b \in \mathbb{R}$ . For each  $n \in \mathbb{N}_0$  and each  $x \in \mathbb{R}$ , we define the  $n$ th degree generalized monic big  $q$ -Laguerre polynomial  $P_n(\cdot; a, b; q)$  by

$$P_n(x; a, b; q) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q (aq^{k+1}; q)_{n-k} (bq^{k+1}; q)_{n-k} q^{k(k+1-2n)/2} (x; q)_k.$$

(iv) *Three term recurrence relation (Proposition 2.1)*

For  $n \in \mathbb{N}_0$  and  $a, b \in \mathbb{R}$ ,

$$P_{n+1}(x; a, b; q) = \left(x - A_n^{(a,b;q)}\right) P_n(x; a, b; q) - B_n^{(a,b;q)} P_{n-1}(x; a, b; q),$$

where

$$\begin{aligned}
 A_n^{(a,b;q)} &= 1 - (1 - aq^{n+1})(1 - bq^{n+1}) - a(-b)q^{n+1}(1 - q^n), \\
 B_n^{(a,b;q)} &= a(-b)q^{n+1}(1 - q^n)(1 - aq^n)(1 - bq^n).
 \end{aligned}$$

When  $a, b \in \{q^{-1}, q^{-2}, \dots\} \cup \{0\}$ , no orthogonality results can be deduced from Favard’s theorem.

(v) *Iterated forward shift operator (Corollary 2.1)*

For  $n \in \mathbb{N}_0, 0 \leq k \leq n + 1$ , and  $a, b \in \mathbb{R}$ ,

$$D_q^k P_n(x; a, b; q) = \frac{(q^{n-k+1}; q)_k}{(1 - q)^k} q^{k(k-n)} P_{n-k}(q^k x; aq^k, bq^k; q).$$

(vi) *Factorization (Proposition 2.3)*

For a fixed  $N \in \mathbb{N}$ , for all  $b \in \mathbb{R}$ , and for each  $n \geq N$ ,

$$P_n(x; q^{-N}, b; q) = (-1)^N q^{N(N+1-2n)/2} (x; q)_N P_{n-N}(q^N x; q^N, bq^N; q). \tag{6.22}$$

(vii) *Evaluations on the lattice points (Corollary 2.3)*

Let  $N \in \mathbb{N}$  and let  $b \in \mathbb{R}$ . For  $n \geq N$ ,

$$D_q^k P_n(q^{-j}; q^{-N}, b; q) = 0, \quad 0 \leq k \leq j \leq N - 1. \tag{6.23}$$

(viii)  *$q$ -Sobolev orthogonality (Theorem 3.1).*

**Theorem 6.1.** *For each positive integer  $N$ , there exists a symmetric and positive definite matrix  $A$ , of order  $N$ , such that the family of generalized monic big  $q$ -Laguerre polynomials  $\{P_n(\cdot; q^{-N}, b; q)\}_{n=0}^{\infty}$ , with  $b < 0$ , is orthogonal with respect to the inner product  $(\cdot, \cdot)_{(b;q)}^{(N;A)}$  defined by*

$$\begin{aligned}
 (p_1, p_2)_{(b;q)}^{(N;A)} &= \left( \left( p_1(q^{-k}) \right)_{k=0}^{N-1} \right) A \left( \left( p_2(q^{-k}) \right)_{k=0}^{N-1} \right)^t \\
 &+ \int_{bq^{N+1}}^q (b^{-1}q^{-N}x; q)_{\infty} (D_q^N p_1(q^{-N}x))(D_q^N p_2(q^{-N}x)) d_q x, \quad p_1, p_2 \in \mathbb{P}.
 \end{aligned}$$



### 6.2. The generalized little $q$ -Laguerre polynomials

Monic little  $q$ -Laguerre polynomials  $p_n(\cdot; a|q)$  are monic little  $q$ -Jacobi polynomials  $p_n(\cdot; a, b|q)$  with  $b = 0$ . Therefore, setting  $b = 0$  in all the results of Sections 4 and 5, we will obtain results for the family of monic little  $q$ -Laguerre polynomials, similar to those obtained for the monic little  $q$ -Jacobi system. As in the previous subsection, we will summarize all these results, that have been previously obtained in [20].

(i) *Definition as a basic series (4.16)*

Let  $a \in \mathbb{R} \setminus \{q^{-1}, q^{-2}, \dots\}$ . For each  $n \in \mathbb{N}_0$  we define the  $n$ th degree monic little  $q$ -Laguerre polynomial  $p_n(\cdot; a|q)$  by

$$p_n(x; a|q) = (-1)^n q^{n(n-1)/2} (aq; q)_n {}_2\phi_1 \left( \begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \middle| q; qx \right), \quad x \in \mathbb{R}.$$

(ii) *Orthogonality relation (4.15)*

For  $0 < a < q^{-1}$ ,

$$\begin{aligned} & \frac{1}{(q; q)_\infty} \int_0^1 (qx; q)_\infty x^{\log_q a} p_m(x; a|q) p_n(x; a|q) d_q x \\ &= \sum_{k=0}^\infty \frac{(aq)^k}{(q; q)_k} p_m(q^k; a|q) p_n(q^k; a|q) = a^n q^{n^2} \frac{(q; q)_n}{(aq^{n+1}; q)_\infty} \delta_{mn}, \quad m, n \in \mathbb{N}_0. \end{aligned}$$

(iii) *Generalized monic little  $q$ -Laguerre polynomials (Definition 4.1)*

Let  $a \in \mathbb{R}$ . For each  $n \in \mathbb{N}_0$  and each  $x \in \mathbb{R}$ , we define the  $n$ th degree generalized monic little  $q$ -Laguerre polynomial  $p_n(\cdot; a|q)$  by

$$p_n(x; a|q) = \sum_{k=0}^n (-1)^n \begin{bmatrix} n \\ k \end{bmatrix}_q (aq^{k+1}; q)_{n-k} q^{(n-k)(n-k-1)/2} (-x)^k.$$

(iv) *Three term recurrence relation (Proposition 4.1)*

For  $n \in \mathbb{N}_0$  and  $a \in \mathbb{R}$ ,

$$p_{n+1}(x; a|q) = \left( x - A_n^{(a; q)} \right) p_n(x; a|q) - B_n^{(a; q)} p_{n-1}(x; a|q),$$

where  $A_n^{(a; q)} = q^n(1 + a - aq^n(1 + q))$  and  $B_n^{(a; q)} = aq^{2n-1}(1 - q^n)(1 - aq^n)$ . When  $a \in \{q^{-1}, q^{-2}, \dots\} \cup \{0\}$ , no orthogonality results can be deduced from Favard’s theorem.

(v) *Iterated forward shift operator (Corollary 4.1)*

For  $n \in \mathbb{N}_0$ ,  $0 \leq k \leq n + 1$ , and  $a \in \mathbb{R}$ ,

$$D_q^k p_n(x; a|q) = \frac{(q^{n-k+1}; q)_k}{(1 - q)^k} p_{n-k}(x; aq^k|q).$$

(vi) *Factorization (Proposition 4.3)*

For a fixed  $N \in \mathbb{N}$ , and for each  $n \geq N$ ,

$$p_n(x; q^{-N}|q) = x^N p_{n-N}(x; q^N|q).$$

(vii) *0 as a  $q$ -zero of order  $N$  (Corollary 4.2)*

Let  $N$  be a fixed positive integer. For  $n \geq N$ ,

$$D_q^k p_n(0; q^{-N}|q) = 0, \quad 0 \leq k \leq N - 1.$$

(viii)  *$q$ -Sobolev orthogonality (Theorem 5.1).*

**Theorem 6.2.** For each positive integer  $N$ , there exists a symmetric and positive definite matrix  $A$ , of order  $N$ , such that the family of generalized monic little  $q$ -Laguerre polynomials  $\{p_n(\cdot; q^{-N}|q)\}_{n=0}^\infty$  is orthogonal with respect to the inner product  $(\cdot, \cdot)_q^{(N;A)}$  defined by

$$\begin{aligned} (p_1, p_2)_q^{(N;A)} &= \left( p_1(0), D_q p_1(0), \dots, D_q^{N-1} p_1(0) \right) A \\ &\quad \times \left( p_2(0), D_q p_2(0), \dots, D_q^{N-1} p_2(0) \right)^t \\ &\quad + \sum_{k=0}^\infty \frac{q^k}{(q; q)_k} \left( D_q^N p_1(q^k) \right) \left( D_q^N p_2(q^k) \right), \quad p_1, p_2 \in \mathbb{P}. \end{aligned} \tag{6.24}$$

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**Appendix. More about factorization results**

In this section we will give some factorization properties for the families of generalized monic big and little  $q$ -Jacobi polynomials, which are closely related to those we have stated above (Propositions 2.3 and 4.3). For some of these results we will need the relation  $P_n(x; a, b, 0; q) = (aq)^n p_n((aq)^{-1}x; b, a|q)$ , that can be rewritten also in the form  $p_n(x; a, b|q) = (bq)^{-n} P_n(bqx; b, a, 0; q)$  (for a different normalization, in the classical setting, see [15, p. 74]).

Using the relation above between big and little monic  $q$ -Jacobi polynomials, and also Proposition 4.3, we can give a factorization result for the monic big  $q$ -Jacobi polynomials when their second parameter equals  $q^{-N}$  and their third parameter vanishes.

**Proposition A.1.** For a fixed  $N \in \mathbb{N}$  and  $a \in \mathbb{R} \setminus \{q^{N-2}, q^{N-3}, \dots\}$ , we have

$$P_n(x; a, q^{-N}, 0; q) = x^N P_{n-N}(x; a, q^N, 0; q), \quad n \geq N.$$

Very similar calculations to those in the proof of Proposition 2.3 led us to the following result, in which we establish the third factorization result for the generalized monic big  $q$ -Jacobi polynomials, now when their third parameter equals  $q^{-N}$ .

**Proposition A.2.** For a fixed  $N \in \mathbb{N}$  and  $a \in \mathbb{R}$  we have, for each  $b \in \mathbb{R}$  such that  $ab \notin \{q^{-2}, q^{-3}, \dots\}$ , and for each  $n \geq N$ ,

$$P_n(x; a, b, q^{-N}; q) = (-1)^N q^{N(N+1-2n)/2} (x; q)_N P_{n-N}(q^N x; aq^N, bq^N, q^N; q).$$

Suitable adaptations of Corollaries 2.2 and 2.3 give us:

**Corollary A.1.** Let  $N$  be a fixed positive integer, let  $a$  be an arbitrary real number, and let  $b$  be a real number such that  $ab \notin \{q^{-2}, q^{-3}, \dots\}$ . For  $n \geq N$  and  $0 \leq k \leq j \leq N - 1$ ,

$$\begin{aligned} P_n(q^{-j}; a, b, q^{-N}; q) &= 0, \\ D_q^k P_n(q^{-j}; a, b, q^{-N}; q) &= 0. \end{aligned}$$

We can also give a factorization result for the monic little  $q$ -Jacobi polynomials, similar to that in Proposition 4.3, but now when their second parameter equals  $q^{-N}$ .

**Proposition A.3.** For a fixed  $N \in \mathbb{N}$  and for  $a \in \mathbb{R} \setminus \{q^{N-2}, q^{N-3}, \dots\}$ , we have

$$p_n(x; a, q^{-N}|q) = (-1)^N q^{-N(N+1-2n)/2} (q^{1-N}x; q)_N p_{n-N}(q^{-N}x; a, q^N|q), \quad n \geq N.$$

**Proof.** Using the relation between big and little monic  $q$ -Jacobi polynomials and Proposition 2.3, we get

$$\begin{aligned} p_n(x; a, q^{-N}|q) &= q^{(N-1)n} P_n(q^{1-N}x; q^{-N}, a, 0; q) \\ &= q^{(N-1)n} (-1)^N q^{N(N+1-2n)/2} (q^{1-N}x; q)_N P_{n-N}(qx; q^N, a, 0; q) \\ &= (-1)^N q^{-N(N+1-2n)/2} (q^{1-N}x; q)_N p_{n-N}(q^{-N}x; a, q^N|q), \quad n \geq N. \quad \square \end{aligned}$$

In closing, we note that from Proposition A.2 we have a new result for the generalized monic big  $q$ -Laguerre polynomials, similar to (6.22) and (6.23), that reads:

**Proposition A.4.** For a fixed  $N \in \mathbb{N}$ , for all reals  $a$ , and for  $n \geq N$ ,

$$\begin{aligned} P_n(x; a, q^{-N}; q) &= (-1)^N q^{N(N+1-2n)/2} (x; q)_N P_{n-N}(q^N x; a q^N, q^N; q), \\ D_q^k P_n(q^{-j}; a, q^{-N}; q) &= 0, \quad 0 \leq k \leq j \leq N-1. \end{aligned}$$

## References

- [1] M. Alfaro, M. Álvarez de Morales, M.L. Rezola, Orthogonality of the Jacobi polynomials with negative integer parameters, *J. Comput. Appl. Math.* 145 (2002) 379–386.
- [2] M. Alfaro, T.E. Pérez, M.A. Piñar, M.L. Rezola, Sobolev orthogonal polynomials: The discrete–continuous case, *Methods Appl. Anal.* 6 (1999) 593–616.
- [3] M. Álvarez de Morales, T.E. Pérez, M.A. Piñar, Sobolev orthogonality for the Gegenbauer polynomials  $\{C_n^{(-N+1/2)}\}_{n \geq 0}$ , *J. Comput. Appl. Math.* 100 (1998) 111–120.
- [4] M. Álvarez de Morales, T.E. Pérez, M.A. Piñar, A. Ronveaux, Non-standard orthogonality for Meixner polynomials, *ETNA, Electron. Trans. Numer. Anal.* 9 (1999) 1–25.
- [5] G.E. Andrews, R. Askey, Enumeration of partitions: The role of eulerian series and  $q$ -orthogonal polynomials, in: M. Aigner (Ed.), *Higher Combinatorics*, Reidel, Boston, MA, 1977, pp. 3–26.
- [6] G.E. Andrews, R. Askey, Classical orthogonal polynomials, in: *Polynômes orthogonaux et applications*, in: *Lecture Notes in Math.*, vol. 1171, Springer, Berlin, New York, 1985, pp. 36–62.
- [7] T.K. Araaya, The Meixner–Pollaczek polynomials and a system of orthogonal polynomials in a strip, *J. Comput. Appl. Math.* 170 (2004) 241–254.
- [8] T.K. Araaya, The symmetric Meixner–Pollaczek polynomials with real parameter, *J. Math. Anal. Appl.* 305 (2005) 411–423.
- [9] N.M. Atakishiyev, A.U. Klimyk, On discrete  $q$ -ultraspherical polynomials and their duals, *J. Math. Anal. Appl.* 306 (2005) 637–645.
- [10] R.S. Costas-Santos, J.F. Sánchez-Lara, Extensions of discrete classical orthogonal polynomials beyond the orthogonality, *J. Comput. Appl. Math.* 225 (2009) 440–451.
- [11] D. Dominici, Some remarks on a paper by L. Carlitz, *J. Comput. Appl. Math.* 198 (2007) 129–142.
- [12] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2nd ed., Cambridge Univ. Press, 2004.
- [13] W. Hahn, Über Orthogonalpolynome die  $q$ -Differenzgleichungen genügen, *Math. Nachr.* 2 (1949) 4–34.
- [14] J. Koekoek, R. Koekoek, A note on the  $q$ -derivative operator, *J. Math. Anal. Appl.* 176 (1993) 627–634.
- [15] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its  $q$ -analogue, Technical Report 98–17, Delft University of Technology, 1998.
- [16] K.H. Kwon, L.L. Littlejohn, The orthogonality of the Laguerre polynomials  $\{L_n^{-k}(x)\}$  for positive integers  $k$ , *Ann. Numer. Math.* 2 (1995) 289–303.

- [17] K.H. Kwon, L.L. Littlejohn, Sobolev orthogonal polynomials and second-order differential equations, *Rocky Mountain J. Math.* 28 (1998) 547–594.
- [18] Samuel G. Moreno, E.M. García-Caballero, Linear interpolation and Sobolev orthogonality, *J. Approx. Theory* (2008) doi:10.1016/j.jat.2008.08.005.
- [19] Samuel G. Moreno, E.M. García-Caballero, The orthogonality of the Meixner–Pollaczek polynomials  $\{P_n^{(\frac{1-N}{2})}(\cdot; \phi)\}_{n=0}^{\infty}$  for positive integers  $N$  (submitted for publication).
- [20] Samuel G. Moreno, E.M. García-Caballero, Non-standard orthogonality for the little  $q$ -Laguerre polynomials, *Appl. Math. Lett.* (in press).
- [21] T.E. Pérez, M.A. Piñar, On Sobolev orthogonality for the generalized Laguerre polynomials, *J. Approx. Theory* 86 (1996) 278–285.