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GENERALIZED FIBONACCI AND LUCAS SEQUENCES AND ROOTFINDING METHODS

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Dedicated to the memory of D. H. Lehmer

ABSTRACT. Consider the sequences $\{u_n\}$ and $\{v_n\}$ generated by $u_{n+1} = pu_n - qu_{n-1}$ and $v_{n+1} = pv_n - qv_{n-1}$, $n \geq 1$, where $u_0 = 0$, $u_1 = 1$, $v_0 = 2$, $v_1 = p$, with p and q real and nonzero. The Fibonacci sequence and the Lucas sequence are special cases of $\{u_n\}$ and $\{v_n\}$, respectively. Define $r_n = u_{n+d}/u_n$, $R_n = v_{n+d}/v_n$, where d is a positive integer. McCabe and Phillips showed that for $d = 1$, applying one step of Aitken acceleration to any appropriate triple of elements of $\{r_n\}$ yields another element of $\{r_n\}$. They also proved for $d = 1$ that if a step of the Newton-Raphson method or the secant method is applied to elements of $\{r_n\}$ in solving the characteristic equation $x^2 - px + q = 0$, then the result is an element of $\{r_n\}$.

The above results are obtained for $d > 1$. It is shown that if any of the above methods is applied to elements of $\{R_n\}$, then the result is an element of $\{r_n\}$. The application of certain higher-order iterative procedures, such as Halley's method, to elements of $\{r_n\}$ and $\{R_n\}$ is also investigated.

Fibonacci and Lucas numbers appear repeatedly in the works of the father of computational number theory, D. H. Lehmer, who contributed also to numerical analysis, notably [5]. To his memory is dedicated this extension of results of McCabe and Phillips [6] and Jamieson [4] about applying iterative formulas for solving nonlinear equations to ratios of generalized Fibonacci numbers.

1. INTRODUCTION

Let p and q be real and nonzero. Define the generalized Fibonacci sequence

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = pu_n - qu_{n-1}, \qquad n \geq 1, \tag{1.1}$$

and the generalized Lucas sequence

$$v_0 = 2, \quad v_1 = p, \quad v_{n+1} = pv_n - qv_{n-1}, \qquad n \geq 1. \tag{1.2}$$

Let d be a natural number. If $u_n \neq 0$, define the ratio

$$r_n = u_{n+d}/u_n. \tag{1.3}$$

If $v_n \neq 0$, define the ratio

$$R_n = v_{n+d}/v_n. \tag{1.4}$$

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Related to the recurrence relation appearing in (1.1) and (1.2) is the characteristic equation

$$(1.5) \qquad x^2 - px + q = 0.$$

If the equation has two real and unequal roots, then when $d = 1$, the sequences of ratios $\{r_n\}$ and $\{R_n\}$ converge to the root of larger modulus. If there is a double root, then the sequences $\{r_n\}$ and $\{R_n\}$ converge to this root. McCabe and Phillips determined the condition for a generalized Fibonacci sequence to have no zero members; a necessary condition is that equation (1.5) have complex roots ([6, p. 554]). Their analysis can be adapted readily to generalized Lucas numbers, by Lemma 3 below.

If α and β are the roots of (1.5), then they satisfy ([3, equation (1.4)])

$$(1.6) \qquad \alpha + \beta = p, \quad \alpha\beta = q, \quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = p^2 - 4q.$$

If $\alpha = \beta$, then

$$(1.7) \qquad 2\alpha = p, \quad \alpha^2 = q = (p/2)^2, \quad p^2 - 4q = 4\alpha^2 - 4\alpha^2 = 0.$$

Lemma 1 ([3, equations (2.6), (2.7)]). *If α and β are the distinct roots of (1.5) and $n \geq 0$, then*

$$u_n = (\alpha^n - \beta^n)/(\alpha - \beta) \quad \textit{and} \quad v_n = \alpha^n + \beta^n.$$

Lemma 2. *If α is the double root of (1.5) and $n \geq 0$, then $u_n = n(p/2)^{n-1}$ and $v_n = 2(p/2)^n$.*

If $d \geq 1$, and the roots of (1.5) are real, then the sequences of ratios $\{r_n = u_{n+d}/u_n\}$ and $\{R_n = v_{n+d}/v_n\}$ will converge to the dth power of a root of (1.5). In other words, the sequences of ratios $\{r_n\}$ and $\{R_n\}$ converge to a root of

$$(1.8) \qquad x^2 - (\alpha^d + \beta^d)x + (\alpha\beta)^d = x^2 - v_d x + q^d = 0,$$

by Lemmas 1 and 2 and (1.6) and (1.7).

Define the Aitken transformation by

$$(1.9) \qquad A(x, x', x'') = (xx'' - x'^2)/(x - 2x' + x'').$$

Define the secant transformation $S(x, x')$ for equation (1.8) by

$$(1.10) \qquad S(x, x') = \frac{x(x'^2 - v_d x' + q^d) - x'(x^2 - v_d x + q^d)}{(x'^2 - v_d x' + q^d) - (x^2 - v_d x + q^d)} = \frac{xx' - q^d}{x + x' - v_d},$$

and the Newton-Raphson transformation $N(x)$ for equation (1.8) by

$$(1.11) \qquad N(x) = x - (x^2 - v_d x + q^d)/(2x - v_d) = (x^2 - q^d)/(2x - v_d).$$

McCabe and Phillips proved that, if $d = 1$, then

(i) $A(r_{n-t}, r_n, r_{n+t}) = r_{2n}$ if $r_{2n} \neq 0$,
(ii) $S(r_n, r_m) = r_{n+m}$ if $r_{n+m} \neq 0$,
(iii) $N(r_n) = r_{2n}$ if $r_{2n} \neq 0$.

It is now possible to state the extensions. As long as division by zero is avoided, then

(i) $A(r_{n-t}, r_n, r_{n+t}) = r_{2n}$, $A(R_{n-t}, R_n, R_{n+t}) = r_{2n}$,
(ii) $S(r_n, r_m) = r_{n+m}$, $S(R_n, R_m) = r_{n+m}$,
(iii) $N(r_n) = r_{2n}$, $N(R_n) = r_{2n}$,

for any natural number d. The idea of considering $d > 1$ is due to Jamieson [4], who applied it only to the ordinary Fibonacci sequence.

The other extension is to apply the Halley transformation $H(x)$, which is a third-order refinement of the Newton-Raphson transformation:

$$H(r_n) = r_{3n}, \qquad H(R_n) = R_{3n}.$$

Note that in the latter case the image is a ratio of generalized Lucas numbers. The Newton-Raphson and Halley transformations are two members of a certain infinite family of transformations; proofs applicable to the infinite family will be given.

Applying any of these transformations to elements of the sequence $\{R_n\}$, where (1.5) has a double root α, gives rise to division by zero. In this situation $R_n = (p/2)^d = \alpha^d$ for every $n \geq 1$; i.e., R_n is the root of (1.8), by Lemma 2 and (1.7). In this case the ratios are constant, so the sequence is trivial. In the sequel the transformations will be applied to R_n under the assumption that (1.5) has distinct roots.

Section 2 contains a list of elementary relationships about generalized Fibonacci and Lucas numbers. In §3 the Aitken transformation is studied. Section 4 is devoted to the secant transformation. Section 5 begins with the presentation of the Halley transformation. Then an infinite family of transformations, which includes those of Newton-Raphson and Halley, is investigated.

2. Properties of generalized Fibonacci and Lucas numbers

For $n > 0$ define $v_{-n} = \alpha^{-n} + \beta^{-n}$. Then by (1.6) and Lemma 1,

$$q^n v_{-n} = (\alpha\beta)^n v_{-n} = \beta^n + \alpha^n = v_n. \tag{2.1}$$

Similarly, if equation (1.5) has distinct roots, define $u_{-n} = (\alpha^{-n} - \beta^{-n})/(\alpha - \beta)$. Then by (1.6) and Lemma 1 ([3, equation (2.17)])

$$q^n u_{-n} = (\alpha\beta)^n u_{-n} = (\beta^n - \alpha^n)/(\alpha - \beta) = -u_n. \tag{2.2}$$

Formula (2.2) is applicable also if equation (1.5) has a double root, for if u_{-n} is defined by $-n(p/2)^{-n-1}$, then $q^n u_{-n} = -n(p/2)^{-n-1}(p/2)^{2n} = -n(p/2)^{n-1} = -u_n$.

It is easy to verify that the recurrence relations in (1.1) and (1.2) are valid also for negative subscripts.

Lemma 3 ([3, equation (4.10)]). *If n is an integer, then $u_{2n} = u_n v_n$.*

Lemma 4. *If n, m, and e are integers, then*

(a) $u_{n+e}u_{n-e} - u_n^2 = -q^{n-e}u_e^2$,
(b) $u_{n+e}u_m - u_n u_{m+e} = -q^m u_e u_{n-m}$,
(c) $u_{n+e}u_{m+e} - q^e u_n u_m = u_e u_{n+m+e}$,
(d) $u_{n+e} - q^e u_{n-e} = v_n u_e$,
(e) $u_{n+e} - v_e u_n = -q^e u_{n-e}$.

On the right side of statements (a)–(d) of the following lemma, there appears the factor $p^2 - 4q$. If (1.5) has a double root, then $p^2 - 4q = 0$, by (1.7). It suffices to show in the case of a double root, accordingly, that the left side of each of these statements vanishes.

Lemma 5. *If* n, m, *and* e *are integers, then*
(a) $v_{n+e}v_{n-e} - v_n^2 = q^{n-e}(p^2 - 4q)u_e^2$,
(b) $v_{n+e}v_m - v_n v_{m+e} = q^m(p^2 - 4q)u_e u_{n-m}$,
(c) $v_{n+e}v_{m+e} - q^e v_n v_m = (p^2 - 4q)u_e u_{n+m+e}$,
(d) $v_{n+e} - q^e v_{n-e} = (p^2 - 4q)u_n u_e$,
(e) $v_{n+e} - v_e v_n = -q^e v_{n-e}$.

Lemma 6. *If* n, m, *and* e *are integers, then* $u_{n+e}v_m - u_n v_{m+e} = q^m u_e v_{n-m}$.

Lemma 7 ([3, equation (4.13)]). *If* n *is an integer, then* $u_n(v_n^2 - q^n) = u_{3n}$.

3. The Aitken transformation

Theorem 1. *Let* $n > t \geq 0$ *be integers, and assume that division by zero does not occur. Then* (A) $A(r_{n-t}, r_n, r_{n+t}) = r_{2n}$; (B) *if equation* (1.5) *has distinct roots, then* $A(R_{n-t}, R_n, R_{n+t}) = r_{2n}$.

Proof. We prove only part (A). The proof of part (B) is similar. By (1.3) and (1.9),

$$\begin{aligned} A(r_{n-t}, r_n, r_{n+t}) &= \frac{r_{n-t}r_{n+t} - r_n^2}{r_{n-t} - 2r_n + r_{n+t}} \\ &= \frac{(u_{n-t+d}/u_{n-t})(u_{n+t+d}/u_{n+t}) - (u_{n+d}/u_n)^2}{u_{n-t+d}/u_{n-t} - 2u_{n+d}/u_n + u_{n+t+d}/u_{n+t}} \\ &= \frac{u_{n-t+d}u_{n+t+d}u_n^2 - u_{n-t}u_{n+t}u_{n+d}^2}{u_n[u_{n-t+d}u_n u_{n+t} - 2u_{n+d}u_{n-t}u_{n+t} + u_{n+t+d}u_{n-t}u_n]} \\ &= \frac{(u_{n-t+d}u_{n+t+d} - u_{n+d}^2)u_n^2 - (u_{n-t}u_{n+t} - u_n^2)u_{n+d}^2}{u_n[(u_{n-t+d}u_n - u_{n+d}u_{n-t})u_{n+t} - (u_{n+d}u_{n+t} - u_{n+t+d}u_n)u_{n-t}]} \\ &= \frac{-q^{n-t+d}u_t^2u_n^2 + q^{n-t}u_t^2u_{n+d}^2}{u_n u_d(q^{n-t}u_t u_{n+t} - q^n u_t u_{n-t})}, \end{aligned}$$

by Lemmas 4(a) and 4(b),

$$= \frac{u_t(u_{n+d}^2 - q^d u_n^2)}{u_n u_d(u_{n+t} - q^t u_{n-t})} = \frac{u_t u_d u_{2n+d}}{u_n u_d v_n u_t},$$

by Lemmas 4(c) and 4(d),

$$= u_{2n+d}/u_{2n} = r_{2n},$$

by Lemma 3 and then (1.3). □

4. The secant transformation

Theorem 2. *Let* n *and* m *be positive integers, and assume that division by zero does not occur. Then* (A) $S(r_n, r_m) = r_{n+m}$; (B) *if equation* (1.5) *has distinct roots, then* $S(R_n, R_m) = r_{n+m}$.

Proof. We prove only part (B). The proof of part (A) is similar. By (1.4) and (1.10),

$$S_d(R_n, R_m) = \frac{R_n R_m - q^d}{R_n + R_m - v_d} = \frac{(v_{n+d}/v_n)(v_{m+d}/v_m) - q^d}{v_{n+d}/v_n + v_{m+d}/v_m - v_d}$$

$$= \frac{v_{n+d}v_{m+d} - q^d v_n v_m}{v_{n+d}v_m + v_n(v_{m+d} - v_d v_m)} = \frac{(p^2-4q)u_d u_{n+m+d}}{v_{n+d}v_m - q^d v_n v_{m-d}},$$

by Lemmas 5(c) and 5(e),

$$= \frac{(p^2-4q)u_d u_{n+m+d}}{(p^2-4q)u_d u_{n+m}} = \frac{u_{n+m+d}}{u_{n+m}} = r_{n+m},$$

by Lemma 5(c) and then (1.3). □

5. The Newton-Raphson and Halley transformations

The Halley transformation for the equation $f(x) = 0$ is given by ([1, p. 131])

$$H(x) = x - f(x)/[f'(x) - f(x)f''(x)/2f'(x)].$$

Applying the Halley transformation to equation (1.8) yields

$$\begin{aligned} H(x) &= x - \frac{x^2 - v_d x + q^d}{(2x - v_d) - (x^2 - v_d x + q^d)/(2x - v_d)} \\ &= \frac{x^3 - 3q^d x + v_d q^d}{3x^2 - 3v_d x + v_d^2 - q^d}. \end{aligned} \tag{5.1}$$

An infinite family of transformations, which includes those of Newton-Raphson and Halley, will now be investigated. To this end, define the homogeneous polynomials in y and z by

$$u_d^h q^{-f} T_{h,f,d}(y, z) = -\sum_{k=0}^{h} \binom{h}{k} (-y)^k z^{h-k} u_{dk-f}. \tag{5.2}$$

Lemma 8. *For* $i = 0, 1, 2, \ldots, h$ *define*

$$E(i) = u_d^i q^{it} \sum_{k=0}^{h-i} \binom{h-i}{k} (-u_t)^k u_{t+d}^{h-i-k} u_{dk-f-it}.$$

Then $E(i)$ *is independent of* i.

Proof. It suffices to show that if $0 \le i \le h-1$, then $E(i) = E(i+1)$. By definition, $\binom{j}{k} = 0$ if $k < 0$ or $k > j$. Thus

$$\begin{aligned} E(i) &= u_d^i q^{it} \sum_{k=0}^{h-i} \left[\binom{h-i-1}{k} + \binom{h-i-1}{k-1} \right] (-u_t)^k u_{t+d}^{h-i-k} u_{dk-f-it} \\ &= u_d^i q^{it} \sum_{k=0}^{h-i-1} \binom{h-i-1}{k} (-u_t)^k u_{t+d}^{h-i-k} u_{dk-f-it} \\ &\quad + u_d^i q^{it} \sum_{j=0}^{h-i-1} \binom{h-i-1}{j} (-u_t)^{j+1} u_{t+d}^{h-i-j-1} u_{dj+d-f-it} \\ &= u_d^i q^{it} \sum_{k=0}^{h-i-1} \binom{h-i-1}{k} (-u_t)^k u_{t+d}^{h-i-k-1} (u_{t+d} u_{dk-f-it} - u_t u_{dk+d-f-it}) \end{aligned}$$

$$= u_d^{i+1} q^{(i+1)t} \sum_{k=0}^{h-i-1} \binom{h-i-1}{k} (-u_t)^k u_{t+d}^{h-i-1-k} u_{dk-f-(i+1)t},$$

by Lemma 4(b),

$$= E(i+1). \quad \square$$

Theorem 3. *If* $u_d \neq 0$, *then* $T_{h,f,d}(u_t, u_{t+d}) = u_{ht+f}$.

Proof. By Lemma 8,

$$u_d^h q^{-f} T_{h,f,d}(u_t, u_{t+d}) = -E(0) = -E(h) = -u_d^h q^{ht} u_{-ht-f}.$$

By (2.2),

$$T_{h,f,d}(u_t, u_{t+d}) = u_{ht+f}. \quad \square$$

Lemma 9. *For* $0 \le i \le h$, i *even, define*

$$F(i) = u_d^i q^{it} \sum_{k=0}^{h-i} \binom{h-i}{k} (-v_t)^k v_{t+d}^{h-i-k} u_{dk-f-it}.$$

For $0 < s \le h$, s *odd, define*

$$G(s) = -u_d^s q^{st} \sum_{k=0}^{h-s} \binom{h-s}{k} (-v_t)^k v_{t+d}^{h-s-k} v_{dk-f-st}.$$

Then $F(i) = G(i+1)$ *if* $i < h$, *and* $G(i+1) = (p^2 - 4q)F(i+2)$ *if* $i < h-1$.

Proof. We have

$$\begin{aligned} F(i) &= u_d^i q^{it} \sum_{k=0}^{h-i} \left[\binom{h-i-1}{k} + \binom{h-i-1}{k-1} \right] (-v_t)^k v_{t+d}^{h-i-k} u_{dk-f-it} \\ &= u_d^i q^{it} \sum_{k=0}^{h-i-1} \binom{h-i-1}{k} (-v_t)^k v_{t+d}^{h-i-k-1} (v_{t+d} u_{dk-f-it} - v_t u_{dk+d-f-it}) \\ &= -u_d^{i+1} q^{(i+1)t} \sum_{k=0}^{h-i-1} \binom{h-i-1}{k} (-v_t)^k v_{t+d}^{h-i-k-1} v_{dk-f-(i+1)t}, \end{aligned}$$

by Lemma 6,

$$= G(i+1).$$

Continuing,

$$\begin{aligned} G(i+1) &= -u_d^{i+1} q^{(i+1)t} \sum_{k=0}^{h-i-1} \left[\binom{h-i-2}{k} + \binom{h-i-2}{k-1} \right] (-v_t)^k v_{t+d}^{h-i-k-1} v_{dk-f-(i+1)t} \\ &= -u_d^{i+1} q^{(i+1)t} \sum_{k=0}^{h-i-2} \binom{h-i-2}{k} (-v_t)^k v_{t+d}^{h-i-k-2} (v_{t+d} v_{dk-f-(i+1)t} - v_t v_{dk+d-f-(i+1)t}) \\ &= u_d^{i+2} q^{(i+2)t} (p^2 - 4q) \sum_{k=0}^{h-i-2} \binom{h-i-2}{k} (-v_t)^k v_{t+d}^{h-i-k-2} v_{dk-f-(i+2)t}, \end{aligned}$$

by Lemma 5(b),

$$= (p^2 - 4q)F(i+2). \quad \square$$

Theorem 4. *Assume* $u_d \neq 0$. *If* h *is even, then*

$$T_{h,f,d}(v_t, v_{t+d}) = (p^2 - 4q)^{h/2} u_{ht+f}.$$

If h *is odd, then*

$$T_{h,f,d}(v_t, v_{t+d}) = (p^2 - 4q)^{(h-1)/2} v_{ht+f}.$$

Proof. Apply Lemma 9 $[h/2]$ times:

If h is even, then

$$\begin{aligned} u_d^h q^{-f} T_{h,f,d}(v_t, v_{t+d}) &= -F(0) = -(p^2-4q)F(2) = -(p^2-4q)^2 F(4) \\ &= \cdots = -(p^2-4q)^{h/2}F(h) = -u_d^h q^{ht}(p^2-4q)^{h/2} u_{-ht-f}. \end{aligned}$$

By (2.2), $T_{h,f,d}(v_t, v_{t+d}) = (p^2-4q)^{h/2} u_{ht+f}$.

If h is odd, then

$$\begin{aligned} u_d^h q^{-f} T_{h,f,d}(v_t, v_{t+d}) &= -F(0) = -(p^2-4q)F(2) \\ &= \cdots = -(p^2-4q)^{(h-1)/2}F(h-1) \\ &= -(p^2-4q)^{(h-1)/2}G(h) = (p^2-4q)^{(h-1)/2} u_d^h q^{ht} v_{-ht-f}. \end{aligned}$$

By (2.1), $T_{h,f,d}(v_t, v_{t+d}) = (p^2-4q)^{(h-1)/2} v_{ht+f}$. $\square$

Define

$$g_h(z/y) = \frac{-q^d \sum_{k=0}^{h} \binom{h}{k} \left(\frac{z}{-y}\right)^{h-k} u_{d(k-1)}}{-\sum_{k=0}^{h} \binom{h}{k} \left(\frac{z}{-y}\right)^{h-k} u_{dk}}.$$

Multiply the numerator and the denominator of the fraction by $u_d^{-h}(-y)^h$:

$$(5.3) \qquad g_h(z/y) = \frac{-u_d^{-h} q^d \sum_{k=0}^{h} \binom{h}{k} (-y)^k z^{h-k} u_{d(k-1)}}{-u_d^{-h} \sum_{k=0}^{h} \binom{h}{k} (-y)^k z^{h-k} u_{dk}} = \frac{T_{h,d,d}(y,z)}{T_{h,0,d}(y,z)}.$$

The immediate consequences of Theorems 3 and 4 are:

Theorem 5. (a) *Assume that* $u_d \neq 0$ *and* $u_{ht} \neq 0$. *Then* $g_h(u_{t+d}/u_t) = u_{ht+d}/u_{ht}$.

(b) *Assume that* $u_d \neq 0$, $v_t \neq 0$, *and* $v_{ht} \neq 0$. *Then*

$$g_h(v_{t+d}/v_t) = \begin{cases} u_{ht+d}/u_{ht}, & h \text{ even}, \\ v_{ht+d}/v_{ht}, & h \text{ odd}. \end{cases}$$

Theorem 6. *If* n *is a positive integer, and division by zero does not occur, then* $N(r_n) = N(R_n) = r_{2n}$.

Proof. In view of Theorem 5, it suffices to show that $g_2(z/y) = N(z/y)$, where $N(x)$ is given by equation (1.11). By (5.3),

$$g_2(z/y) = \frac{-q^d(z^2 u_{-d} + y^2 u_d)}{-(-2yzu_d + y^2 u_{2d})} = \frac{z^2 u_d - q^d y^2 u_d}{2yzu_d - y^2 u_d v_d},$$

by (2.2) and Lemma 3,

$$= \frac{(z/y)^2 - q^d}{2z/y - v_d} = N(z/y). \quad \square$$

Theorem 7. *If* n *is a positive integer, and division by zero does not occur, then* $H(r_n) = r_{3n}$ *and* $H(R_n) = R_{3n}$.

Proof. In view of Theorem 5, it suffices to show that $g_3(z/y) = H(z/y)$, where $H(x)$ is given by equation (5.1). By (5.3),

$$\begin{aligned} g_3(z/y) &= \frac{-q^d(z^3u_{-d} + 3y^2zu_d - y^3u_{2d})}{-(-3yz^2u_d + 3y^2zu_{2d} - y^3u_{3d})} \\ &= \frac{z^3u_d - 3y^2zq^du_d + y^3q^du_dv_d}{3yz^2u_d - 3y^2zu_dv_d + y^3u_d(v_d^2 - q^d)}, \end{aligned}$$

by (2.2), Lemma 3, and Lemma 7,

$$= \frac{(z/y)^3 - 3q^d(z/y) + q^dv_d}{3(z/y)^2 - 3(z/y)v_d + v_d^2 - q^d} = H(z/y). \quad \square$$

Remark. Theorem 3, with $f = 0$ and $d = 1$, resembles a formula given by H. Siebeck, cited in [2, p. 394].

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